Equidistribution speed for endomorphisms of projective spaces

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Abstract

Let f be a non-invertible holomorphic endomorphism of $\mathbb{P}^k$, f^n its iterate of order n and μ the equilibrium measure of f. We estimate the speed of convergence in the following known result. If a is a Zariski generic point in $\mathbb{P}^k$, the probability measures, equidistributed on the preimages of a under f^n, converge to μ as n goes to infinity. We give a similar result for preimages of analytic sets of arbitrary codimension, but in that case the result holds only for a Zariski dense set of holomorphic endomorphisms of $\mathbb{P}^k$. This uses the theory of Super-Potentials introduced by the authors.

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Let $\mathbb{P}^k$ denote the complex projective space of dimension k. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. The map f defines a ramified covering of degree d^k over $\mathbb{P}^k$. Let $\omega_{\rm FS}$ denote the Fubini-Study form on $\mathbb{P}^k$ normalized so that $\omega_{\rm FS}^k$ is a probability measure. Let $f^n := f \circ \cdots \circ f$ (n times) be the iterate of order n of f. It is well-known that $d^{-kn}(f^n)^*(\omega_{\rm FS}^k)$ converge to a probability measure μ which is totally invariant: $d^{-k}f^*(\mu) = f_*(\mu) = \mu$. The measure μ is called *the equilibrium measure*. We refer to the surveys [9, 18] for the basic dynamical properties of such maps.

Consider a point a in $\mathbb{P}^k$. We are interested on the asymptotic distribution of the fibers $f^{-n}(a)$ of a when n goes to infinity. More precisely, if δ_a denotes the Dirac mass at a, $d^{-kn}(f^n)^*(\delta_a)$ is the probability measure which is equidistributed on the fiber $f^{-n}(a)$. The points in $f^{-n}(a)$ are counted with multiplicities. The following results are proved in two articles by the authors [8] and [10].

Theorem 1. *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let μ be the equilibrium measure of f and $1 < \lambda < d$ be a constant. There is* *an invariant proper analytic subset E_λ, possibly empty, of $\mathbb{P}^k$ such that if a is a point out of E_λ and if φ is a $\mathcal{C}^\alpha$ function on $\mathbb{P}^k$ with $0 < \alpha \leq 2$, then*

$$|\langle d^{-kn}(f^n)^*(\delta_a) - \mu, \varphi\rangle| \leq A\Big[1 + \log^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\Big]^{\alpha/2} \|\varphi\|_{\mathcal{C}^\alpha} \lambda^{-\alpha n/2},$$

where $A > 0$ is a constant independent of n, a, φ and $\log^+(\cdot) := \max(0, \log(\cdot))$.

Note that the distance $\mathrm{dist}(a, E_\lambda)$ is with respect to the Fubini-Study metric on $\mathbb{P}^k$. When E_λ is empty, by convention, this distance is the diameter of $\mathbb{P}^k$ which is a finite number. A priori, the constant A depends on λ and α. Note also that we have the estimate

$$A\Big[1 + \log^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\Big]^{\alpha/2} \leq A\Big[1 + \log^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\Big]$$

for $0 < \alpha \leq 2$.

Let $\mathcal{H}_d(\mathbb{P}^k)$ denote the set of holomorphic endomorphisms of algebraic degree d of $\mathbb{P}^k$. We can identify it with a Zariski open set of a projective space. It is shown in [8, Lemma 5.4.5], that for f in a dense Zariski open set $\mathcal{H}_d^\lambda(\mathbb{P}^k)$ in $\mathcal{H}_d(\mathbb{P}^k)$, the multiplicities of points in the fibers of f^{n_0} are smaller than $d^{n_0}\lambda^{-n_0}$ for some $n_0 \geq 1$. In this case, E_λ is empty. The following consequence of Theorem 1 gives a precise version of [8, prop. 5.4.13].

Corollary 2. *Let $1 < \lambda < d$ be a constant and f an element of $\mathcal{H}_d^\lambda(\mathbb{P}^k)$. Then for every a in $\mathbb{P}^k$ and for φ a $\mathcal{C}^\alpha$ function on $\mathbb{P}^k$ with $0 < \alpha \leq 2$, we have*

$$|\langle d^{-kn}(f^n)^*(\delta_a) - \mu, \varphi\rangle| \leq A\|\varphi\|_{\mathcal{C}^\alpha} \lambda^{-\alpha n/2},$$

where $A > 0$ is a constant independent of n, a and φ.

Recall that for f in $\mathcal{H}_d(\mathbb{P}^k)$, there is a maximal proper analytic subset $\mathcal{E}$ of $\mathbb{P}^k$ which is totally invariant under f, i.e. $f^{-1}(\mathcal{E}) = f(\mathcal{E}) = \mathcal{E}$.

Theorem 3. *Let f, μ and $\mathcal{E}$ be as above. There is a constant $\lambda > 1$ such that if a is a point out of $\mathcal{E}$ and if φ is a $\mathcal{C}^\alpha$ function on $\mathbb{P}^k$ with $0 < \alpha \leq 2$, then*

$$|\langle d^{-kn}(f^n)^*(\delta_a) - \mu, \varphi\rangle| \leq A\Big[1 + \log^+ \frac{1}{\mathrm{dist}(a, \mathcal{E})}\Big]^{\alpha/2} \|\varphi\|_{\mathcal{C}^\alpha} \lambda^{-\alpha n/2},$$

where $A > 0$ is a constant independent of n, a and φ. In particular, $d^{-kn}(f^n)^*(\delta_a)$ converge to μ locally uniformly for $a \in \mathbb{P}^k \setminus \mathcal{E}$.

The following result is a direct consequence of Theorem 3.

Corollary 4. *Let f, μ and $\mathcal{E}$ be as above. Then $d^{-kn}(f^n)^*(\delta_a)$ converge to μ if and only if $a \notin \mathcal{E}$.*

Note that in dimension 1, this result was proved by Brolin [3] for polynomials, by Lyubich [17] and Freire-Lopes-Mañé [16] for general maps. A version of Theorem 3 for points in the support of μ, was proved by Drasin-Okuyama in [11] using Nevanlinna theory. See also Denker-Przytycki-Urbański [4], Favre and Rivera-Letelier [12], Fornæss-Sibony [14] and the references therein.

In higher dimension, a partial result in this direction was obtained by Fornæss and the second author in [14]. It shows the equidistribution property for a outside a set of zero Lebesgue measure (a pluripolar set). Corollary 4 was announced by Briend-Duval in [1]. Their proof shows the equidistribution property for a outside a countable union of hypersurfaces (the orbit of the critical values of f). The first complete proof of this corollary was given by the authors in [6], see also [7].

Our proof there is valid for a much more general setting (polynomial-like maps and maps on singular varieties) and is separated into two parts. The first one improves a geometrical method due to Lyubich in dimension 1 and developed by Briend-Duval [1] in higher dimension. The second part, quite different from the dimension 1 case, shows the existence of an analytic exceptional set $\mathcal{E}$ and some extra properties which are useful in the proof. This exceptional set is still not well understood, its study seems to require new ideas. Very recently, Briend-Duval [2] showed that using a dynamical argument, as in our work [5, 6, 7], one can obtain a short proof of Corollary 4, see also [5, Prop. 2.4] for a more general setting.

In this paper, we will use known properties of the exceptional set but we will replace the geometrical method with a pluripotential one which seems to be more powerful. The strategy was already introduced by Fornæss and the second author in [15] for the equidistribution of hypersurfaces. In [8], we showed that this strategy can be extended to the case of varieties of arbitrary dimension, in particular, for the equidistribution of points that we consider. The novelty here is that we obtain precise quantitative results. We can obtain some results on the speed of convergence for analytic sets of codimension p, and even for currents. The result uses heavily the theory of superpotentials developped by the authors. Consider a holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree $d \geq 2$. Recall that f^* acts continuously on positive closed currents of any bidegree. It is well-known that $d^{-n}(f^n)^*(\omega)$ converge to a positive closed $(1,1)$-current T with Hölder continuous quasi-potentials. One deduces from the intersection theory of currents that $d^{-pn}(f^n)^*(\omega^p)$ converge to T^p, see [13, 18] for the first stages of the theory. The current T^p is *the Green current of order* p and its super-potentials are *the Green super-functions of order* p of f. The following result is proved in [8].

Theorem 5. *Let $1 < \lambda < d$ be a constant. There is an open Zariski dense set $\mathcal{H}_d^\lambda(\mathbb{P}^k)$ in $\mathcal{H}_d(\mathbb{P}^k)$ such that given f in $\mathcal{H}_d^\lambda(\mathbb{P}^k)$, we have the following property. For any $0 \leq \alpha \leq 2$, there is a constant $c > 0$ such that if S is a positive closed (p,p)-current of mass 1 and Φ is a test $(k-p, k-p)$-form of class $\mathcal{C}^\alpha$, then*

$$|\langle d^{-pn}(f^n)^*(S) - T^p, \Phi\rangle| \leq c\lambda^{-n\alpha/2}\|\Phi\|_{\mathcal{C}^\alpha}.$$

In particular, if φ is a $\mathcal{C}^\alpha$ function such that $\langle \mu, \varphi \rangle = 0$, then

$$\|d^{-kn}(f^n)_*(\varphi)\|_\infty \leq c\lambda^{-n\alpha/2}\|\varphi\|_{\mathcal{C}^\alpha}.$$

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