WEIGHTED GREEN FUNCTIONS OF POLYNOMIAL SKEW PRODUCTS ON $\mathbb{C}^2$

KOHEI UENO

In this report we consider the dynamics of polynomial skew products on $\mathbb{C}^2$. We introduce the weighted Green function of a polynomial skew product f, a generalization of the Green function of f. Moreover, we consider the dynamics of the extension of f to a holomorphic map on the weighted projective space.

1. Introduction

We consider the dynamics of a polynomial skew product on $\mathbb{C}^2$ of the form $f(z,w) = (p(z), q(z,w))$, where $p(z)$ and $q(z,w)$ are polynomials such that $p(z) = z^d + O(z^{d-1})$ and $q(z,w) = w^d + O_z(w^{d-1})$. We assume that $d \geq 2$. Main topics in this report are an investigation into the existence of the Green function of f, which measures the rate of convergence of points to infinity, and an introduction of a generalized Green function of f which suits to the dynamics of f. Moreover, we extend f to a holomorphic map on the weighted projective space, and exhibit a relation between the dynamics of the extension of f and the generalized Green function of f.

There is a nice class of polynomial skew products, which is called the class of regular polynomial skew products. Many researches have studied the dynamics of regular polynomial skew products (e.g. [1] and [2]). We say that a polynomial map is regular if it extends to a holomorphic map on the projective space $\mathbb{P}^2$, or equivalently, if it extends to a non-degenerate homogeneous map on $\mathbb{C}^3$. The polynomial skew product f above is regular if and only if the algebraic degree of f is d. Let f be a regular polynomial skew product of degree d and f^n the n-th iterate of f. It is known that the Green function G_f of f,

$$G_f(z,w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z,w)|,$$

where $\log^+ = \max\{\log, 0\}$ and $|(z,w)| = \max\{|z|,|w|\}$, is well behaved on $\mathbb{C}^2$, that is, it is well-defined, continuous and plurisubharmonic on $\mathbb{C}^2$. Moreover, it coincides with the pluricomplex Green function of K_f with pole at infinity, where $K_f = \{\{f^n(z,w)\}_{n\geq 1} \text{ bounded}\}$.

Our results indicate that similar things hold for polynomial skew products without the assumption of regularity. In addition, our results are fruitful even for regular polynomial skew products. Let f be a polynomial skew product. We give a region where G_f or, more precisely, the vertical Green function of f is well behaved, and examples of polynomial skew products whose Green functions are not well behaved on $\mathbb{C}^2$ in Section 2. Our main idea to deal with the dynamics of f is putting some weight determined by f to the first or second coordinate. Using the weight above, we introduce a generalization of G_f which is well behaved on $\mathbb{C}^2$ and coincides with G_f on some region in Section 3. We call this generalized Green function of f the weighted Green function of f. Again, by using the same weight above, it follows that f extends to a holomorphic map on the weighted projective space. We show that the Fatou and Julia sets of the extension of f are determined by the weighted Green function of f in Section 4.

2. Dynamics of polynomial skew products

Let $f(z, w) = (p(z), q(z, w))$ be a polynomial skew product such that $p(z) = z^d + O(z^{d-1})$ and $q(z, w) = w^d + O_z(w^{d-1})$. We assume that $d \geq 2$, and consider the existence of the following function:

$$G_f(z, w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z, w)|.$$

In general, the algebraic degree of f may be grater than d. However, the dynamical degree of f, $\lim_{n\to\infty} \sqrt[n]{\deg(f^n)}$, is equal to d. Here $\deg f$ denotes the algebraic degree of f.

To consider the existence of G_f, let us recall the dynamics of polynomial skew products. Roughly speaking, the dynamics of polynomial skew products consists of the dynamics on the base space and the dynamics on the vertical lines. The first component p defines the dynamics on the base space $\mathbb{C}$. Define $K_p = \{z : \{p^n(z)\}_{n\geq 1} \text{ bounded}\}$. Note that f preserves the set of vertical lines in $\mathbb{C}^2$. In this sense, we often use the notation $q_z(w)$ instead of $q(z, w)$. The restriction of f^n to a line $\{z\} \times \mathbb{C}$ can be viewed as the composition of n polynomials on $\mathbb{C}$, $q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$.

A useful tool in the study of the dynamics of p on the base space is the Green function G_p of p, defined by

$$G_p(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |p^n(z)|.$$

It is known that G_p is well behaved on $\mathbb{C}$, that is, it is well-defined, continuous and subharmonic on $\mathbb{C}$. More precisely, G_p is harmonic and positive on $\mathbb{C} - K_p$ and zero on K_p, and $G_p(z) = \log |z| + o(1)$ as

z tends to infinity. By definition, $G_p(p(z)) = dG_p(z)$. Note that G_p coincides with the Green function of K_p with a pole at infinity, which is determined only by the compact set K_p. In a similar fashion, we consider the vertical Green function $G_z(w)$ of f,

$$G_z(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |Q_z^n(w)|,$$

where $Q_z^n(w) = q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z(w)$. Since G_p is well behaved on $\mathbb{C}$, the existence of $G_z(w)$ implies that of G_f.

Now we give the definition of the weight of f and a region where $G_z(w)$ is well behaved. We define the most important number k in this report as

$$\min \left\{ l \in \mathbb{R} \;\middle|\; \begin{array}{l} ld \geq n_j + lm_j \text{ for any integers } n_j \text{ and } m_j \text{ s.t.} \\ c_j z^{n_j} w^{m_j} \text{ is a term in } q(z,w) \text{ for some } c_j \neq 0 \end{array} \right\}$$

if $\deg_z q > 0$ and as 1 if $\deg_z q = 0$. Since q has only finitely many terms, one can take the minimum. Indeed, k is equal to

$$\max \left\{ \frac{n_j}{d - m_j} \;\middle|\; \begin{array}{l} c_j z^{n_j} w^{m_j} \text{ is a term in } q(z,w), \\ \text{with } c_j \neq 0, \text{ which is not } w^d \end{array} \right\}.$$

Thus k is a rational number. Note that f is regular if and only if $k \leq 1$. The explanation of the weight k is as follows. If f is regular, then w^d is a term of highest degree in q. Generally, if we define the weight of a monomial $z^n w^m$ as $n + km$, then w^d is a term of highest weight in q. It follows that $k \leq \deg_z q$ and $k < \deg q$ from the definition of k. Moreover, if $k \geq 1$, then $d^n \leq \deg(f^n) \leq kd^n$ for any integer n.

Let $W_R = \{|w| > R|z|^k, |w| > R^{k+1}\}$ and $A_f = \bigcup_{n=0}^{\infty} f^{-n}(W_R)$ for large $R > 0$. Then f preserves W_R and $G_z(w)$ is well behaved on A_f.

Theorem 2.1. *The vertical Green function $G_z(w)$ is well-defined, continuous and pluriharmonic on A_f. Moreover, $G_z(w)$ tends to $kG_p(z)$ as (z,w) in A_f tends to ∂A_f.*

Hence G_f is also well behaved on A_f. A proof of the first statement in this theorem is similar to the proof of Lemma 2.3 in [3].

Remark 2.2. *Even if f is regular, the theorem above is fruitful. Because G_f is well behaved on $\mathbb{C}^2$, it is well known that $G_z(w)$ is well behaved on $\{G_f(z,w) > G_p(z)\}$. On the other hand, it follows that $G_z(w)$ is well behaved on $\{G_f(z,w) > kG_p(z)\}$ from Theorem 2.1. More precisely, $G_z(w)$ is pluriharmonic on the region above. If $k < 1$, then our region is bigger than the well known region.*

Now we know the existence of $G_z(w)$ and G_f on A_f. However, these functions may not exist on $\mathbb{C}^2 - A_f$. Indeed, we give polynomial skew products whose Green functions are not well behaved on $\mathbb{C}^2$, which are semiconjugate to polynomial products. One may be able to calculate the Green functions of such polynomial skew products using an analogy of the Green functions of polynomials.

Example 2.3. *Let* $f(z,w) = (z^2, w^2 + cz^4)$*. Then* f *is semiconjugate to a polynomial product* $(z^2, w^2 + c)$ *by* $\pi(z,w) = (z, z^2 w)$*. Hence* $Q_z^n(w) = z^{2^{n+1}} q_c^n(z^{-2}w)$*, where* $q_c(w) = w^2 + c$*. Let* $\tilde{G}_c(w) = \lim_{n\to\infty} 2^{-n} \log |q_c^n(w)|$*. If* 0 *is a periodic point of period* $l > 1$*, then* $\tilde{G}_c$ *does not converge on* $\{0, q_c(0), \cdots, q_c^{l-1}(0)\}$*. Hence* $G_z(w)$ *and* G_f *are not well-defined on the curves* $\{w = q_c^j(0)z^2, |z| > 1\}_{j=0}^{l-1}$*.*

Example 2.4. *Let* $f(z,w) = (z^2, w^2 + z^2 w)$*. Then* f *is semiconjugate to a polynomial product* $(z^2, w^2 + w)$ *by* $\pi(z,w) = (z, z^2 w)$*. Hence* $Q_z^n(w) = z^{2^{n+1}} q^n(z^{-2}w)$*, where* $q(w) = w^2 + w$*. Let* $\tilde{G}_q(w) = \lim_{n\to\infty} 2^{-n} \log |q^n(w)|$*. Note that* 0 *is a parabolic fixed point of* q*, and* $\tilde{G}$ *is not continuous at* 0*. Hence* $G_z(w)$ *and* G_f *are not continuous on the curve* $\{w = 0, |z| > 1\}$*.*

3. Weighted Green functions

For polynomial skew product $f(z,w) = (p(z), q(z,w))$ as before, we consider the following function:

$$G_f^k(z,w) = \begin{cases} G_z(w) & \text{on } A_f, \\ kG_p(z) & \text{on } \mathbb{C}^2 - A_f. \end{cases}$$

It follows that this function is well behaved on $\mathbb{C}^2$ from Theorem 2.1. The explanation of the function above is as follows. In the case when f is regular, it is known that G_f is well behaved on $\mathbb{C}^2$. However, $G_z(w)$ may not exist on $\mathbb{C}^2 - A_f$. Roughly speaking, $G_f(z,w)$ is the maximum of $G_p(z)$ and $G_z(w)$. Thus, G_p hides the region where $G_z(w)$ may not be well behaved. We do a similar thing for f which may not be regular: hide the region where $G_z(w)$ may not be well behaved with $kG_p(z)$. Note that

$$G_f^k(z,w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z,w)|_k,$$

where $|(z,w)|_k = \max\{|z|^k, |w|\}$. Hence the function G_f^k is a generalization of the Green function of f, and we call this function the weighted Green function of f.

Corollary 3.1. *The weighted Green function G_f^k is well-defined, continuous and plurisubharmonic on $\mathbb{C}^2$. More precisely, G_f^k is pluriharmonic and positive on A_f and on $\mathbb{C}^2 - \overline{K_f \cup A_f}$, and zero on K_f. The convergence to G_f^k is uniform on $\mathbb{C}^2$, and $G_f^k(f(z,w)) = dG_f^k(z,w)$.*

The dynamics of f near infinity is as follows. Let h be the weighted homogeneous part of q that contains w^d, that is, h is the polynomial consisting of all terms of highest weight kd in q. First, let us consider the case when k is an integer. Put $w = cz^k$, then $h(z, cz^k) = h(1, c)z^{kd}$. For any fixed c, $q(z, cz^k)$ is a polynomial in z. By letting $h(c) = h(1, c)$, it follows that $h(c)z^{kd}$ is the homogeneous part of $q(z, cz^k)$ of degree kd. Note that $h^n(c)z^{kd^n}$ is the homogeneous part of $Q_z^n(cz^k)$ of degree kd^n. We expect that the dynamics of $Q_z^n(cz^k)$ should be controlled by that of $h^n(c)z^{kd^n}$ as z tends to infinity. If k is not an integer, then z^k is not a well-defined function. However, we get the following estimate.

Proposition 3.2. $G_f^k(z,w) = \log|(z,w)|_k + \rho_h(z^{-k}w) + o(1)$ *as* (z,w) *tends to infinity, where* $\rho_h(c) = G_h(c) - \log^+|c|$.

If k is not an integer, then the polynomial h and the Green function G_h of h have some symmetries with respect to the denominator of k. Hence $\rho_h(z^{-k}w)$ is well-defined.

4. Dynamics on weighted projective spaces

Let f be the polynomial skew product as before. As we saw in Section 3, if we put $w = cz^k$ for a fixed number c, then $Q_z^n(cz^k) \sim h^n(c)z^{kd^n}$ as z tends to infinity. We want to compactify $\mathbb{C}^2$ so that each curve $\{(z, cz^k) : z \in \mathbb{C}\}$ converges to one point as z tends to infinity. The required space is the weighted projective space $\mathbb{P}(r, s, 1)$, where r and s are the denominator and numerator of k respectively. The weighted projective space $\mathbb{P}(r, s, 1)$ is a quotient space of $\mathbb{C}^3 - \{0\}$,

$$\mathbb{P}(r, s, 1) = \mathbb{C}^3 - \{0\} \ / \sim,$$

where $(z, w, t) \sim (\lambda^r z, \lambda^s w, \lambda t)$ for any λ in $\mathbb{C} - \{0\}$. We insist that $f(z,w) = (p(z), q(z,w))$ extends to a holomorphic map $\tilde{f}$ on $\mathbb{P}(r, s, 1)$,

$$\tilde{f}[z : w : t] = \left[p\left(\frac{z}{t^r}\right) t^{dr} : q\left(\frac{z}{t^r}, \frac{w}{t^s}\right) t^{ds} : t^d \right],$$

or equivalently, it extends to a non-degenerate weighted homogeneous map F on $\mathbb{C}^3$,

$$F(z, w, t) = \left(p\left(\frac{z}{t^r}\right) t^{dr}, q\left(\frac{z}{t^r}, \frac{w}{t^s}\right) t^{ds}, t^d \right).$$

Proposition 4.1. *The extension $\tilde{f}$ is holomorphic on $\mathbb{P}(r, s, 1)$.*

We define the Fatou set of $\tilde{f}$ as the maximal open set of $\mathbb{P}(r,s,1)$ where the iterates $\{\tilde{f}^n\}$ is a normal family. The Julia set of $\tilde{f}$ is defined by the complement of the Fatou set of $\tilde{f}$. We insist that the weighted Green function of f determines the Fatou and Julia sets of $\tilde{f}$. Recall that $A_f = \{G_f^k > kG_p\}$ and $K_f = \{G_f^k = 0\}$. Define $B_f = \mathbb{C}^2 - (K_f \cup A_f)$. Note that $B_f = \{G_f^k = kG_p > 0\}$. Let $\tilde{K_f} = \text{int}K_f$, $\tilde{A_f} = \text{int}\overline{A_f}$ and $\tilde{B_f} = \text{int}\overline{B_f}$, where the closures are taken in $\mathbb{P}(r,s,1)$.

Theorem 4.2. *The Fatou set of $\tilde{f}$ coincides with the union of $\tilde{K_f}$, $\tilde{A_f}$ and $\tilde{B_f}$. In other words, The Julia set of $\tilde{f}$ coincides with the closure of the set where G_f^k is not pluriharmonic.*

The dynamics of $\tilde{f}$ on its Fatou set is as follows. The dynamics of $\tilde{f}$ on $\tilde{K_f}$ is induced by the dynamics of f on K_f. Since the fixed point $p = [0:1:0]$ is superattracting, $\tilde{A_f}$ is the attracting basin of p. Since the line at infinity is attracting, any point in $\tilde{B_f}$ is attracted to the line at infinity by iterates of $\tilde{f}$. Finally, the dynamics on the line at infinity, which is induced by the weighted homogeneous part h of q, should determine the dynamics on $\tilde{B_f}$.

References

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Department of Mathematics, Kyoto University

E-mail address: `kueno@math.kyoto-u.ac.jp`