THE GEOMETRY OF THE SPACE OF BRODY CURVES AND MEAN DIMENSION

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Let $z = x + \sqrt{-1}y \in \mathbb{C}$ be the standard coordinate of the complex plane $\mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic map. We define a norm $|df|(z) \geq 0$ at each point $z \in \mathbb{C}$ by setting

$$|df|^2(z) := \frac{1}{4\pi}\Delta \log(|f_0|^2 + |f_1|^2 + \cdots + |f_N|^2), \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

where $f = [f_0 : f_1 : \cdots : f_N]$ (each f_i is a holomorphic function). We call f a Brody curve if it satisfies $|df|(z) \leq 1$ for all $z \in \mathbb{C}$. Let $\mathcal{M}(\mathbb{C}P^N)$ be the space of all Brody curves in $\mathbb{C}P^N$. We consider the compact open topology on $\mathcal{M}(\mathbb{C}P^N)$. Then $\mathcal{M}(\mathbb{C}P^N)$ becomes a compact metrizable infinite dimensional space. It admits the following natural $\mathbb{C}$-action:

$$\mathcal{M}(\mathbb{C}P^N) \times \mathbb{C} \to \mathcal{M}(\mathbb{C}P^N), \quad (f(z), a) \mapsto f(z+a).$$

Then we can consider its mean dimension $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$. Mean dimension is a notion defined by Gromov [2]. Intuitively,

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) = \frac{\dim \mathcal{M}(\mathbb{C}P^N)}{\mathrm{vol}(\mathbb{C})}.$$

For a Brody curve $f : \mathbb{C} \to \mathbb{C}P^N$, we define the Shimizu-Ahlfors characteristic function $T(r, f)$ by setting

$$T(r, f) := \int_1^r \frac{dt}{t} \int_{|z| \leq t} |df|^2 dxdy \quad (r \geq 1).$$

Since $|df| \leq 1$, we have $T(r, f) \leq \pi r^2/2$. We set

$$e(f) := \limsup_{r\to\infty} \frac{2}{\pi r^2} T(r, f) \in [0, 1].$$

We define $e(\mathbb{C}P^N)$ as the supremum of $e(f)$ over $f \in \mathcal{M}(\mathbb{C}P^N)$. We have $0 \leq e(\mathbb{C}P^N) \leq 1$ by the definition. Actually we can prove ([5, 7])

$$0 < e(\mathbb{C}P^N) < 1.$$

We call $f \in \mathcal{M}(\mathbb{C}P^N)$ an elliptic Brody curve if there exists a lattice $\Lambda \subset \mathbb{C}$ such that $f(z + \lambda) = f(z)$ for all $\lambda \in \Lambda$. Let $e(\mathbb{C}P^N)_{ell}$ be the supremum of $e(f)$ over all elliptic Brody curves f in $\mathbb{C}P^N$. We have $0 < e(\mathbb{C}P^N)_{ell} \leq e(\mathbb{C}P^N) < 1$ and we can prove ([6, 7])

$$\lim_{N\to\infty} e(\mathbb{C}P^N)_{ell} = \lim_{N\to\infty} e(\mathbb{C}P^N) = 1.$$

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The main result of this talk is the following theorem ([6]).

Theorem 1.

$$2(N+1)e(\mathbb{C}P^N)_{ell} \le \dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \le 4Ne(\mathbb{C}P^N).$$

When $N = 1$, this becomes

$$4e(\mathbb{C}P^1)_{ell} \le \dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) \le 4e(\mathbb{C}P^1).$$

From this we propose the following conjecture.

Conjecture 2.

$$e(\mathbb{C}P^1)_{ell} = e(\mathbb{C}P^1).$$

If this is true, then we get

$$\dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = 4e(\mathbb{C}P^1).$$

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