# Proceedings of Hayama Symposium on Complex Analysis in Several Variables 2005

December 18—21

Shonan Village

# Preface

This proceedings contains summaries of the papers which were lectured at (the 10th) Hayama Symposium on Complex Analysis in Several Variables 2005, held at Shonan Village Center, Japan. Although the purpose of the symposium is to recognize and communicate the main ideas in recent activity in several complex variables without selecting a special topic, each year has some tendency here. This time, there were fewer aspects towards algebraic geometry compared to the usual year. An explanation for that is that there was held another workshop a week ago, specialized to the Bergman kernel, and 20 people had attended it among the whole 73 participants in the Hayama conference.

This symposium was mainly supported by Grand-in-Aid for Scientific Research in 2005:

(S) 17104001 represented by Junjiro Noguchi (Tokyo University)

Organizers
A. Fujiki
A. Kodama
J. Noguchi
T. Ohsawa (management)
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# Complex Analysis in Several Variables 2005
## Shonan Village Center, December18-21

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# Compact non-kähler threefolds associated to hyperbolic 3-manifolds

Akira FUJIKI (Osaka University)

I would like to talk about a class of three dimensional non-kähler compact complex manifolds which are almost homogeneous with respect to the special linear group $SL_2(\mathbf{C})$. These manifolds are related to hyperbolic manifolds and Kleinian groups .

In general my interest is in finding methods for constructing compact complex manifolds which are non-kähler or, more generally which are not in class $\mathcal{C}$. (A compact complex manifold is said to be in class $\mathcal{C}$ if it is bimeromorphic to a compact Kähler manifold.) So far quite a few methods are known, but still very much in sporadic ways.

Now it is known for long time that a class of compact non-kähler manifolds are provided by homogeneous manifolds, especially a complex parallelizable manifold which is characterized by the following equivalent properties [5]:

(1) its (holomorphic) tangent bundle is trivial, and

(2) it is of the form $X = G/\Gamma$,

where $G$ is a complex Lie group and $\Gamma$ is a cocompact discrete subgroup of $G$. Suppose for example that $G$ is a simple linear Lie group. Then we can see that $G/\Gamma$ is not Kähler, or more strongly, is not in $\mathcal{C}$.

Given a compact complex manifold a way of constructing new manifolds is to consider its deformations. In the case under consideration, however, it is known by Raghunathan that $X$ is rigid under local deformations unless $G$ is locally isomorphic to $SL_2(\mathbf{C})$, while Ghys [1] has constructed the Kuranishi family of deformations of $X$ in the latter case, which is non-trivial for a general choice of the discrete group $\Gamma$.

We then ask an almost complex analogue for the above manifolds, namely we can ask if for some discrete group $\Gamma$ (which is not cocompact, but infinite), there exists an equivariant compactification $X$ of the quotient $G/\Gamma$ which is non-kähler, or is not in class $C$. So far no examples of such manifolds seem to have been known; in any case it is easy to see that the resulting manifolds satisfy the necessary non-kähler properties:

**Proposition 1.** *Any such $X$ is not in class $C$ and its Kodaira dimension $\kappa(X) = -\infty$.*

In this talk we shall show that such equivariant compactifications exist for a class of discrete subgroups of $G = PSL_2(\boldsymbol{C})$, and study some of the basic properties of these compactified manifolds. Recall that a discrete subgroup of $G := PSL_2(\boldsymbol{C})$ is called a *Kleinian group* and our manifolds should be related with real 3-dimensional hyperbolic manifolds.

Recall that $G$ is identified with the group of oritentation-preserving isometries of the hyperbolic upper-half space $H^3; G \cong Isom^+ H^3$, and $H^3$ becomes a homogeneous space of $G$; $H^3 \cong K \setminus G$, where $K = PSU(2)$. Now we take and fix a non-trivial torsionfree Kleinian group $\Gamma$. Then we have a commutative diagram of quotients

$$\begin{array}{ccc} U: & = & G/\Gamma \\ \downarrow & & \downarrow \\ M: & = & K \setminus G/\Gamma = H^3/\Gamma \end{array}$$

where $K = PSU(2)$, the vertical arrows are natural projections of our complex manifold $U$ to the 3-dimensional complete hyperbolic manifold $M$.

Here an observation is that often there exists a natural compactification of the base $M$. For instance for many knots in the 3-sphere $S^3$, its complement $M := S^3 - K$ is a complete hyperbolic manifold as above so that $M$ is compactified canonically to the 3-sphere $S^3$. One of the most typical one is the case where $K$ is the figure eight knot, in which case $\Gamma$ in $M = H^3/\Gamma$ is the arithmetic Kleinian group explicitly determined as the Bianchi group $PSL_2(O_3)$ associated to the maximal order $O_3$ of the real quadratic field $\boldsymbol{Q}(\sqrt{3})$ (Riley '82)[4]. Especially, $M$ is volume finite and $M$ has a unique cusp.

In any case one may ask if this compactification $M \subseteq N := S^3$ can be lifted to a (natural) equivariant compactification $U \subseteq X$. But this turns out to be impossible:

**Proposition 2**. Suppose that $G/\Gamma$ is volume finite and admits a cusp. Then $G/\Gamma$ admits no equivariant compactifications.

In view of this result we have to try another direction. $H^3$ admits a natural compactification $\bar{H}^3$ as a 3-manifold with boundary by adding to it the sphere at $\infty$, denoted by $bH^3$. Thus we have $\bar{H} = H \cup bH$ omitting the superscript. The action of $G$ extends naturally up to the boundary. Let $\Gamma$ be a finitely generated torsion-free Kleinian group. $\Gamma$ then admits the (maximal) domain of discoinituity $\Omega \subseteq bH^3$ on the boundary, which we assume to be non-empty. Then $M = H^3/\Gamma$ is partially compactified to a 3-manifold with boundary $N := M \cup (\Omega/\Gamma) = (H^3 \cup \Omega)/\Gamma$, called a *Kleinian manifold* (cf. [3]).

Then our basic results are as follows.

**Theorem** *The situation and the assumptions being as above suppose further that $N$ is compact, or equivalently, $C := \Omega/\Gamma$ is compact (in general disconnected). Then there exists an equivariant compactification $G/\Gamma \subseteq X$ fitting into the commutative diagram*

$$\begin{array}{ccc} G/\Gamma & \subseteq & X \\ \downarrow & & \downarrow \\ M & \subseteq & N \end{array}$$

*such that*

*1) the vertical maps are the quotient map by $K$,*

*2) $b_1(X) = b_1(M)$, and $b_2(X) = b_2(M)+$ the number of connected components of $C$, where $b_i$ denotes the i-th betti number.*

*3) $S := X - G/\Gamma$ is isomorphic to the product $C \times \boldsymbol{P}^1$, where $\boldsymbol{P}^1$ is the complex projective line.*

*4) $-K_X = 2[S]$ and $N_{S/X} = -K_S$, where $K_X$ and $N_{S/X}$ are the canonical bundle of $X$ and the normal bundle of $S$ in $X$ respectively.*

*5) there exists a four dimensional covering family of $\boldsymbol{P}^1$ with normal bundle $O(1) \oplus O(1)$; in fact $X$ is a manifold of class L in the sense of Kato [2].*

*6) the algebraic dimension $a(X) = 0$ unless $\Gamma$ is elementary, i.e., $\#(bH \setminus \Omega) \leqq 2$; in the latter case $\Gamma \cong Z$, $a(X) = 2$ and in fact $X$ is the twistor space of a diagonal Hopf surface.*

**Example.** 1) The case $\Gamma \subsetneq PSL_2(\boldsymbol{R}) \subsetneq PSL_2(\boldsymbol{C})$ is a cocompact Fuchsian group. In this case $\Omega = H^+ \coprod H^-$ and the universal covering $\tilde{X}$ of $X$ is the twistor space of $S^4 - S^1$ with the induced metric. Since $N$ is known to be homeomorphic to $C \times I$ [3] , we have $b_1(X) = 2g$ and $b_2(X) = 3$.

2) The case $\Gamma$ is the Schottoky group of rank $g \geqq 1$. In this case $N$ is the handlebody of genus $g$ [1] ; thus we have $b_1(X) = g$.

We shall also disucss the deformation problem for such manifolds in analogy with the results of Ghys [1]. It is interesting to ask if any higher dimensional analogue of the above manifolds.

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# EXPLICIT EXAMPLE OF MOISHEZON TWISTOR SPACES AND THEIR MINITWISTOR SPACES

NOBUHIRO HONDA

In this report I would like to explain our recent results in [2, 3, 4] about constructions and classifications of (mini)twistor spaces associated to self-dual metrics on $3\mathbf{CP}^2$, the connected sum of three complex projective planes. The main result is the following

**Theorem 1.** [2] *Let $g$ be a self-dual metric on $3\mathbf{CP}^2$ satisfying the following 3 conditons: (i) the scalar curvature of $g$ is positive, (ii) $g$ admits a non-zero Killing field, (iii) $g$ is not conformally isometric to self-dual metrics constructed by C. LeBrun in* [6]. *Then the twistor space of $g$ is obtained as a small resolution of a double covering of $\mathbf{CP}^3$ branched along certain singular quartic surface whose defining equation can be explicitly given (cf. below). Conversely, the complex 3-fold obtained by the above construction is always a twistor space of $3\mathbf{CP}^2$ and the corresponding self-dual metric satisfies (i), (ii) and (iii).*

The equation of the branch quartic surface is explicitly given by

$$\{y_2y_3 + Q(y_0, y_1)\}^2 = y_0y_1(y_0 + y_1)(y_0 - \alpha y_1), \tag{1}$$

where $(y_0, y_1, y_2, y_3)$ is a homogeneous coordinate on $\mathbf{CP}^3$, $Q(y_0, y_1)$ is a homogeneous quardratic polynomial with real coefficients, and $\alpha$ is a positive real number. Moreover, $Q$ and $\alpha$ satisfy the condition that $Q(y_0, y_1)^2 - y_0y_1(y_0 + y_1)(y_0 - \alpha y_1)$ has a unique double root which is a real number. Under these conditions, the quartic surace (1) becomes birational to an elliptic ruled surface and has just 3 isolated singular points which are two simple elliptic singularities (of type $\tilde{E}_7$) and one ordinary double point. Correspondingly, there are a lot of small resolutions of the double covering. It is possible to give a small resolution explicitly which actually yields a twistor space. We remark that the twistor spaces do not admit a Kähler metric (by a theorem of Hitchin [1]), so that the projectivity is lost through small resolution of the double covering.

The 'converse part' of Theorem 1 means that the quartic surface (1) naturally determines a self-dual metrics on $3\mathbf{CP}^2$. This enable us to determine a global structure of a moduli space of self-dual metrics on $3\mathbf{CP}^2$ as follows:

**Corollary 2.** *Let* $\mathscr{M}$ *be the set of all conformal classes* $[g]$ *on* $3\mathbf{CP}^2$*, where* $g$ *is a self-dual metric on* $3\mathbf{CP}^2$ *satisfying (i), (ii) and (iii) of Theorem 1. Then* $\mathscr{M}$ *can be naturally identified with* $\mathbf{R}^3/G$*, where* $G$ *is a reflection of* $\mathbf{R}^3$ *having 2-dimensional fixed locus. In particular,* $\mathscr{M}$ *is non-empty and connected.*

Let us explain backgrounds related to these results. In general, it is known that if $[g]$ is a self-dual conformal class on a compact 4-manifold $M$, and if the scalar curvature of $[g]$ is of positive type, then $M$ must be homeomorphic to $S^4$ or $n\mathbf{CP}^2$ for some $n \geq 1$. It is known that for $S^4$ and $\mathbf{CP}^2$, the standard metrics are unique self-dual structure respectively (although strictly speaking, for $\mathbf{CP}^2$, one has to suppose that the scalar curvature is positive.) In a cerebrated work [7] Y. S. Poon constructed a family of self-dual metrics on $2\mathbf{CP}^2$ of positive scalar curvature and also showed that every such metrics $2\mathbf{CP}^2$ belongs to his family. For $n \geq 3$, LeBrun [6] and Joyce [5] constructed families of self-dual metrics on $n\mathbf{CP}^2$ with posibive scalar curvature, for arbitrary $n$. Significant feature of their metrics is that, they admit a semi-free $U(1)$-isometry for LeBrun metrics, and $U(1) \times U(1)$-isometry for Joyce metrics. Moreover, these properties characterizes their metrics respectively.

However, it is readily seen (by deformation theory applied to the twistor space) that for any $n \geq 3$ there are many self-dual metrics on $n\mathbf{CP}^2$ which are different from LeBrun or Joyce metrics. Corollary 2 classifies such self-dual metrics on $3\mathbf{CP}^2$, under the condition that they admit a non-trivial Killing field. We remark that the existence of such metric is never trivial.

A relation between our new self-dual metrics and LeBrun or Joyce metrics on $3\mathbf{CP}^2$ is as follows:

**Theorem 3.** [3] *Our self-dual metrics on* $3\mathbf{CP}^2$ *obtained in Theorem 1 can be smoothly deformed into LeBrun metrics via Joyce metrics, where the self-duality and the existence of a non-zero Killing field are kept through deformations. In other words, the moduli space of all self-dual conformal classes satisfying (i) and (ii) in Theorem 1 is connected.*

Thus a global picture of the moduli space of all self-dual metrics on $3\mathbf{CP}^2$ with a non-zero Killing field (and with positive scalar curvature) became well understood. We remark that when our self-dual metric is deformed into LeBrun metric, a Killing field (or generated $U(1)$-action) must be exchanged when passing through a Joyce metric (namely, a LeBrun metric with torus action). We also remark that the main result of [3] determines all $U(1)$-subgroups of the torus for which one can obtain equvariant deformations of LeBrun metric with torus action, for arbitrary $n\mathbf{CP}^2$. Moreover, the dimensions of the moduli space of such new metrics with $U(1)$-action are also calculated.

Next we explain a result in [4] which describes the structure of minitwistor spaces associated to our twistor space in Theorem 1. For this, let $\Sigma_2$ be the Hirzebruch surface of degree 2 and $\overline{\Sigma}_2$ the surface obtained from $\Sigma_2$ by contracting the $(-2)$-section of the ruling. (So $\overline{\Sigma}_2$ has a unique ordinary node.)

**Theorem 4.** [4] *Let $\mathscr{T}$ be the minitwistor space of the twistor space in Theorem 1, which is by definition a quotient space of the twistor space by the $\mathbf{C}^*$-action, where the action is the one coming from the Killing field. Then $\mathscr{T}$ has a structure of the double covering of $\overline{\Sigma}_2$ branched along a smooth elliptic curve that is an anticanonical curve of $\overline{\Sigma}_2$ not going through the node. Moreover, general minitwistor line (namely the image of twistor lines by the quotient map) are anticanonical curves of $\mathscr{T}$ which has a unique ordinary node.*

We mention that the period of the branch elliptic curve is the same as the period of the elliptic curve of the branch quartic elliptic ruled surface. (So it is determined by $\alpha$ in (1).)

Theorem 4 shows that the structure of our minitwistor space is quite different from that of LeBrun metrics. Namely, for LeBrun metrics, the minitwistor space is $\mathbf{CP}^1 \times \mathbf{CP}^1$ and general minitwistor lines are curves of bidegree $(1,1)$; in particular, they are smooth.

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# An application of the Hamiltonian flow to the $\overline{\partial}$ equation

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This paper is a series of our study of the mixed Hodge structure (Rumin complex) for the case hypersurface isolated singularities. We take a complex euclidean space $\mathbf{C}^{n+1}$, and take a holomorphic function $f$, which satisfies : $df(p) \neq 0$, if $p$ is not the origin. And consider analytic space $V = \{z : z \in \mathbf{C}^{n+1},\ f(z) = 0\}$. This isolated singularity is well studied by several mathematians, but from the point of view of CR structures, even the case hypersurface isolated singularities is not well understood. About 10 years ago, we were trying to obtain a CR analogy for smoothness of the versal deformation of complex structures(at that time, Tian and Todorov gave a simple proof for smoothness of the versal family for *compact* Calabi-Yau manifolds). And we found that $\partial\overline{\partial}$ lemma is a quite important property(in the case compact Kaehler manifolds, this holds, but, otherwise, it is not valid). Concerning $\partial\overline{\partial}$ lemma, today we discuss; on

$$V_{a,b} = \{z : z \in \mathbf{C}^{n+1}, f(z) = 0, a <| z |< b\},$$

" Is the $(T'V_{a,b})^*$-valued Dolbeault cohomology, represented by the harmonic forms ?" . Namely, does the following isomorphism hold ?

$$H^1(V_{a,b}, \wedge^{n-1}(T'V_{a,b})^*) \simeq$$

$$\{\phi : \phi \in \Gamma(V_{a,b}, \wedge^{n-1}(T'V_{a,b})^* \bigwedge \wedge(T''V_{a,b})^*)), \overline{\partial}\phi = 0, \delta''\phi = 0\},$$

where $\delta''$ is the formal adjoint of $\overline{\partial}$ with respect to the Kaehler metric, induced by the standard Kaehler metric on $\mathbf{C}^{n+1}$. We must mention that in [O], in a general setting, by using the functional analysis method, it is discussed. But, in our case, $V_{a,b}$ is no longer strongly pseudo convex. Rather, here, we use the Hamiltonian flow, and the Euler vector field (we discuss only $A_l$ - singularities).

## 1 Geometrical meaning of $H^1(V_{a,b}, \wedge^{n-1}(T'V_{a,b})^*)$

$V_{a,b}$ is an open Calabi-Yau manifold(this means that our $V_{a,b}$ admits a non-vanishing holomorphic $(n, 0)$ form. So,

$$H^1(V_{a,b}, T'_{V_{a,b}}) \simeq H^1(V_{a,b}, \wedge^{n-1}(T'V_{a,b})^*).$$

And we know that the left hand side is the infinitesimal deformation space of complex structures over $V_{a,b}$. This versal family is explicitly written as follows. On $\mathbf{C}^{n+1} \times \mathbf{C}^l$,

$$\tilde{V} = \{(z, t), : F(z, t) = 0\},$$

where $F(z,t) = f(z) + \sum_{i=1}^{l} t_i g_i(z)$, $l = dim_{\mathbf{C}} H^1(V_{a,b}, T'_{V_{a,b}})$, and $\{g_i\}_{1 \le i \le l}$ means representatives of the moduli algebra,

$$\frac{\mathbf{C}[z_1, \ldots, z_{n+1}]}{(\frac{\partial f}{\partial z_i}, f)}.$$

Let $(\tilde{V}, \pi(z), S)$ be a family of deformations of complex structures of $V_{a,b}$. Here $S$ is an analytic space with the origin, and $\pi(z)$ is a smooth map from $\tilde{V}$ to $S$. So we have the Kodaira Spencer map

$$\rho_o : T_o S \to H^1(V_{a.b}, T'V_{a,b}) \tag{1.1}$$

Assume that there is a holomorphic $(n,0)$ form, $\omega$, which is non-vanishing on $V_{a,b}$, and it can be extended to $\tilde{V}$ holomorphically(we use the notation $\tilde{\omega}$ for this extension). In this situation, we write down the Kodaira-Spencer class. We take a $C^\infty$ diffeomorphism map from $V_{a,b} \times S$ to $\tilde{V}$,

$$\begin{array}{ccc} V_{a,b} \times S & \xrightarrow{i_s} & \tilde{V} \\ \text{the projection to the second factor} \downarrow & & \downarrow \pi(z) \\ S & \xrightarrow{\text{identity map}} & S \end{array}$$

Then, we have

**Theorem 1.**

$$\{\frac{\partial}{\partial s} i_s^*(\tilde{\omega} \mid_{X_s}) \mid_{s=o}\}^{(n-1,1)} = \omega \,\bar{\wedge}\, \rho_o((\frac{\partial}{\partial s})_o)$$

*Here,* $\{\frac{\partial}{\partial s} i_s^*(\tilde{\omega} \mid_{X_s}) \mid_{s=o}\}^{(n-1,1)}$ *means the* $(n-1,1)$ *part of* $\frac{\partial}{\partial s} i_s^*(\tilde{\omega} \mid_{V_s}) \mid_{s=o}$ *and* $\bar{\wedge}$ *means the inner product, and the equality means in* $H^1(V_{a,b}, \wedge^{n-1}(T'V_{a,b})^*)$.

In this note, for brevity, we take a constant function 1 in the moduli algebra, as a represetative (the other representatives, the same argument follows), and consider the corresponding family of deformations $\{\tilde{V}, \pi(z), \mathbf{C}\}$ of $V_{a,b}$, where $\tilde{V} = \mathbf{C}^{n+1}$, and $\pi(z) = f(z)$, $\pi^{-1}(t) = \{z : z \in \mathbf{C}^{n+1}, f(z) = t\}$. For this deformation, we have a corresponding Kodaira-Spencer class.By choosing, a proper $C^\infty$-trivialization and a holomorphic $(n,0)$ form, we look for the corresponding harmonic form. In Sect.2, we set $C^\infty$-trivialization, and in Sect.3, we discuss a holomorphic $(n,0)$ form.

## 2 $C^\infty$- trivialization

For the family, constructed in Sect.1, $\{\tilde{V}, \pi(z), \mathbf{C}\}$, we construct a $C^\infty$ trivialization, which preserves the standard induced Kaehler metric. First, we set a Hamiltonian vector field $X_f$(for the convenient, we take a $(1,0)$ part) by;

$$\Omega_{\mathbf{C}^{n+1}}(X_f, W) = d\overline{f}(W), \quad W \in T''\mathbf{C}^{n+1}.$$

This flow, generated by $X_f$, preserves $\Omega_{\mathbf{C}^{n+1}}$. Our $X_f$ is explicitly written as follows.

$$X_f = \sum_{i=1}^{n+1} \overline{(\frac{\partial f}{\partial z_i})} \frac{\partial}{\partial z_i}.$$

Set $X'_f = (\frac{1}{X_f f}) X_f$. This vector field makes sense out side of the origin. Now consider the flow, generated by $X'_f$. This means that: in mod $(t^2, \bar{t})$, we consider the $C^\infty$ map $i_t$;

$$z_i \to z_i + (X'_f z_i)t \quad \text{in mod } (t^2, \bar{t}).$$

Then,

$$f(z_i + (X'_f z_i)t) \equiv f(z_i) + t \quad \text{in mod } (t^2, \bar{t}).$$

Our $X'_f$ is not a Hamiltonian vector filed, but still satisfies

$$i_t^*(\Omega \mid_{V_t}) \equiv \Omega \mid_{V_{a,b}} \quad \text{mod} \quad (t^2, \bar{t}).$$

Here $V_t = \pi^{-1}(t)$.

# 3 Holomorphic $(n, 0)$ forms along the parameter space

The $C^\infty$ trivialization map $i_t$ is determined in Sect.2. In this section, we find a holomorphic $(n, o)$ form. Let $\omega' = X'_f \rfloor (dz_1 \wedge \cdots \wedge dz_{n+1})$. Then,

**Proposition 2.**

$$dz_1 \wedge \cdots \wedge dz_{n+1} = df \wedge \omega' \quad on \quad \mathbf{C}^{n+1} - o.$$

For the proof, see [AG ]. Especially. this proposition means that our $\omega'$ is holomorphic along the fiber. While, for any vector field of type $(1, 0)$( we write it by $Y$), satisfying $Yf = 0$ on $V_{a,b}$,

$$\omega'(Y) = (\frac{1}{(X_f + Y)f})(X_f + Y) \rfloor (dz_1 \wedge \cdots \wedge dz_{n+1})$$

also satisfies

$$dz_1 \wedge \cdots \wedge dz_{n+1} = df \wedge \omega'(Y) \quad \text{on a neighborhood of } V_{a,b} \text{ in } \quad \mathbf{C}^{n+1} - o.$$

Suppose that : there is a type $(1, 0)$ vector field $Y$ satsisfying $Yf = 0$, and and

$$\omega'(Y)_1 \text{ is purely of type } (n-1, 1), \tag{3.1}$$

where $\omega'(Y)_1$ is defined by;

$$i_t^* \omega'(Y) \equiv \omega(Y) + \omega'(Y)_1 t, \quad \text{mod} \quad (t^2, \bar{t}),$$

Then, as our $\omega'(Y)$ is $d$-closed on $V_{a,b}$, we have

$$\bar{\partial} \omega'(Y)_1 = 0, \quad \partial \omega'(Y)_1 = 0.$$

Furthermore as $i_t$ preserves $\Omega\mid_{V_{a,b}}$, our $\omega'(Y)_1$ satisfies

$$\Omega\mid_{V_{a,b}}\wedge\omega'(Y)_1=0.$$

We recall some the Hodge identities, $\sqrt{-1}\delta''=[\Lambda,\partial]$, and for the middle dimension, $[L,\Lambda]=0$. So, we have

$$\delta''\omega'(Y)_1=0.$$

Hence the problem is to find such a type $(1,0)$ vector field $Y$ satisfying (3.1).

# 4 The ordinary double points

Let

$$f=z_1^2+\cdots+z_{n+1}^2.$$

And consider

$$V_{a,b}=\{z;z\in C^{n+1},z_1^2+\cdots+z_{n+1}=0,a<\mid z\mid b\}.$$

In this case, the situation is quite simple. In fact, take the $C^\infty$-trivialization, $i_t$, defined in Sect.2, and take the holomorphic $(n,0)$ form $\omega'$, defined in Sect.3. Consider

$$i_t^*\omega'\equiv\omega'+\omega_1't \mod (t^2,\bar{t}).$$

Then our $\omega_1'$ is purely of $(n-1,1)$-type(see [AGL]). Therefore this $\omega_1'$ is automatically a harmonic form by the Hodge identities. However, for $A_l$-singualrities($l\neq 1$), this is not true. In the next section, we sketch how to remedy this point.

# 5 The Euler vector field

For $A_l$ singularities, there is the Euler vector field. Let $f=z_1^2+\cdots+z_n^2+z_{n+1}^{l+1}$. Then on $V_{a,b}=\{z:z\in C^{n+1},f(z)=0,a<\mid z\mid<b\}$, there is the Euler vector field

$$E=\frac{1}{2}\sum_{i=1}^{l}\frac{\partial}{\partial z_i}+\frac{1}{l+1}\frac{\partial}{\partial z_{n+1}}.$$

We adopt $gE$ as for $Y$ in Sect.3. Namely, consider the holomorphic $(n,0)$ form

$$\omega'(g)=\frac{1}{(X_f+gE)f}(X_f+gE)\rfloor(dz_1\wedge\cdots\wedge dz_{n+1}).$$

Here $g$ is an arbitrary complex valued $C^\infty$ function. We choose a $C^\infty$ function $g$ satisfying (3.1)(in the notation in Sect.3, $Y=gE$).

**Remark** The holomorphic $(n,0)$ form $E\rfloor(dz_1\wedge\cdots\wedge dz_{n+1})$, restricted to $V_{a,b}$, vanishes. And $E\rfloor(dz_1\wedge\cdots\wedge dz_{n+1})$, on $f(z)=t$, is a closed $(n,0)$ form. This corresponds to a vanishing cycle(Lagarange submanifold).

**Theorem 3.** *Let $f=z_1^2+\cdots+z_n^2+z_{n+1}^{l+1}$. Then,*

$$H^1(V_{a,b},\wedge^{n-1}(T'V_{a,b})^*)\simeq$$

$$\{\phi:\phi\in\Gamma(V_{a,b},\wedge^{n-1}(T'V_{a,b})^*\bigwedge\wedge(T''V_{a,b})^*)),\bar{\partial}\phi=0,\delta''\phi=0\}.$$

The $(n,0)$ part of the coefficient of $t$ of $i_t^*(E\rfloor(dz_1 \wedge \cdots \wedge dz_{n+1}))$, is proportional to $\omega'$. We write it by; $h\omega'$(here $h$ is a $C^\infty$ function on $V_{a,b}$). If $h$ never vanishes, then by taking a suitable $g$, we can control $(n,0)$ part. For $A_l$ singularities, this $C^\infty$ function $h$ does not vanish on $V_{a,b}$(this is proved by a direct computation). So, taking a proper $g$, we can cancel the $(n,0)$ part of the coefficient of $t$ of

$$i_t^*\omega'(g).$$

Hence, we have a type $(n-1,1)$ differential form which satisfies (3.1), and corresponds to the Kodaira-Spencer class of the family of deformations $(\tilde{V}, \pi(z), \mathbf{C})$. The other deformations are the same.

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# DISCRETE SPECTRUM OF QUANTUM TUBES

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A quantum tube is essentially a tubular neighborhood about an immersed complete manifold in some Euclidean space. To be more precise, let $\Sigma \hookrightarrow \mathbb{R}^{n+k}$, $k \geq 1$, $n = dim(\Sigma)$, be an isometric immersion, where $\Sigma$ is a complete, noncompact, orientable manifold. Then consider the resulting normal bundle $T^{\perp}\Sigma$ over $\Sigma$, and the submanifold $F = \{(x, \xi) | x \in \Sigma, |\xi < r\} \subset T^{\perp}\Sigma$ for $r$ small enough. The quantum tube is defined as the Riemannian manifold $(F, f^*(ds_E^2))$, where $ds_E^2$ is the Euclidean metric in $\mathbb{R}^{n+k}$ and the map $f$ is defined by $f(x, \xi) = x + \xi$. If $k = 1$, then the quantum tube is also called the quantum layer. The immersion of $\Sigma$ means that the resulting image of $F$ under $f$ in $\mathbb{R}^{n+k}$ can have intersections. Moreover, since $\Sigma$ can have quite complicated topology in general, $f(F)$ can too. However, by doing our analysis on $F$ directly (with the pull-back metric), these complications are naturally bypassed (cf. [1, 5]).

Although on noncompact, noncomplete manifolds there is no unique self-adjoint extension of the Laplacian acting on compactly supported functions, we can always, via the Dirichlet quadratic form define the *Dirchlet Laplacian* $\Delta_D$, which is the self-adjoint extension that reduces to the self-adjoint Laplacians defined on complete manifolds and compact manifolds with Dirichlet boundary conditions. Therefore we can proceed to perform spectral analysis, in particular, on the quantum tube. Geometers, like physicists, are first and foremost interested in the existence and distribution of the discrete spectrum. For noncompact manifolds this is in general not an easy task at all. However, using standard variational techniques, the authors Duclos, Exner, and Krejčiřík were able to, in an interesting paper [2], prove the existence of discrete spectra for the quantum layer (corresponding to $n = 2$ and

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$k = 1$ in our defintion) under certain integral-curvature conditions on $\Sigma$. Since the discrete spectrum are isolated eigenvalues of finite multiplicity, their result is even better, especially in the physical sense since the discrete spectrum is composed of energy levels of bound states of a nonrelativistic particle. Our definition of the quantum tube improved theirs in [2] and we were able to generalize the same existence result to the quantum tube. The challenges in our attempt at generalization were mainly geometrical, as we sought to replace the necessary geometric conditions with appropriate higher dimensional analogs so that similar variational techniques from [2] can be applied meaningfully. One notable observation that arised is the sharp contrast between parabolic and non-parabolic manifolds.

The main result in [5] is as follows:

**Theorem 1.** *Let $n \geq 2$ be a natural number. Suppose $\Sigma \subset R^{n+1}$ is a complete immersed parabolic hypersurface such that the second fundamental form $A \to 0$ at infinity. Moreover, we assume that*

$$(1) \qquad \sum_{k=1}^{[n/2]} \mu_{2k} \mathrm{Tr}(\mathcal{R}^k) \text{ is integrable and } \qquad \int_\Sigma \sum_{k=1}^{[n/2]} \mu_{2k} \mathrm{Tr}(\mathcal{R}^k) d\Sigma \leq 0,$$

*where $\mu_{2k} > 0$ for $k \geq 1$ are positive computable coefficients; $[n/2]$ is the integer part of $n/2$, and $\mathcal{R}^k$ is the induced endomorphism of $\Lambda^{2k}(T_x\Sigma)$ by the curvature tensor $\mathcal{R}$ of $\Sigma$. Let $a$ be a positive real number such that $a||A|| < C_0 < 1$ for a constant $C_0$. If $\Sigma$ is not totally geodesic, then the ground state of the quantum layer $\Omega$ exists.*

In [6], we generalized the above results to high codimensional cases:

**Theorem 2.** *Let $(F, f^*(ds_E^2))$ be an order-$k$ quantum tube with radius $r$ and base manifold $\Sigma$ of dimension $n$ such that the second fundamental form goes to zero at infinity. Moreover, we assume that $\Sigma$ is a parabolic manifold, $\sum_{p=1}^{[n/2]} \mu_{2p} \mathrm{Tr}(\mathcal{R}^k)$ integrable, and*

$$(2) \qquad \int_\Sigma \sum_{p=1}^{[n/2]} \mu_{2p} \mathrm{Tr}(\mathcal{R}^k) d\Sigma \leq 0,$$

*If $\Sigma$ is not totally geodesic, then the ground state of the quantum tube from $\Sigma$ exists.*

By applying the above result into two dimensional case, we get

**Corollary 1.** *Suppose that $\Sigma$ is a complete immersed surface of $R^{n+1}$ such that the second fundamental form $A \to 0$. Suppose that the Gauss curvature is integrable and suppose that*

$$e(\Sigma) - \sum \lambda_i \leq 0, \tag{3}$$

*where $e(\Sigma)$ is the Euler characteristic number of $\Sigma$; $\lambda_i$ is the isoperimetric constant at each end of $\Sigma$, defined as*

$$\lambda_i = \lim_{r\to\infty} \frac{\text{vol}(B(r))}{\pi r^2}$$

*at each end $E_i$. Let $a$ be a positive number such that $a||A|| < C_0 < 1$. If $\Sigma$ is not totally geodesic, then the ground state of the quantum layer $\Omega$ exists. In particular, if $e(\Sigma) \leq 0$, then the ground state exists.*

We remark here that in the proof of Theorem 2 (and so as in Theorem 1 and the analogous result in [2]), the asymptotically flat condition on $\Sigma$ ensures that we get a lower bound on the bottom of the essential spectrum, while condition 2 (along with parabolicity) enabled us to show that such a bound is also a strict upper bound for the total spectrum. In this way, we were able to conclude that the discrete spectrum must be non-empty. It seems intuitive that the asymptotically flat condition on $\Sigma$ is essential for there to be discrete spectra, since only the "relatively-curved part of $\Sigma$" located in the "interior" of $\Sigma$ will trap a particle. If $\Sigma$ is curved more-or-less the same everywhere, then a particle may be equally likely to be anywhere since the "terrain" is more-or-less indistinguishable everywhere. The preceding is of course a physical intuition coming from the interpretation of our problem as a problem in non-relativistic quantum mechanics, however, it serves to motivate the idea that other global curvature assumptions similar to (2) may also provide the existence of ground state on quantum tubes.

From Corollary 1 (and the result in [2]), it is natural to make the following

**Conjecture.** *Suppose $\Sigma$ is an embedded asymptotically flat surface in $R^3$ which is not totally geodesic and the Gauss curvature is integrable. Then the ground state of the quantum layer built from $\Sigma$ exists.*

We have partial results in this direction [8]:

**Theorem 3** (Lu). *Suppose $\Sigma$ is asymptotically flat but not totally geodesic in $R^3$. If the Gauss curvature of $\Sigma$ is positive, then the ground state exists for the quantum layer.*

In general, we have the following result:

**Theorem 4** (Lu). *Suppose $\Sigma$ is asymptotically flat but not totally geodesic in $R^3$ and suppose the Gauss curvature is integrable. Let $H$ be the mean curvature. If there is an $\varepsilon > 0$ such that*

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \left| \int_{B(r)} H d\Sigma \right| \geq \varepsilon, \tag{4}$$

*then the ground state of the quantum layer exists.*

Let's make some remarks on the above results. By the work of [2], we only need to prove the conjecture under the assumption that

$$\int_\Sigma K d\Sigma > 0.$$

By a result of Hartman [4], we know that

$$\frac{1}{2\pi} \int_\Sigma K d\Sigma = e(\Sigma) - \sum \lambda_i. \tag{5}$$

Thus we have $e(\Sigma) > 0$, or $e(\Sigma) \geq 1$. Let $g(\Sigma)$ be the genus of $\Sigma$, we then know $g(\Sigma) = 0$ and $\Sigma$ must be differmorphic to $\mathbb{R}^2$, which is a very strong topological restriction.

On the other hand, we have the following lemma:

**Lemma 1.** *Under the assumption that $\int_\Sigma K d\Sigma > 0$, there is an $\varepsilon > 0$ such that*

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \int_{B(r)} |H| d\Sigma \geq \varepsilon.$$

**Proof.** Since $\Sigma$ is differmorphic to $\mathbb{R}^2$, by (5)

$$0 < \int_\Sigma K d\Sigma \leq 2\pi < 4\pi.$$

Thus by a theorem of White [9], we get the conclusion.

□

We believe (4) is true under the same assumption as in the Lemma.

The above results confirmed the belief that the spectrum of the quantum tube only depends on the geometry of $\Sigma$, its base manifold. With regard to the geometry of $\Sigma$ (or any complete, noncompact manifold for that matter), the volume growth (of geodesic balls) is an important geometric property. Roughly speaking, complete, noncompact manifolds can be separated into those with at most quadratic volume growth and those with faster volume growth. They are termed (very roughly) parabolic and non-parabolic, respectively. It is the property of parabolicity assumed on $\Sigma$ that allowed us to prove the existence of discrete spectra on quantum tubes. However, if one looks at the hypothesis of Theorem 2, where $\Sigma$ is required to have vanishing curvature at infinity while being immersed in Euclidean space, it is highly likely that $\Sigma$ will not be of at most quadratic volume growth if $\dim(\Sigma) > 2$, hence unlikely to be parabolic. However, one can be sure that the set of base manifolds satisfying the hypothesis of Theorem 2 is not empty, due to an example provided in [5]. Nevertheless, it is clear that if one were to maintain the assumption of asymptotic flatness of $\Sigma$, then one should begin paying attention to the situation when $\Sigma$ is non-parabolic.

Although we do not yet have a result specifically for quantum tubes over non-parabolic manifolds, there is the following preliminary result for general (possibly non-parabolic) base manifolds (see [7]):

**Theorem 5.** *Suppose $\Sigma$ is not totally geodesic, satisfies the volume growth $V(r) \leq Cr^m$, and whose second fundamental form $\vec{A}$ goes to zero at infinity and decays like $r^2\|\vec{A}\| \to 0$ as $r \to \infty$. Moreover, suppose*

$$\lim_{R\to\infty} \frac{1}{R^{m-2}} \int_{B(R)} \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p} \tag{6}$$

*exists(possibly $-\infty$) and strictly less than $-\frac{1}{4}CC_1m^2e^2$, where $C_1$ is an explicit constant that depends on the dimension of $\Sigma$, radius of the quantum tube, and the upper bound on the curvature of $\Sigma$. Then the discrete spectrum of the quantum tube with base manifold $\Sigma$ is non-empty.*

The result above is certainly an overkill if $\Sigma$ is parabolic. Thus we should think of applying it only to the case of non-parabolic $\Sigma$, where $m > 2$. The direct application of the volume growth hypothesis allows one to use polynomially decaying test functions to obtain the condition on (6), and in turn obtain the upper-bound for the bottom of the total spectrum.

Theorem 5 is only a first step towards generalizing the phenomenon of localization (we mean this to be the existence of ground state) to quantum tubes over non-parabolic manifolds with similar non-positivy assumptions on curvature as the parabolic case. One clearly cites the technical assumption on the decay rate of the second fundamental form, and one would like to remove it. In addition, the negativity condition on (6) is very strong. We do not yet know if weaker assumptions such as (2) are applicable to the case where $\Sigma$ is non-parabolic.

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# Multiplier Ideals and $b$-Function

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Let $X$ be a complex manifold, and $D$ a divisor defined by $f$. The multiplier ideal $\mathcal{J}(X, \alpha D)$ is defined by the local integrability of $|g|^2/|f|^{2\alpha}$ for $\alpha > 0$, $g \in \mathcal{O}_X$, see [17]. This is also defined by using an embedded resolution of $(X, D)$, and there are positive rational numbers $\alpha_1 < \cdots < \alpha_j < \cdots$ such that $\mathcal{J}(X, \alpha D) = \mathcal{J}(X, \alpha_j D)$ for $\alpha \in [\alpha_j, \alpha_{j+1})$ and $\mathcal{J}(X, \alpha_j D) \supsetneq \mathcal{J}(X, \alpha_{j+1} D)$, where $\mathcal{J}(X, \alpha_0 D) = \mathcal{O}_X$ with $\alpha_0 = -\infty$, see [12]. These $\alpha_j$ $(j > 0)$ are called the jumping coefficients.

Let $b_f(s)$ be the $b$-function (i.e. the Bernstein-Sato polynomial) of $f$, see e.g. [9]. By definition, it is the monic generator of the ideal satisafying the relation

$$b_f(s) f^s = P f^{s+1} \quad \text{in} \quad \mathcal{O}_X[f^{-1}][s],$$

where $P \in \mathcal{D}_X[s]$. Let $\mathcal{B}_f$ be the direct image as $\mathcal{D}$-module of the structure sheaf $\mathcal{O}_X$ by the graph embedding $i_f : X \to X \times \mathbb{C}$. This is free over $\mathcal{O}_X[\partial_t]$ with the canonical generator $\delta(f - t)$. M. Kashiwara [10] and B. Malgrange [15] constructed the $V$-filtration on $\mathcal{B}_f$ and proved the canonical isomorphism

$$\mathrm{DR}_X(\textstyle\bigoplus_{0<\alpha\le 1} \mathrm{Gr}_V^\alpha \mathcal{B}_f) = \psi_f \mathbb{C}_X[\dim X - 1],$$

such that the action of $\exp(-2\pi i \partial_t t)$ on the left-hand side corresponds to that of the monodromy $T$ on the right-hand side, where $\mathrm{DR}_X$ denotes the associated de Rham complex [4]. Here $\psi_f = \mathbf{R}\rho_* \mathbb{C}_{X_t}$ with $\rho : X_t \to X_0 = D$ a good retraction, which can be constructed by using an embedded resolution of $D$, see [5]. It is well known that $f^s$ and $s$ can be identified with $\delta(f - t)$ and $-\partial_t t$ respectively so that $\mathcal{D}_X[s] f^s$ is identified with the $\mathcal{D}_X[s]$-submodule of $\mathcal{B}_f$ generated by $\delta(f - t)$. This implies the well-known relation between the roots of the $b$-function and the eigenvalues of the Milnor monodromy. By [3] we have

$$\mathcal{J}(X, \alpha D) = V^\alpha \mathcal{O}_X \quad \text{if } \alpha \text{ is not a jumping coefficient},$$

where the filtration $V$ on $\mathcal{O}_X$ is induced by the $V$-filtration on $\mathcal{B}_f$. If $\alpha$ is a jumping coefficient (or actually, for any $\alpha$), we have for $0 < \varepsilon \ll 1$

$$\mathcal{J}(X, \alpha D) = V^{\alpha+\varepsilon} \mathcal{O}_X, \quad V^\alpha \mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon) D).$$

The proof can be reduced to the normal crossing case using the theory of bifiltered direct images. (By [2] this is generalized to the case of arbitrary subvarieties.) This gives another proof of a theorem of L. Ein, R. Lazarsfeld, K.E. Smith, and D. Varolin [7] that any jumping coefficients which are less than 1 are roots of $b_f(-s)$. It is well known that the minimal jumping coefficient $\alpha_f$ coincides with the minimal root of $b_f(-s)$, see [11].

For $x \in D$, we define $b_{f,x}(s)$, $\alpha_{f,x}$ by replacing $X$ with a sufficiently small open neighborhood of $x$. For $\alpha > 0$ with $0 < \varepsilon \ll 1$, the graded pieces are defined by

$$\mathcal{G}(X, \alpha D) = \mathcal{J}(X, (\alpha - \varepsilon) D) / \mathcal{J}(X, \alpha D) \, (= \mathrm{Gr}_V^\alpha \mathcal{O}_X).$$

We say that $\alpha$ is a local jumping coefficient of $D$ at $x$ if $\mathcal{G}(X, \alpha D)$ does not vanish at $x$. We have a partial converse of their theorem as follows (see [18]):

**Theorem 1.** *Let $\alpha$ be a root of $b_{f,x}(-s)$ contained in $(0,1)$. Assume*
(i) *$\xi f = f$ for a holomorphic vector field $\xi$.*
(ii) *$\alpha < \beta_{f,x} := \min\{\alpha_{f,y} \,|\, y \neq x$ is sufficiently near $x\}$.*
*Then $\alpha$ is a local jumping coefficient of $D$ at $x$.*

This does not hold if either of the two conditions is not satisfied. Condition (ii) is satisfied if $\exp(-2\pi i\beta)$ is not an eigenvalue of the Milnor monodromy of $f$ at $y \neq x$ for any $\beta \in [\alpha_{f,x}, \alpha]$. By definition, the jumping coefficients have a periodicity so that $\alpha > 0$ is a jumping coefficient if and only if $\alpha + 1$ is. However, the roots of $b_f(-s)$ do not have such a periodicity and we have to restrict to $(\alpha_{f,x}, 1)$.

As for the relation with the spectrum ([19], [20]), N. Budur [1] proved that, if $\mathcal{G}(X, \alpha D)$ is supported on a point $x$ of $D$ with $\alpha \in (0,1)$, then the coefficient $m_\alpha$ of the spectrum $\mathrm{Sp}(f,x) = \sum_\beta m_\beta t^\beta$ is given by

$$m_\alpha = \dim \mathcal{G}(X, \alpha D)_x. \tag{0.1}$$

Indeed, under the above hypothesis, $\mathcal{G}(X, \alpha D)\, (= \mathrm{Gr}_V^\alpha \mathcal{O}_X)$ is identified with the Hodge filtration $F^{n-1}$ on the $\lambda$-eigenspace $H^{n-1}(F_x, \mathbb{C})_\lambda$ for the Milnor monodromy, where $\lambda = \exp(-2\pi i\alpha)$, $n = \dim X$, and $F_x$ denotes the Milnor fiber around $x$. Note that the spectrum is defined by

$$\begin{gathered} m_\alpha = \textstyle\sum_j (-1)^{j-n+1} \dim \mathrm{Gr}_F^p \widetilde{H}^j(F_x, \mathbb{C})_\lambda \\ \text{with } p = [n - \alpha],\ \lambda = \exp(-2\pi i\alpha), \end{gathered} \tag{0.2}$$

In the isolated singularity case, (0.1) is closely related to [13], [14], [21], [22]. We have a generalization of a result of Malgrange [14] as follows (see [18]):

**Theorem 2.** *There is a filtration $\widetilde{P}$ on $H^{n-1}(F_x, \mathbb{C})_\lambda$ stable by the monodromy, containing the Hodge filtration $F$, and having the following property: If $\lambda = \exp(-2\pi i\alpha)$ is not an eigenvalue of the Milnor monodromy at $y \neq x$ sufficiently near $x$, then $\alpha$ is a root of $b_{f,x}(-s)$ if and only if $\mathrm{Gr}_{\widetilde{P}}^p H^{n-1}(F_x, \mathbb{C})_\lambda \neq 0$ with $p = [n - \alpha]$. Moreover the multiplicity of the root coincides with the degree of the minimal polynomial of the action of the monodromy on $\mathrm{Gr}_{\widetilde{P}}^p H^{n-1}(F_x, \mathbb{C})_\lambda$.*

This property of the roots of $b_f(-s)$ is similar to the definition of the spectrum (0.2), replacing $\widetilde{P}$ with $F$ and the minimal polynomial with the characteristic polynomial. If $f$ is a homogeneous polynomial, then $\widetilde{P}$ coincides with the pole order filtration $P$ defined by using a meromorphic connection on $\mathbb{P}^{n-1}$ calculating $H^{n-1}(F_x, \mathbb{C})_\lambda$, see also [6].

We can give a formula for $\mathcal{J}(X, \alpha D)$ if $D$ is locally conical along a stratification, i.e. if $D$ is locally defined by a weighted homogeneous function with nonnegative weights and the zero weight part, which is the limit of the (local) $\mathbb{C}^*$-action, is given by the stratum passing through the point [18]. This generalizes a formula of Mustaţă [16] for a hyperplane arrangement with a reduced equation. A similar formula has been known for a function with nondegenerate Newton boundary [8].

For a divisor $D$ on a complex manifold, let $\alpha_D = \min\{\alpha_{f,x} : x \in D\}$ where $D$ is locally defined by $f$. By a similar argument we have

**Proposition 1.** *Assume $X = \mathbb{C}^n$ and $D$ is the affine cone of a divisor $Z$ of degree $d$ on $\mathbb{P}^{n-1}$. Let $\mathcal{I}_0$ be the ideal sheaf of $\{0\} \subset \mathbb{C}^n$. Then we have for $\alpha < \alpha_Z$*

$$\mathcal{J}(X, \alpha D) = \mathcal{I}_0^k \quad \textit{with} \quad k = [d\alpha] - n + 1.$$

*In particular, $j/d$ is a local jumping coefficient of $D$ at 0 if $n \le j < d\alpha_Z$.*

In general $\alpha_Z \le 1$, and $\alpha_Z = 1$ if $Z$ is a reduced divisor with normal crossings, e.g. if $D$ is a generic hyperplane arrangement. Since $\dim \mathcal{I}_0^k/\mathcal{I}_0^{k+1} = \binom{n+k-1}{n-1}$, we deduce that if $Z$ is a reduced divisor with normal crossings on $\mathbb{P}^{n-1}$, then the coefficients $m_\alpha$ and $m_{n-\alpha}$ of the spectrum $\mathrm{Sp}(f, 0)$ are $\binom{j-1}{n-1}$ for $\alpha = j/d < 1$. This is the same for homogeneous polynomials with isolated singularity.

In the case of generic central hyperplane arrangements (with reduced equations), the $b$-function is determined by U. Walther [23] (except for the multiplicity of the root $-1$):

$$b_f(s) = (s+1)^{n-1} \prod_{j=n}^{2d-2} (s + \frac{j}{d}),$$

where $d = \deg f > n$. Here generic central means that it is the cone of a projective arrangement with normal crossings in $\mathbb{P}^{n-1}$. His formula can be reduced to the assertion that the roots of $b_f(-s)$ is strictly smaller than 2 using the above calculation of the spectrum, see [18]. Walther's formula shows that, without restricting to the interval $(0, 1)$, there is no relation between the spectrum and the roots of $b_f(-s)$ (contrary to the case of a homogeneous polynomial with an isolated singularity). This comes from the difference between the Hodge and pole order filtrations on the Milnor cohomology in Theorem 2.

Finally, the jumping coefficients and the spectrum of a hyperplane arrangement are determined by the combinatorial data, as conjectured by Mustaţă [16].

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# SOME RESULTS CONCERNING HYPERBOLICITY IN ALMOST COMPLEX MANIFOLDS

HERVÉ GAUSSIER

This Note introduces some recent results concerning a geometric study of almost complex manifolds. Our main focus, around hyperbolicity, is to exhibit some properties which particularize nonintegrable structures from complex structures. Most of the results were obtained in joint works with A.Sukhov [6, 7] and with J.Byun-K.H.Lee [3]. The corresponding references are stated before each corresponding result.

We recall that an almost complex manifold is a pair $(M, J)$ where $M$ is a real manifold and $J$ is a (continuous) $(1,1)$ tensor $J$ on $M$, satisfying $J^2 = -Id$.

The local existence of pseudoholomorphic curves was proved by A.Nijenhuis-W.Woolf [14] for Hölderian structures :

**Theorem 1.** *Let $(M, J)$ be an almost complex manifold, where $J$ is Hölderian with exponent $\alpha$ $(0 < \alpha < 1)$. Then for every $p \in M$ there is a neighborhood $U$ of $(p, 0)$ in $TM$ such that :*

$$\forall (q, v) \in U, \exists f : (\Delta, J_{st}) \to (M, J), f(0) = q, \ df(0)(\partial/\partial x) = v.$$

This statement deserves some comments.

(*i*) The condition "$f : (\Delta, J_{st}) \to (M, J)$" means that $f$ is a pseudoholomorphic disc in $M$, i.e. satisfying $df \circ J_{st} = J \circ df$. Here $J_{st}$ denotes the standard complex structure on $\mathbb{C}$, and more generallly on $\mathbb{C}^n$, $n \geq 1$.

(*ii*) From classical elliptic theory, every pseudoholomorphic disc is of class $\mathcal{C}^{k+1,\alpha}$ whenever $J$ is of class $\mathcal{C}^{k,\alpha}$, $k \in \mathbb{N}\backslash\{0\}$, $0 < \alpha < 1$.

(*iii*) A.Nijenhuis-W.Woolf also proved the persistence of "small" pseudoholomorphic discs under deformation of the structure : if $J'$ is an almost complex structure on $M$ such that $\|J' - J\|_{\mathcal{C}^\alpha} << 1$, then there exists $f' : (\Delta, J_{st}) \to (M, J')$ such that $\|f' - f\|_\infty << 1$.

Thanks to the local existence of pseudoholomorphic discs one may define the Kobayashi-Royden pseudonorm $K_{(M,J)}$ in $(M, J)$ for a Hölderian structure $J$ :

**Definition 2.** *For every $p \in M$ and for every $v \in T_pM$, we set :*

$$K_{(M,J)} := inf\{\alpha > 0 \ / \ \exists f : (\Delta, J_{st}) \to (M, J), f(0) = p, \ df(0)(\partial/\partial x) = v\}.$$

The upper semi continuity of $K_{(M,J)}$, proved by H.L.Royden [17] in complex manifolds, relies on the persistence of pseudoholomorphic discs under perturbation of the parameters $p$ and $v$. This stability result is proved in the almost complex setting by B.Kruglikov [10] for smooth $\mathcal{C}^\infty$ structures and by S.Ivashkovich-J.P.Rosay [9] for $\mathcal{C}^{1,\alpha}$ structures. Finally, the upper semi continuity fails for Hölderian structures; S.Ivashkovich-S.Pinchuk-J.P.Rosay [8] gave an example of a disc that cannot be deformed.

From now on we will only consider smooth $\mathcal{C}^{1,\alpha}$ almost complex structures.

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By analogy with complex manifolds, the Kobayashi pseudodistance may be defined by integration of the Kobayashi-Royden pseudonorm :

**Definition 3.** *(i) For every $x, y \in M$ the Kobayashi pseudodistance between $x$ and $y$ is given by $d_{(M,J)}(x,y) = inf\{\int_0^1 K_{(M,J)}(\gamma(t), \gamma'(t))dt\}$, where the infimum is taken over all $C^1$ paths joining $x$ and $y$.*

*(ii) $(M, J)$ is (Kobayashi) hyperbolic if $d_{(M,J)}$ is a distance (this will induce the usual topology on $M$)*

*(iii) $(M, J)$ is complete hyperbolic if the metric space $(M, d_{(M,J)})$ is complete.*

Our first result (Corollary 1 in [6]) concerns the local hyperbolicity of almost complex manifolds :

**Theorem 4.** *Every point in $(M, J)$ admits a basis of complete hyperbolic neighborhoods.*

This result is classical in complex manifolds. The Kobayashi pseudodistance and the Poincaré metric being equal on Euclidean balls, such balls are therefore complete hyperbolic and they provide the desired basis of neighborhoods.

The natural approach in the almost complex setting consists in viewing a nonintegrable structure on sufficiently small balls, imbedded in $\mathbb{C}^n$, as a deformation of the standard integrable structure. For such small deformations, Euclidean balls are defined by a strictly plurisubharmonic function. The problem relies consequently on estimating the Kobayashi-Royden pseudonorm on a domain $D = \{\rho < 0\}$ where $\rho$ is strictly $J$-plurisubharmonic. This is given by Theorem 1 in [6], firstly by proving an attraction property for pseudoholomorphic discs whose center is close to a bounday point of $D$ and secondly by using a blow-up technique. This scaling method, initiated by S.Pinchuk [15] in $\mathbb{C}^n$, has a new feature in the almost complex setting since this involves a deformation both of the domain and of the almost complex structure.

The scaling process reflects the local geometry of the domain $D$ and emphasizes the osculation of $\partial D$ by spheres. The most striking fact is the convergence of the associated dilated almost complex structures to "model structures", owing particular properties. To present them we first realize the almost complex structure as a $C^{1,\alpha}$ almost complex deformation of $J_{st}$ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, with a special choice of complex coordinates (fitted to the geometry of $D$). For a positive real number $\tau$ let $\Lambda^\tau$ be the dilation map defined on $\mathbb{C}^n$ by $\Lambda^\tau('z, z_n) = (\tau^{-1/2}\, 'z, \tau^{-1} z_n)$, where $('z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. If $J_\tau := (\Lambda^\tau)_*(J) = \Lambda^\tau \circ J \circ (\Lambda^\tau)^{-1}$ on $\Lambda^\tau(\mathbb{B}_n)$ then $\lim_{\tau \to 0} J_\tau = J_0$, uniformly on compact subsets of $\mathbb{C}^n$, where $J_0$ is an almost complex structure defined on $\mathbb{C}^n$ by the matricial representation :

$$J_0(z) = J_{st} + L('z, 0). \tag{0.1}$$

Here $L('z, 0)$ is a matrix with $L_{kj} = 0$ for $k = 1, \ldots, n-1$, $j = 1, \ldots, n$, $L_{nn} = 0$ and $L_{nj}('z, 0)$ are real linear forms in $'z$ for $j = 1, \ldots, n-1$.

Intuitively the scaling process reduces the local study of $(D, J)$ to the global study of $(\mathbb{H}, J_0)$, where $\mathbb{H}$ is the Siegel half space.

The main properties of the "model almost complex manifold" $(\mathbb{H}, J_0)$, studied in [7] and by K.H.Lee in [12], are summerized in the following proposition :

**Proposition 5.** *(iii) Nonintegrable structures define an open dense subset in the space of almost complex structures defined by (0.1),*

*(i) The model almost complex manifold $(\mathbb{H}, J_0)$ is hyperbolic and strictly pseudoconvex,*

*(ii) The automorphism group of $(\mathbb{H}, J_0)$ is transitive.*

We recall that a domain $D$ in an almost complex manifold $(M, J)$ is strictly pseudoconvex if every point in $\partial D$ has a neighborhood $U$ (imbedded in $\mathbb{C}^n$) such that $D \cap U = \{z \in U : \rho(z) < 0\}$, where the Lvi form $-d(J^* d\rho)$ of $\rho$ is positive on $T(D \cap U)$. Finally, an automorphism of an almost complex manifold $(M, J)$ is a diffeomorphism of $M$ satisfying $df \circ J = J \circ df$.

Proposition 5 cancels the Wong-Rosay Theorem in almost complex manifolds. In complex manifolds this is stated as follows (see [18, 16, 15, 5] :

**Theorem 6.** *Let $D$ be a domain in a complex manifold of dimension $n$. Assume that there is a point $p \in \partial D$, a point $q \in D$ and automorphisms $\varphi^\nu$ of $D$ such that $\lim_{\nu \to \infty} \varphi^\nu(q) = p$. If $D$ is strictly pseudoconvex at $p$, then $D$ is biholomorphic to the unit ball $\mathbb{B}$ in $\mathbb{C}^n$.*

Consider a nonintegrable structure $J_0$ by Statement (i) of Proposition 5. This prevents from the existence of a biholomorphism between $(\mathbb{H}, J_0)$ and $(\mathbb{B}, J_{st})$. By Statement (ii), $\mathbb{H}$ is strictly pseudoconvex at the origin and by Statement (iii) there is an orbit of the automorphism group $Aut(\mathbb{H}, J_0)$ which accumulates at the origin in $\partial\mathbb{H}$.

This new phenomenon may be explained by viewing model nonintegrable almost complex manifolds as degenerate in the following sense. The Cayley transform $('z, z_n) \mapsto (2'z/(z_n - 1), (z_n + 1)/(z_n - 1))$ transforms $\bar{\mathbb{H}} \cup \{\infty\}$ biholomorphically onto $\bar{\mathbb{B}}$. This is a particularity of the standard complex structure. One can indeed prove the following [3] :

**Proposition 7.** *If the model structure $J_0$ is not integrable there is no strongly pseudoconvex relatively compact domain $D$ (possibly with a singularity) in an almost complex manifold $(M, J)$ such that $(D, J)$ is biholomorphic to $(\mathbb{H}, J_0)$.*

As a corollary of Proposition 7 we have the following version of the Wong-Rosay Theorem, stated as a generic compactness phenomenon for the automorphism group of almost complex manifolds [3] :

**Theorem 8.** *Let $D$ be a strictly pseudoconvex, relatively compact domain in an almost complex manifold $(M, J)$. If $(D, J)$ is not biholomorphic to $(\mathbb{B}, J_{st})$ then the automorphism group $Aut(D, J)$ is compact.*

To prove Theorem 8 we first establish that if an orbit of the automorphism group accumulates at a strictly pseudoconvex point in the boundary of a domain, then this domain is biholomorphic to a model almost complex manifold (see [7, 12]). Then we may apply Proposition 7.

Several articles deal with the persistence either of hyperbolicity or, on the opposite, of foliations by entire pseudoholomorphic curves under deformation of the almost complex structure. For instance, J.Duval proved an almost complex version of a theorem by M.Green, stating that the complement of five lines in general position in an almost complex projective space is hyperbolic. On the opposite, the stability of generic foliation by entire pseudoholomorphic curves in a complex torus was obtained by J.Moser [13] for small deformation of a complex standard structure. The non hyperbolicity of such a torus, equipped with an almost complex structure tamed by a standard symplectic form, was proved by V.Bangert [1].

We recall that an almost complex manifold $(M, J)$ is Brody-hyperbolic if this does not contain any nontrivial entire curve, namely a map $f : (\mathbb{C}, J_{st}) \to (M, J)$. Every (Kobayashi) hyperbolic manifold is clearly Brody-hyperbolic, since the Kobayashi pseudodistance vanishes identically on $(\mathbb{C}, J_{st})$ and decreases under the action of pseudoholomorphic maps. The converse, for compact manifolds, is due to Brody [2] in complex manifolds and can be carried out to almost complex manifolds. This was pointed out by B.Kruglikov-M.Overholt [11] and it implies the following stability result :

**Proposition 9.** *Let $(M,J)$ be a compact hyperbolic almost complex manifold. If $J'$ is an almost complex structure on $M$ satisfying $\|J'-J\|_{\mathcal{C}^{1,\alpha}(M)} << 1$, then $(M,J')$ is hyperbolic.*

For convenience we give a sketch of the proof.

Assume by contradiction that there is a sequence $(J_\nu)_\nu$ of almost complex structures on $M$ such that $\|J'-J\|_{\mathcal{C}^{1,\alpha}(M)} \to_{\nu\to\infty} 0$ and such that $(M,J_\nu)$ is not hyperbolic. Consider for every $\nu$ a nonconstant entire pseudoholomorphic curve $f^\nu : (\mathbb{C}, J_{st}) \to (M, J_\nu)$. Since $J_\nu$ is of class $\mathcal{C}^{1,\alpha}$, $f^\nu$ is at least of class $\mathcal{C}^2$. Let $g$ be any Riemannian metric on $M$ and let $\|.\|$ be the associated norm. We can assume that $\|df^\nu(0)(\partial/\partial x)\|$ is different from zero for every $\nu$ and so, by isotropic dilations, that $\|df^\nu(0)(\partial/\partial x)\| \to \infty$ when $\nu \to \infty$. Let $\Delta_\nu := \{\lambda \in \mathbb{C} : |\lambda| < \|df^\nu(0)(\partial/\partial x)\|/2\}$ and let $g^\nu : (\Delta_\nu, J_{st}) \to (M, J^\nu)$ be the $J_\nu$-holomorphic map defined by :

$$g^\nu(\lambda) = f^\nu\left(\frac{2\lambda}{\|df^\nu(0)(\partial/\partial x)\|}\right).$$

Then $g^\nu$ satisfies, for $\lambda \in \Delta_\nu$ :

$$\begin{cases} \|dg^\nu(0)(\partial/\partial x)\| &= 1 \\ |dg^\nu(\lambda)(\partial/\partial x)\| &\leq \dfrac{\|df^\nu(0)(\partial/\partial x)\|^2}{\|df^\nu(0)(\partial/\partial x)\|^2 - 4|\lambda|^2}. \end{cases}$$

This inequality is the key point in the Brody reparametrization Lemma [2].

By the classical Ascoli theorem we extract from the sequence $(g^\nu)_\nu$ a subsequence, still denoted by $(g^\nu)$, that converges uniformly on compact subsets of $\mathbb{C}$ to a map $g : \mathbb{C} \to M$. It follows from the $\mathcal{C}^{1,\alpha}$ convergence of $J_\nu$ to $J$ and from the quasi-ellipticity of $J$, that $g$ is a $J$-holomorphic curve. Moreover the maps $g^\nu$ converge to $g$, uniformly with their first derivatives, by "elliptic bootstrapping". This contradicts the hyperbolicity of $(M,J)$ since $\|dg^\nu(0)(\partial/\partial x)\| = 1$. □

As an application of the proofs of Theorem 4 and of Proposition 9 we have ([3]) :

**Proposition 10.** *Let $D$ be a relatively compact strongly pseudoconvex domain in an almost complex manifold $(M,J)$. If $(D,J)$ is hyperbolic, then $(D,J')$ is complete hyperbolic for every almost complex structure $J'$ satisfying $\|J'-J\|_{\mathcal{C}^{1,\alpha}(\bar{D})} << 1$.*

We point out that the assumptions in Proposition 10 are not redundant. Indeed, in contrast with the complex case, there exist non hyperbolic strongly pseudoconvex domains in almost complex manifolds (see [9]).

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# On surfaces of class $\mathrm{VII}_0^+$ with a cycle of rational curves - Application to bihermitian surfaces

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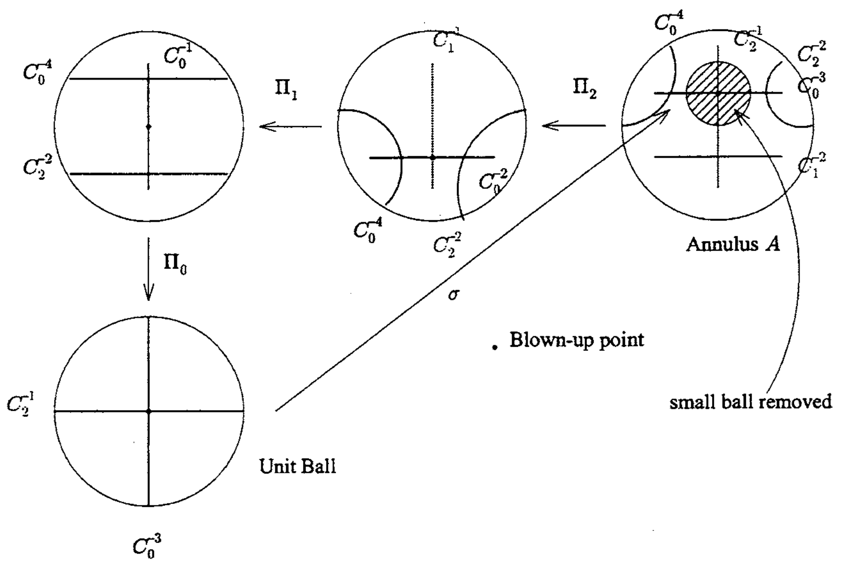
## 1 Introduction

We want to report on non-kählerian part of Kodaira's classification of compact complex surfaces. More precisely we are interested in the following situation:
A minimal compact complex surface $S$ is said to be of the **class $\mathbf{VII_0}$** of Kodaira if the first Betti number satisfies $b_1(S) = 1$. A surface $S$ is of **class $\mathbf{VII_0^+}$** if moreover $n := b_2(S) > 0$; these surfaces admit no nonconstant meromorphic functions.
The major problem in classification of non-kählerian surfaces is to achieve the classification of surfaces $S$ of class $\mathrm{VII}_0^+$. All known surfaces of this class contain Global Spherical Shells (GSS), i.e. admit a biholomorphic map $\varphi : U \to V$ from a neighbourhood $U \subset \mathbb{C}^2 \setminus \{0\}$ of the sphere $S^3 = \partial B^2$ onto an open set $V$ such that $\Sigma = \varphi(S^3)$ does not disconnect $S$. For example Hopf surfaces or blown-up Hopf surfaces contain GSS. But many minimal examples may obtained: For instance the following surface $S$ with $b_2(S) = 3$ (the second Betti number $b_2(S)$ equals the number of rational curves), where $\Pi_i$, $i = 0, 1, 2$ are blowing-ups and $\sigma$ is a biholomorphic map onto its image.

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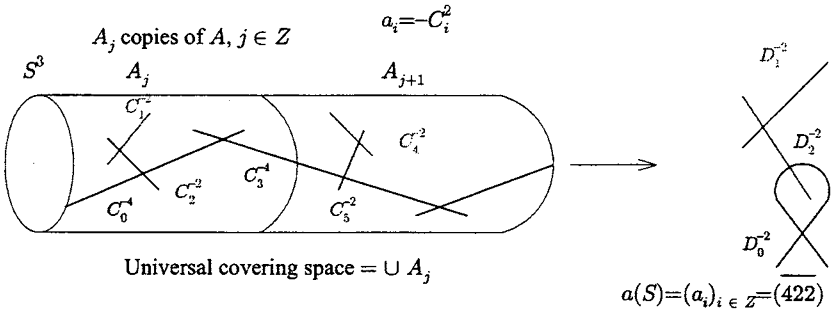
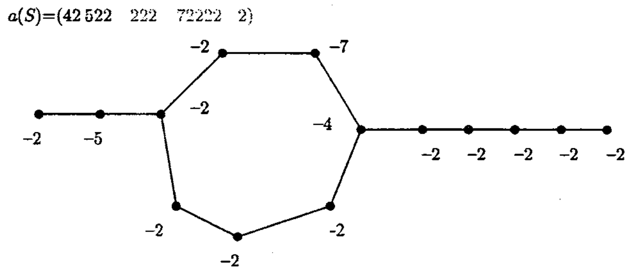



Another example with 14 curves:




Are there other surfaces ?

## 2 Surfaces with GSS and foliations

All surfaces with GSS $S$ admit at least one (singular) holomorphic foliation. We denote by $n = b_2(S)$ the second Betti number, $D = D_0 + \cdots + D_{n_1}$ the maximal reduced divisor, by $M(S)$ the intersection matrix of the curves and by $\sigma_n(S) = -\sum_{i=0}^{n-1} D_i^2$ the opposite sum of self-intersections.

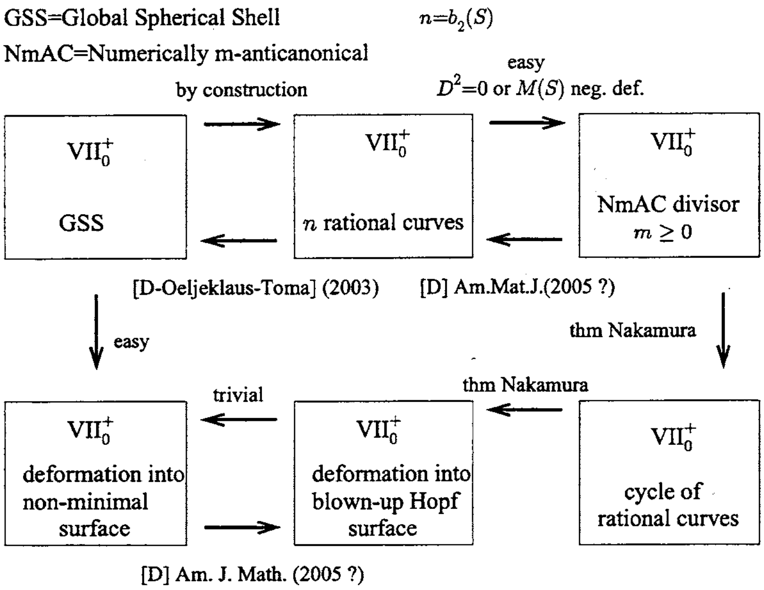
**Theorem 2. 1** **([4, 10, 9])** *If $\sigma_n(S) < 3n$, then there is a unique foliation defined by a closed twisted logarithmic 1-form $\omega \in H^0(S, \Omega^1(LogD)\otimes L^k)$, where $L^k \in H^1(S, \mathbb{C}^\star) \simeq \mathbb{C}^\star$ is a flat line bundle with $k = k(S) = \sqrt{|\det M(S)|} + 1 \in \mathbb{N}^\star$. Moreover $k(S) = 1$ if and only if $\sigma_n(S) = 2n$.*
*If $\sigma_n(S) = 3n$ (i.e. $S$ is a Inoue-Hirzebruch surface), there are exactly two foliations defined by twisted logarithmic 1-forms.*

It is an open question to know if a surface of class $VII_0^+$ admitting a foliation contains a GSS, however, if the foliation is induced by a non-trivial vector field, it is the case [5].

## 3 Results and conjectures in the general case

**Definition 3. 2** *Let $S$ be a compact complex surface and $m$ an integer $m \geq 0$. We say that $S$ admits a numerically $m$-anticanonical divisor (NmAC divisor) if there exists a flat line bundle $F \in H^1(S, \mathbb{C}^\star)$ such that $H^0(S, -mK_S \otimes F) \neq 0$.*

The following implications are known, it is conjectured that all these conditions are equivalent.




The main problem is to show the existence of curves. A curve $C$ gives a line bundle $[C]$ and a line bundle $L$ gives a Chern class $c_1(L) \in H^2(S, \mathbb{Z})$. The idea is to try to do the converse:
By index theorem, $b_2^- = b_2(S)$, then a theorem of Donaldson [7] gives a $\mathbb{Z}$-base $(E_i)$ of $H^2(S,\mathbb{Z})/\text{Torsion}$, such that $E_iE_j = -\delta_{ij}$. It is known that $p_g = h^2(S, \mathcal{O}) = 0$ hence the exponential exact sequence implies that these cohomology classes can be represented by line bundles $L_i$ such that $K_SL_i = L_i^2 = -1$. Indeed, these line bundles generalize exceptional curves of the first kind, and since $S$ is minimal, they have no section. Over the versal deformation $\mathcal{S} \to B$ of $S$ these line bundles form families $\mathcal{L}_i$. We propose the following conjecture which can be easily checked for surfaces with GSS:

**Conjecture 1**: Let $S$ be a surface in class $\text{VII}_0^+$ and $\mathcal{S} \to B$ be the versal deformation of $S \simeq S_0$ over the ball of dimension $h^1(S, \Theta)$. Then there exists $u \in \Delta$, $u \neq 0$, and flat line bundles $F_i$ such that $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$ for $i = 0, \ldots, n-1$.

We have

**Theorem 3. 3** *Let $S$ be a surface in class $VII_0^+$ and $\mathcal{S} \to B$ its versal deformation. If there exists $u \in B$ and flat line bundles $F_i \in H^1(S, \mathbb{C}^\star)$ such that $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$ for $i = 0, \ldots, n-1$, then there is a non empty Zariski open set $U \subset B$ such that for all $u \in U$, $S_u$ is a blown-up Hopf surface. In particular, $S$ is a degeneration of blown-up Hopf surfaces.*

If a surface is a degeneration of blown-up Hopf surfaces, the fundamental group of a fiber is isomorphic to $\mathbb{Z} \times \mathbb{Z}_l$, hence taking a finite covering, once obtains a surface obtained by degeneration of blown-up primary Hopf surfaces. Notice that a finite quotient of a surface of class $\mathrm{VII}_0^+$ containing a GSS still contains a GSS [3].

**Conjecture 2**: Let $S$ be a surface of class $\mathrm{VII}_0^+$. If $S$ is a degeneration of blown-up primary Hopf surfaces, then $S$ contains a cycle of rational curves.

A surface admitting a NmAC divisor, contains a cycle of rational curves.

**Theorem 3. 4** *Let $S$ be a surface of class $VII_0^+$. If $S$ admits a NmAC divisor, then $S$ contains a GSS.*

The proof relies on [6]. It is a weak version of

**Conjecture 3 (Nakamura [12]).** Let $S$ be a surface of class $\mathrm{VII}_0^+$. If $S$ contains a cycle $C$ of rational curves, $S$ contains a GSS.

The proof is based on the fact that, if $H_1(C,\mathbb{Z}) = H_1(S,\mathbb{Z})$, a curve is equivalent in $H^2(S,\mathbb{Z})$ to a class of the form $L_i - \sum_{j\in I} L_j$, with $I \neq \emptyset$. Intuitively $L_i$ represents an exceptional curve of the first kind and $C$ is then equivalent to an exceptional curve of the first kind blown-up several times (Card($I$) times). It explains why curves have self-intersection $\leq -2$. We recover a characterization of Inoue-Hirzebruch surfaces by Oeljeklaus, Toma & Zaffran [11]:

**Theorem 3. 5** *Let $S$ be a surface of class $VII_0$ with $b_2(S) > 0$. Then $S$ is a Inoue-Hirzebruch surface if and only if there exists two flat line bundles $F_1$, $F_2$, two twisted vector fields $\theta_1 \in H^0(S, \Theta \otimes F_1)$, $\theta_2 \in H^0(S, \Theta \otimes F_2)$, such that $\theta_1 \wedge \theta_2(p) \neq 0$ at at least one point $p \in S$.*

## 4 Bihermitian surfaces

We apply these results to complete the classification of bihermitian 4-manifolds $M$ (see [1],[2] [13]), when $b_1(M) = 1$ and $b_2(M) > 0$: A bihermitian surface is a riemannian oriented connected 4-manifold $(M, g)$ endowed with two integrable almost complex structures $J_1$, $J_2$ inducing the same orientation, orthogonal with respect to $g$ and independent i.e. $J_1(x) \neq \pm J_2(x)$ for at least one point $x \in M$. This structure depends only on the conformal class $c$ of $g$. A bihermitian surface is strongly bihermitian if $J_1(x) \neq \pm J_2(x)$ for every point $x \in M$. We denote

$$\mathcal{D}_+ = \{x \in M \mid J_1(x) = J_2(x)\}, \quad \mathcal{D}_- = \{x \in M \mid J_1(x) = -J_2(x)\}, \quad \mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-.$$

The key observation is that under these assumptions, $(M, J_i)$, $i = 1, 2$ admit a numerically anticanonical divisor, more precisely there exists a flat line bundle $L^t$ with $t \in \mathbb{R}_+^\star$, such that $H^0(S, -K \otimes L^t) \neq 0$. Remark that for an odd Inoue-surface $S$, i.e. with one cycle $C$ of rational curves, $-K + F = C$, with $F^{\otimes 2} = \mathcal{O}$, therefore $F = L^{-1}$ and $S$ cannot have a bihermitian structure.

**Theorem 4. 6** *Let* $(M, c, J_1, J_2)$ *be a compact bihermitian surface with odd first Betti number.*
*1) If* $(M, c, J_1, J_2)$ *is strongly bihermitian (i.e* $\mathcal{D} = \emptyset$*), then the complex surfaces* $(M, J_i)$ *are minimal and either a Hopf surface covered by a primary one associated to a contraction* $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *of the form*

$$F(z_1, z_2) = (\alpha z_1 + s z_2^m, a\alpha^{-1} z_2),$$

$$\text{with} \quad a, s \in \mathbb{C},\ 0 < |\alpha|^2 \leq a < |\alpha| < 1,\ (a^m - \alpha^{m+1})s = 0,$$

*or else* $(M, J_i)$ *are Inoue surfaces* $S^+_{N,p,q,r;t}$, $S^-_{N,p,q,r}$.
*2) If* $(M, c, J_1, J_2)$ *is not strongly bihermitian, then* $\mathcal{D}$ *has at most two connected components,* $(M, J_i)$, $i = 1, 2$, *contain GSS and the minimal models* $S_i$ *of* $(M, J_i)$ *are*

- *Surfaces with GSS of intermediate type if* $\mathcal{D}$ *has one connected component*
- *Hopf surfaces of special type (see* [13] 2.2*), Inoue (parabolic) surfaces or Inoue-Hirzebruch surfaces if* $\mathcal{D}$ *has two connected components.*

*Moreover, the blown-up points belong to the NAC divisors.*

If moreover the metric $g$ is anti-self-dual (ASD), we obtain

**Corollary 4. 7** *Let* $(M, c, J_1, J_2)$ *be a compact ASD bihermitian surface with odd first Betti number. Then the minimal models of the complex surfaces* $(M, J_i)$, $i = 1, 2$, *are*

- *Hopf surfaces of special type (see [13] 2.2),*
- *(parabolic) Inoue surfaces or*
- *even Inoue-Hirzebruch surfaces.*

*Moreover, the blown-up points belong to the NAC divisors.*

**Details may be found on Arxiv and will be published in Am. J. Math 2005**

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# Saddle measures for holomorphic endomorphisms of $\mathbb{CP}^2$

Henry de Thélin

Let $f$ be a holomorphic endomorphism of $\mathbb{CP}^2$ of algebraic degree $d \geq 2$.

J.E. Fornæss and N. Sibony defined the Green current $T$ of $f$. We can obtain it as follows. Let $L$ be a generic projective line (the genericity means outside an algebraic subset of the dual of $\mathbb{CP}^2$). Then $f^{-n}(L)$ is an algebraic curve of degree $d^n$ and we have :

**Theorem.** *(see [6], [7] and [5])*

*There is only one limit for the sequence $\frac{[f^{-n}(L)]}{d^n}$. This limit is the Green current $T$.*

The support of this current is exactly the Julia set of $f$.

**Example.** *If we consider $f([z:w:t]) = [z^2 : w^2 : t^2]$ in the chart $t \neq 0$ then :*

$$T = \int [\{e^{i\theta}\} \times D] d\lambda(\theta) + \int [D \times \{e^{i\theta}\}] d\lambda(\theta) + \int [V_\theta] d\lambda(\theta)$$

*where $\lambda$ is the Lebesgue measure on $[0, 2\pi]$, $D$ the unit disk in $\mathbb{C}$ and $V_\theta = \{(z,w),\ z = e^{i\theta}w,\ |z| \geq 1\}$.*

The current $T$ has a continuous potential : in particular, we can define the Green measure $\mu$ with the formula $\mu := T \wedge T$ (see [7] and [8]).

J.-Y. Briend and J. Duval gave an other construction for this measure $\mu$ :

**Theorem.** *(see [2])*

*Let $\gamma$ be a generic point of $\mathbb{CP}^2$. Then the sequence of probabilities $\frac{1}{d^{2n}} \sum_{f^n(\gamma_i)=\gamma} \delta_{\gamma_i}$ converges to the measure $\mu$.*

In the theorem the genericity means that $\gamma$ is outside an algebraic subset of $\mathbb{CP}^2$ but when $f$ is generic the convergence to $\mu$ holds for all $\gamma \in \mathbb{CP}^2$.

The dynamical properties of $\mu$ are given by the following result of J.-Y. Briend and J. Duval :

**Theorem.** *(see [1] and [2])*

*The measure $\mu$ is the unique measure of maximal entropy $2\log(d)$ and its Lyapounov exponents are greater or equal to $\frac{\log(d)}{2}$.*

The second part of the previous theorem means that almost every point for $\mu$ has two expanding directions and the fact that $\mu$ is the unique measure of maximal entropy implies that $\mu$ is the measure with the richest dynamics for $f$.

In this talk, we will describe the dynamics of $f$ outside the support of $\mu$. The topological entropy of $f$ outside this support is smaller than $\log(d)$ (see [3]), so, our aim is to have measures $\nu$ of maximal entropy (i.e. $\log(d)$) outside the support of $\mu$, and to evaluate their Lyapounov exponents.

Let $L$ be a projective line of $\mathbb{CP}^2$ and $S$ be a limit of the sequence $S_m = \frac{1}{m}\sum_{i=0}^{m-1}\frac{[f^i(L)]}{d^i}$. We have $f_*S = dS$ and the intersection of $S$ with the Green current $T$ gives a measure $\nu = T \wedge S$ which is invariant by $f$ (in fact the limit of $S_m$ isn't unique, so we may have a lot of different measures $\nu$).

When $f$ is hyperbolic, J.E. Fornæss and N. Sibony proved in [9] that these measures $\nu$ are saddle (i.e. they admit a positive and a negative Lyapounov exponent) and that their supports are outside the support of the Green measure $\mu$.

The goal of this talk is to give the dynamics of these measures in the general case (i.e. for all $f$). First of all, we have (see [4]) :

**Theorem 1.**
*The entropy of $\nu$ is greater or equal to $\log(d)$.*

In particular when the support of $\nu$ is outside the support of $\mu$ (it happens often) then $\nu$ is a measure of maximal entropy outside the support of $\mu$. So these measures are the good ones in order to describe the dynamics of $f$ outside the support of $\mu$. It remains to evaluate their Lyapounov exponents.

In the general case, $\nu$ isn't ergodic. In particular, we can't deduce from the Ruelle's inequality that $\nu$ has a positive exponent. However, if we use geometric arguments we can prove (see [4]) :

**Theorem 2.**
*For $\nu$ almost every point $x$, the highest Lyapounov exponent of $\nu$ at $x$ is greater or equal to $\frac{\log(d)}{2}$.*

Here the highest Lyapounov exponent at the point $x$ is equal to $\lim_n \frac{1}{n}\log\|D_xf^n\|$ and so this theorem means that for almost every point there is an expanding direction.

The bound of this theorem is sharp : to see this take a polynomial endomorphism $f$ such that the restriction of $f$ on the line $L$ at infinity is a Lattès map. In this case the measure $\nu$ is exactly the equilibrium measure of the Lattès map and so the highest Lyapounov exponent is equal to $\frac{\log(d)}{2}$.

For the smallest Lyapounov exponent, we have the following theorem (see [4]) :

**Theorem 3.**
*Suppose that $\nu$ has no mass on algebraic curves. Then for $\nu$ almost every point $x$ outside the support of $\mu$, the smallest Lyapounov exponent is non-positive.*

In particular the measures $\nu$ are saddle in a weak sense : we have an expanding and a non-expanding direction for almost every point outside the support of $\mu$.

The hypothesis on $\nu$ (i.e. the fact that $\nu$ does not put mass on algebraic curves) in the last theorem is generic on $f$. Moreover the bound on the smallest exponent is sharp.

**Ideas for the proof of the theorem 2 :**

We will explain the origin of the dilation.

**First step**

We give here the construction of a subdivision of $f^i(L)$ into reasonable disks.

Let $\gamma$ be a generic point in $\mathbb{CP}^2$ and $N$ be a projective line. We can define the projection $\pi : \mathbb{CP}^2 - \{\gamma\} \mapsto N$ by using the pencil of lines through the point $\gamma$. The restriction of $\pi$ on $f^i(L)$ gives a ramified covering of degree $d^i$ between $f^i(L)$ and $N$ (it is the Bezout's theorem because the degree of $f^i(L)$ is $d^i$).

The disks in $f^i(L)$ will be preimages of squares of $N$ by the map $\pi_{|f^i(L)}$.

We take a subdivision of $N$ into squares. If we consider a preimage by $\pi_{|f^i(L)}$ of a square $c$, we may have a graph (we call it "good component") or a ramification point (a "bad component").

We want to show that the number of good components is greater than the number of bad components. But we know that the number $r$ of bad components is roughly speaking equal to the number of ramifications for $\pi_{|f^i(L)}$. So, by the Riemann-Hurwitz's formula, we have :

$$r + \chi(f^i(L)) = \chi(N)d^i$$

i.e.

$$r = 2d^i - 2.$$

So the number of good components (which is equal to Card. of the subdivision of $N \times d^i - (2d^i - 2)$) is greater than the number of bad components and the difference grows up when the subdivision of $N$ gets smaller.

To simplify, we suppose now that we have $d^i$ good disks (i.e. graphs) $\Delta$ on $f^i(L)$ with radius 1.

**Second step**

We take the preimages of these $d^i$ disks by $f^i$ which are in $L$. We obtain $d^i$ disks $\Delta_j$ in $L$ and the number of $\Delta_j$ with area greater than $Cd^{-i}$ is smaller than $d^i/C$ (because the area of $L$ is 1). So, for almost every preimage $\Delta_j$ we have $\text{area}(\Delta_j) \leq Cd^{-i}$.

Now if we reduce a little the disks $\Delta$ and if we use an estimate of J.-Y. Briend and J. Duval (see the appendix of [2]), we obtain :

$$\text{diam}(\Delta_j)^2 \leq K\text{area}(\Delta_j) \leq KCd^{-i}$$

for almost all preimage of the $d^i$ disks of $f^i(L)$ in $L$.

By using the Cauchy's formula (and by reducing a little more the disks $\Delta$), we obtain :

$$\|D_x f^i\| \geq \frac{d^{i/2}}{(KC)^{1/2}}$$

for $x$ in $\Delta_j$.

It implies that we have a lot of points $x$ for which :

$$\lim_n \frac{1}{n} \log \|D_x f^n\| \geq \frac{\log(d)}{2}.$$

This is the estimate that we expected.

In conclusion, the dilation comes from the fact that the area of $L$ is equal to 1 and the area of $f^i(L)$ is equal to $d^i$ (i.e. comes from cohomological reasons).

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# Fefferman-Graham metric for even dimensional conformal structures

Kengo Hirachi

In the paper, Fefferman [F] initiated a program of studying local invariants of CR manifold of dimension $2n-1$ by using a Ricci-flat Lorentz-Kähler metric on an $n+1$ dimensional complex manifold. This program was later generalized by Fefferman-Graham [FG] to the case of conformal manifold $(M^n,[g])$; the associated Ricci-flat Lorentz metric $\widetilde{g}$ is defined on an $n+2$ dimensional real manifold. The metric $\widetilde{g}$ is now called the *ambient metric* or *Fefferman-Graham metric.* The ambient metric becomes a standard tool in CR and conformal geometries, but in CR and even-dimensional conformal cases the construction of the ambient metric is obstructed at a finite jet and thus the ambient metric construction of CR/conformal invariants are not complete.

In this note, I describe how to improve the construction of the ambient metric with the intention to get all conformal invariants out of the metric. (See [H] for the case of CR geometry.) This is an interim report on a joint project with Robin Graham; but I am responsible for any error in this note.

**1. Conformal invariant.** Let $g=\sum_{i,j=1}^n g_{ij}(x)dx^i dx^j$ be a metric defined on a neighborhood of $0\in\mathbb{R}^n$. We want to write down all conformally invariant expressions in the $g_{ij}(x)$ and their derivatives of all orders. To start with, we consider scalar conformal invariant. Thus a conformal invariant $I(g)$ is a polynomial in $(\det g_{ij})^{-1}$, and the derivatives $\partial^\alpha g_{ij}$, satisfying two invariance properties:

(1) $I(g)$ is independent of the choice of the coordinate that represent $g_{ij}$ and its derivatives.

(2) There is a constant $w\in\mathbb{R}$ (called the weight of $I$) such that $I(e^{2f}g)=e^{2wf}I(g)$ for any smooth function $f$.

The first condition says that $I(g)$ is a Riemannian invariant and the second says that it is covariant under scaling of the metric. It is well-know that the Weyl tensor $W_{ijkl}$, the trace free part of the Riemannian curvature, is a local conformally invariant tensor. Thus any $O(n)$-invariant homogeneous polynomial of $W_{ijkl}$ gives a conformal invariant. However, it is not easy to give even one example of conformal conformal invariant that contain higher derivatives of the metric.

**2. Ambient metric.** Let $(M, [g])$ be an $n$-dimensional conformal manifold and $\pi : \mathcal{G} = \{(x, t^2 g(x)) : x \in M, t > 0\} \to M$ be the metric bundle, which admits an $\mathbb{R}_+$-action $\delta_s(x, g) = (x, s^2 g)$ on each fiber. The tautological 2-tensor $\widetilde{g}_0$ on $\mathcal{G}$ is defined by

$$\widetilde{g}_0(X, Y) = g(\pi_* X, \pi_* Y) \quad \text{for } X, Y \in T_{(x,g)}\mathcal{G}.$$

The ambient space $\widetilde{M}$ of $M$ is an $(n+2)$-dimensional manifold that contains the metric bundle $\mathcal{G}$ as a hypersurface and admits an $\mathbb{R}_+$-action extending that on $\mathcal{G}$. The generator of the $\mathbb{R}_+$-action is denoted by $X$ and the inclusion is denoted by $\iota : \mathcal{G} \to \widetilde{M}$.

An ambient metric for a conformal structure $[g]$ is a Lorentzian metric $\widetilde{g}$ on $\widetilde{M}$ that solves $\mathrm{Ricci}(\widetilde{g}) = 0$ (to $\infty$-jets along $\mathcal{G}$) such that

(i) $\iota^* \widetilde{g} = g_0$

(ii) $\widetilde{g}$ is homogeneous of degree 2, that is, $\delta_s^* \widetilde{g} = s^2 \widetilde{g}$.

**Theorem ([FG]).** *If $n$ is odd, the ambient metric $\widetilde{g}$ exists uniquely up to $\mathbb{R}_+$-equivariant diffeomorphisms that fix $\mathcal{G}$.*

In odd-dimensions, we can construct all conformal invariant — see Theorem ([FG], [BEG]) below. However, if $n$ is even, the equation for the ambient metric may not have a smooth solution. We thus consider ambient metrics with singularities: An (singular) ambient metric for a conformal structure $[g]$ is a formal solution to $\mathrm{Ricci}(\widetilde{g}) = 0$ along $\mathcal{G}$ such that:

(i) $\iota^* \widetilde{g} = g_0$

(ii) Let $Q = \widetilde{g}(X, X)$, then $\widetilde{g}$ admits an expansion

$$\widetilde{g} = g^{(0)} + \sum_{k=1}^{\infty} g^{(k)} (Q^{n/2} \log Q)^k,$$

where each $g^{(k)}$ has homogeneous degree $2 - nk$.

(iii) $\nabla_X X = X$ (it means that $\mathbb{R}_+$-orbits are parametrized geodesics.)

Note that (iii) implies that $Q$ is a smooth function even thought $\widetilde{g}$ is not. Such $\widetilde{g}$ always exits and then $Q$ is shown to be smooth. If $n$ is odd, singular ambient metric is shown to be smooth and we get nothing new (The condition (iii) follows from other ones).

**Theorem.** *If $n$ is even, the (singular) ambient metric $\widetilde{g}$ exists for any conformal structure $[g]$.*

In even-dimensions, the ambient metric is not unique (even up to diffeomorphisms). However, we can parametrize the family of ambient metrics for a fixed $[g]$ by a two tensor appears in the $n/2$-jet of $g^{(0)}$.

**3. Invariant theory.** We now construct conformal invariants by using the ambient metric. If $n$ is odd, we define $R^{(p)} = \nabla^p R$ be the $p$-th iterated covariant derivative of the curvature of $\widetilde{g}$. If $n$ is even, we define $R^{(p)}$ to be that for the smooth part $g^{(0)}$ of $\widetilde{g}$. Let $\widetilde{\varepsilon}$ be the volume form of $\widetilde{g}$ and set

$$\widetilde{\varepsilon}_0 = X \lrcorner \widetilde{\varepsilon}.$$

Then we define scalar valued functions by taking the following complete contractions (with respect to $\widetilde{g}$):

$$\begin{aligned} &\mathrm{contr}(R^{(p_1)} \otimes \cdots \otimes R^{(p_d)}), \\ &\mathrm{contr}(R^{(p_1)} \otimes \cdots \otimes R^{(p_d)} \otimes \widetilde{\varepsilon}), \\ &\mathrm{contr}(R^{(p_1)} \otimes \cdots \otimes R^{(p_d)} \otimes \widetilde{\varepsilon}_0). \end{aligned}$$

Such a contraction defines an $\mathbb{R}_+$-homogenous function $\widetilde{W}$ on $\widetilde{M}$. Since a metric $g \in [g]$ defines a section $S_g$ of $\mathcal{G} \to M$, the composition $W := \widetilde{W} \circ S_g$ gives a function on $M$. We call such functions *Weyl invariants.* If $n$ is odd, we can shown that $W$ depends polynomially on the jets of the metric and thus define a conformal invariant of weight $\sum_{j=1}^{d}(-p_j - 2)$.

**Theorem ([FG], [BEG]).** *If $n$ is odd, all Weyl invariants are conformal invariants, and all conformal invariants are given as linear combinations of Weyl invariants.*

If $n$ is even, the ambient metric is not unique and thus Weyl invariant may not determined by the conformal structure. However, there are many linear combinations of Weyl invariants that are independent of the choice of the ambient metric. In fact, we have

**Theorem.** *If n, each conformal invariant is written as a linear combination of Weyl invariants and exceptional invariants.*

Exceptional invariants are another class of invariants studied by Bailey and Gover [BG]. They exit only when $n$ is divisible by 4 and weight is $-n$, and their construction in terms of $R^{(p)}$ is well-understood.

This theorem does not tell which linear combination of Weyl invariants gives conformal invariants. It is the main open problem and we only have some sufficient conditions. For example,

**Proposition.** *Let $n$ be even. Weyl invariants of degree $d$ of weight $> -2(d-1)-n$ are conformal invariants.*

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# THE ASYMPTOTIC EXPANSION OF THE BERGMAN KERNEL ON NON-COMPACT MANIFOLDS

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The asymptotic of the Bergman kernel on high tensor powers of a line bundle has attracted a lot of attention recently.

Let $(X, \Theta)$ be a Hermitian manifold of dimension $n$, where $\Theta$ is the $(1,1)$ form associated to a hermitian metric on $X$. Given Hermitian holomorphic line bundle $L$ on $X$, we consider the space of $L^2$ holomorphic sections $H^0_{(2)}(X, L^p)$ in the tensor powers $L^p = L^{\otimes p}$.

Let $P_p(x, x')$, $(x, x' \in X)$ be the Schwartz kernel of the orthogonal projection $P_p$ from the space $L^2(X, L^p)$ of $L^2$ sections of $L^p$ onto $H^0_{(2)}(X, L^p)$ with respect to the Riemannian volume form $dv_X(x')$ associated to $(X, \Theta)$. Then by the ellipticity of the Kodaira-Laplacian and Schwartz kernel theorem, we know $P_p(x, x')$ is $\mathscr{C}^\infty$. Choose an orthonormal basis $(S_i^p)_{i=1}^{d_p}$ $(d_p \in \mathbb{N} \cup \{\infty\})$ of $H^0_{(2)}(X, L^p)$. The Bergman kernel can then be expressed as

$$P_p(x, x') = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(x'))^* \in (L^p)_x \otimes (L^p)^*_{x'}.$$

Moreover its restriction on the diagonal has the form

$$B_p(x) = P_p(x, x) = \sum_{i=1}^{d_p} \left|S_i^p(x)\right|^2 \in \mathbb{R}.$$

We denote by $K_X = \det(T^{*(1,0)}X)$ the canonical line bundle of $X$ and by $R^{\det}$ the Ricci curvature of $\Theta$ (i.e. the curvature of $K_X$ induced by $\Theta$). The line bundle $L$ is supposed to be positive and we set

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L.$$

We denote by $g^{TX}_\omega$ the Riemannian metric associated to $\omega$ and by $r^X_\omega$ the scalar curvature of $g^{TX}_\omega$. Moreover, let $a_1 \leqslant \ldots \leqslant a_n$ be the eigenvalues of $\omega$ with respect to $\Theta$. The torsion of $\Theta$ is defined by $T = [i(\Theta), \partial\Theta]$, where $i(\Theta)$ is the contraction with $\Theta$. We have the following result.

**Theorem 1** ([2]). *Assume that $(X, \Theta)$ is a complete Hermitian manifold of dimension $n$. Suppose that there exist $\varepsilon > 0$, $C > 0$ such that one of the following assumptions holds true:*

$$\sqrt{-1}R^L \geqslant \varepsilon\Theta, \quad \sqrt{-1}R^{\det} \geqslant -C\Theta, \quad |T| \leqslant C\Theta \tag{1}$$

*Then the kernel $P_p(x, x')$ has a full off-diagonal asymptotic expansion uniformly on compact sets of $X \times X$. Especially, there exist coefficients $b_r \in \mathscr{C}^\infty(X)$, $r \in \mathbb{N}$, such that for any compact set $K \subset X$, any $k, l \in \mathbb{N}$, there exists $C_{k,l,K} > 0$ such that for $p \in \mathbb{N}$,*

$$\Big|\frac{1}{p^n}B_p(x) - \sum_{r=0}^{k} b_r(x)p^{-r}\Big|_{\mathscr{C}^l(K)} \leqslant C_{k,l,K}\, p^{-k-1}.$$

*Moreover, $b_0 = a_1 \cdots a_n$ and*

$$b_1 = \frac{a_1 \cdots a_n}{8\pi}\Big[r^X_\omega - 2\Delta_\omega\Big(\log(a_1 \cdots a_n)\Big) + 4\sum_{j=1}^{n} R^E(v_{\omega,j}, \overline{v}_{\omega,j})\Big],$$

*where $\{v_{\omega,j}\}$ is an orthonormal basis of $(T^{(1,0)}X, g^{TX}_\omega)$.*

Let us remark that if $L = K_X$, the first two conditions in (1) are to be replaced by

$$h^L \text{ is induced by } \Theta \text{ and } \sqrt{-1}R^{\det} < -\varepsilon\Theta. \tag{2}$$

Moreover, if $(X, \Theta)$ is Kähler, the condition on the torsion is trivially satisfied.

The proof is based on the observation that the Kodaira-Laplacian $\square_p = \overline{\partial}^*\overline{\partial}$ acting on $\boldsymbol{L}^2(X, L^p)$ has a spectral gap of the form

$$\operatorname{Spec}\square_p \subset \{0\} \cup [2p\mu_0 - C_L, \infty)$$

where $\mu_0 = \inf_{x\in X} a_1(x)$ and $C_L$ is a constant which depends on the geometry of $L$ and $X$. The technique from [2] apply then and deliver the result.

Theorem 1 has several applications e.g. holomorphic Morse inequalities on non-compact manifolds (as the well-known results of Nadel-Tsuji [6], see also [2, 8]) or Berezin-Toeplitz quantization (see [4] or the fothcommimg [3]).

We will emphazise in the sequel the Bergman kernel for a singular metric. Let $X$ be a compact complex manifold. A *singular Kähler metric* on $X$ is a closed, strictly positive $(1,1)$-current $\omega$. If the cohomology class of $\omega$ in $H^2(X, \mathbb{R})$ is integral, there exists a holomorphic line bundle $(L, h^L)$, endowed with a singular Hermitian metric, such that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$ in the sense of currents. We call $(L, h^L)$ a *singular polarization* of $\omega$.

If we change the metric $h^L$, the curvature of the new metric will be in the same cohomology class as $\omega$. In this case we speak of a polarization of $[\omega] \in H^2(X, \mathbb{R})$. Our purpose is to define an appropriate notion of polarized section of $L^p$, possibly by changing the metric of $L$, and study the associated Bergman kernel.

**Corollary 2.** *Let $(X,\omega)$ be a compact complex manifold with a singular Kähler metric with integral cohomology class. Let $(L,h^L)$ be a singular polarization of $[\omega]$ with strictly positive curvature current having singular support along a proper analytic set $\Sigma$. Then the Bergman kernel of the space of polarized sections*

$$H^0_{(2)}(X\smallsetminus\Sigma,L^p)=\left\{u\in L^{0,0}_2(X\smallsetminus\Sigma,L^p\,,\,\Theta_P\,,h^L_\varepsilon):\overline{\partial}^{L^p}u=0\right\}$$

*has the asymptotic expansion as in Theorem 1 for $X\smallsetminus\Sigma$, where $\Theta_P$ is a generalized Poincaré metric on $X\smallsetminus\Sigma$ and $h^L_\varepsilon$ is a modified Hermitian metric on $L$.*

Using an idea of Takayama [7], Corollary 2 gives a proof of the Shiffman-Ji-Bonavero-Takayama criterion, about the characterization of Moishezon manifolds by $(1,1)$ positive currents.

We mention further the Berezin-Toeplitz quantization. For a complex manifold $X$, let $\mathscr{C}^\infty_{const}(X)$ denote the algebra of smooth functions of $X$ which are constant outside a compact set. For any $f\in\mathscr{C}^\infty_{const}(X)$ we denote for simplicity the operator of multiplication with $f$ still by $f$ and consider the linear operator

$$T_{f,p}:\boldsymbol{L}^2(X,L^p)\longrightarrow\boldsymbol{L}^2(X,L^p)\,,\quad T_{f,p}=P_p\,f\,P_p\,. \tag{3}$$

The family $(T_{f,p})_{p\geqslant1}$ is called a Toeplitz operator. The following result generalizes [1] to non-compact manifolds.

**Corollary 3.** *We assume that $(X,\Theta)$ and $(L,h^L)$ satisfy the same hypothesis as in Theorem 1. Let $f,g\in\mathscr{C}^\infty_{const}(X)$. The product of the two corresponding Toeplitz operators admits the asymptotic expansion*

$$T_{f,p}T_{g,p}=\sum_{r=0}^{\infty}p^{-r}T_{C_r(f,g),p}+\mathscr{O}(p^{-\infty})$$

*where $C_r$ are differential operators. More precisely,*

$$C_0(f,g)=fg\,,\quad C_1(f,g)-C_1(g,f)=\frac{1}{\sqrt{-1}}\{f,g\}$$

*where the Poisson bracket is taken with respect to the metric $2\pi\omega$. Therefore*

$$[T_{f,p},T_{g,p}]=p^{-1}T_{\frac{1}{\sqrt{-1}}\{f,g\},p}+\mathscr{O}(p^{-2})$$

If we wish to consider more general class of functions as $\mathscr{C}^\infty_{const}(X)$ we have to impose some restrictions on the geometry of $X$ at infinity.

Finally, we refer the reader to the contribution of Ma and Zhang [5] for further aspects of the Bergman kernels.

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# BERGMAN KERNELS AND GEOMETRIC QUANTIZATION

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In this talk, we explain our recent results on the asymptotic expansion of the Bergman kernel and its relation to the geometric quantization [8]. The talk of Marinescu [7] gives further aspects of the Bergman kernel. The interested readers may find complete references in [3], [5], [8], especially in the forthcoming book [6].

Let $(X, \omega)$ be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle $L$ over $X$ endowed with a Hermitian connection $\nabla^L$ with the property that $\frac{\sqrt{-1}}{2\pi} R^L = \omega$, where $R^L = (\nabla^L)^2$ is the curvature of $(L, \nabla^L)$. Let $(E, h^E)$ be a Hermitian vector bundle on $X$ equipped with a Hermitian connection $\nabla^E$ and $R^E$ denotes the associated curvature.

Let $g^{TX}$ be a Riemannian metric on $X$. Let $\mathbf{J} : TX \to TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u, v) = g^{TX}(\mathbf{J}u, v) \tag{0.1}$$

for $u, v \in TX$. Let $J$ be an almost complex structure such that $g^{TX}(Ju, Jv) = g^{TX}(u, v)$, $\omega(Ju, Jv) = \omega(u, v)$, and that $\omega(\cdot, J\cdot)$ defines a metric on $TX$. Then $J$ commutes with $\mathbf{J}$ and $J = \mathbf{J}(-\mathbf{J}^2)^{-1/2}$. Let $\nabla^{TX}$ be the Levi-Civita connection on $(TX, g^{TX})$ with curvature $R^{TX}$ and scalar curvature $r^X$, and $\nabla^{TX}$ induces a natural connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ with curvature $R^{\det}$, and the Clifford connection $\nabla^{\mathrm{Cliff}}$ on the Clifford module $\Lambda(T^{*(0,1)}X)$ with curvature $R^{\mathrm{Cliff}}$. The spin$^c$ Dirac operator $D_p$ acts on $\Omega^{0,\bullet}(X, L^p \otimes E) = \bigoplus_{q=0}^{n} \Omega^{0,q}(X, L^p \otimes E)$, the direct sum of spaces of $(0,q)$–forms with values in $L^p \otimes E$. We denote by $D_p^+$ the restriction of $D_p$ on $\Omega^{0,\mathrm{even}}(X, L^p \otimes E)$.

Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\dim G = n_0$. Suppose that $G$ acts on $X$ and its action on $X$ lifts on $L$ and $E$. Moreover, we assume the $G$-action preserves the above connections and metrics on $TX, L, E$ and $J$. Then $\mathrm{Ind}(D_p^+)$ is a virtual representation of $G$. Denote by $(\mathrm{Ker}\, D_p)^G$, $\mathrm{Ind}(D_p^+)^G$ the $G$-trivial components of $\mathrm{Ker}\, D_p$, $\mathrm{Ind}(D_p^+)$ respectively.

The $G$-invariant Bergman kernel is $P_p^G(x, x')$ $(x, x' \in X)$, the smooth kernel of $P_p^G$, the orthogonal projection from $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle \ \ \rangle)$ on $(\mathrm{Ker}\, D_p)^G$, with respect to the Riemannian volume form $dv_X(x')$. The purpose of this paper is to study the asymptotic expansion of the $G$-invariant Bergman kernel $P_p^G(x, x')$ as $p \to \infty$, and we will relate it to the asymptotic expansion of the Bergman kernel on the symplectic reduction $X_G$.

**Theorem 0.1.** *For any open $G$-neighborhood $U$ of $P$ in $X$, $\varepsilon_0 > 0$, $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ (depend on $U$, $\varepsilon_0$) such that for $p \geq 1$, $x, x' \in X, d(Gx, x') \geq \varepsilon_0$ or*

*$x, x' \in X \setminus U$,*

$$|P_p^G(x,x')|_{\mathscr{C}^m} \leq C_{l,m} p^{-l}. \tag{0.2}$$

*where $\mathscr{C}^m$ is the $\mathscr{C}^m$-norm induced by $\nabla^L, \nabla^E$, $\nabla^{TX}$, $h^L, h^E, g^{TX}$.*

Assume for simplicity that $G$ acts freely on $P$. Let $U$ be an open $G$-neighborhood of $\mu^{-1}(0)$ such that $G$ acts freely on $U$. For any $G$-equivariant vector bundle $(F, \nabla^F)$ on $U$, we denote by $F_B$ the bundle on $U/G = B$ induced naturally by $G$-invariant sections of $F$ on $U$. The connection $\nabla^F$ induces canonically a connection $\nabla^{F_B}$ on $F_B$. Let $R^{F_B}$ be its curvature. We denote also $\mu^F(K) = \nabla^F_K - L_K \in \mathrm{End}(F)$ for $K \in \mathfrak{g}$. Note that $P_p^G \in (\mathscr{C}^\infty(U \times U, \mathrm{pr}_1^* E_p \otimes \mathrm{pr}_2^* E_p^*))^{G\times G}$, thus we can view $P_p^G(x,x')$ as a smooth section of $\mathrm{pr}_1^*(E_p)_B \otimes \mathrm{pr}_2^*(E_p^*)_B$ on $B \times B$.

Let $g^{TB}$ be the Riemannian metric on $U/G = B$ induced by $g^{TX}$. Let $\nabla^{TB}$ be the Levi-Civita connection on $(TB, g^{TB})$ with curvature $R^{TB}$. Let $N_G$ be the normal bundle to $X_G$ in $B$. We identify $N_G$ with the orthogonal complement of $TX_G$ in $(TB|_{X_G}, g^{TB})$. Let $g^{TX_G}$, $g^{N_G}$ be the metrics on $TX_G$, $N_G$ induced by $g^{TB}$ respectively. Let $P^{TX_G}$, $P^{N_G}$ be the orthogonal projections from $TB|_{X_G}$ on $TX_G$, $N_G$ respectively. Set

$$\begin{aligned} &\nabla^{N_G} = P^{N_G}(\nabla^{TB}|_{X_G})P^{N_G}, \quad \nabla^{TX_G} = P^{TX_G}(\nabla^{TB}|_{X_G})P^{TX_G}, \\ &{}^0\nabla^{TB} = \nabla^{TX_G} \oplus \nabla^{N_G}, \quad A = \nabla^{TB} - {}^0\nabla^{TB}. \end{aligned} \tag{0.3}$$

Then $\nabla^{N_G}, {}^0\nabla^{TB}$ are Euclidean connections on $N_G$, $TB|_{X_G}$ on $X_G$, $\nabla^{TX_G}$ is the Levi-Civita connection on $(TX_G, g^{TX_G})$, and $A$ is the associated second fundamental form. We denote by $\mathrm{vol}(Gx)$ ($x \in U$) the volume of the orbit $Gx$ equipped with the metric induced by $g^{TX}$. Let $h(x)$ be the function on $U$ defined by

$$h(x) = (\mathrm{vol}(Gx))^{1/2}. \tag{0.4}$$

Then $h$ reduces to a function on $B$. We denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X) \otimes E$ onto $\mathbb{C} \otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X) \otimes E = \mathbb{C} \otimes E \oplus \Lambda^{>0}(T^{*(0,1)}X) \otimes E$, and $I_{\mathbb{C}\otimes E_B}$ the corresponding projection on $B$.

In the whole note, for any $x_0 \in X_G$, $Z \in T_{x_0}B$, we write $Z = Z^0 + Z^\perp$, with $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$. Let $\tau_{Z^0} Z^\perp \in N_{G, \exp^{X_G}_{x_0}(Z^0)}$ be the parallel transport of $Z^\perp$ with respect to the connection $\nabla^{N_G}$ along the geodesic in $X_G$, $[0,1] \ni t \to \exp^{X_G}_{x_0}(tZ^0)$. For $\varepsilon_0 > 0$ small enough, we identify $Z \in T_{x_0}B$, $|Z| < \varepsilon_0$ with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp) \in B$, then for $x_0 \in X_G$, $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| < \varepsilon_0$, the map $\Psi : TB|_{X_G} \times TB|_{X_G} \to B \times B$,

$$\Psi(Z, Z') = (\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp), \exp^B_{\exp^{X_G}_{x_0}(Z'^0)}(\tau_{Z'^0}Z'^\perp))$$

is well defined. We identify $(E_p)_{B,Z}$ to $(E_p)_{B,x_0}$ by using parallel transport with respect to $\nabla^{(E_p)_B}$ along $[0,1] \ni u \to uZ$. Let $\pi_B : TB|_{X_G} \times TB|_{X_G} \to X_G$ be the natural projection from the fiberwise product of $TB|_{X_G}$ on $X_G$ onto $X_G$. From Theorem 0.1, we only need to understand $P_p^G \circ \Psi$, and under our identification, $P_p^G \circ \Psi(Z, Z')$ is a smooth section of $\pi_B^*(\mathrm{End}(E_p)_B) = \pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$ on $TB|_{X_G} \times TB|_{X_G}$. Let $|\ |_{\mathscr{C}^{m'}(X_G)}$ be the $\mathscr{C}^{m'}$-norm on $\mathscr{C}^\infty(X_G, \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$ induced by $\nabla^{\mathrm{Cliff}_B}$,

$\nabla^{E_B}$, $h^E$ and $g^{TX}$. The norm $|\ |_{\mathscr{C}^{m'}(X_G)}$ induces naturally a $\mathscr{C}^{m'}$-norm along $X_G$ on $\mathscr{C}^\infty(TB|_{X_G}\times TB|_{X_G}, \pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B))$, we still denote it by $|\ |_{\mathscr{C}^{m'}(X_G)}$.

Let $dv_{X_G}$, $dv_{N_G}$ be the Riemannian volume forms on $(X_G, g^{TX_G})$, $(N_G, g^{N_G})$ respectively. Let $\kappa\in\mathscr{C}^\infty(TB|_{X_G},\mathbb{R})$, with $\kappa=1$ on $X_G$, be defined by that for $Z\in T_{x_0}B$, $x_0\in X_G$,

$$dv_B(x_0,Z)=\kappa(x_0,Z)dv_{T_{x_0}B}(Z)=\kappa(x_0,Z)dv_{X_G}(x_0)dv_{N_{G,x_0}}. \tag{0.5}$$

The following result is one of our main results.

**Theorem 0.2.** *Assume that $G$ acts freely on $\mu^{-1}(0)$ and $\mathbf{J}=J$ on $\mu^{-1}(0)$. Then there exist $\mathscr{Q}_r(Z,Z')\in\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}$ $(x_0\in X_G, r\in\mathbb{N})$, polynomials in $Z, Z'$ with the same parity as $r$, such that if we denote by*

$$P^{(r)}_{x_0}(Z,Z')=\mathscr{Q}_r(Z,Z')P(Z,Z'),\quad \mathscr{Q}_0(Z,Z')=I_{\mathbb{C}\otimes E_B}, \tag{0.6}$$

*with*

$$\begin{aligned}P(Z,Z')=&\exp\Big(-\frac{\pi}{2}|Z^0-Z'^0|^2-\pi\sqrt{-1}\langle J_{x_0}Z^0,Z'^0\rangle\Big)\\ &\times 2^{\frac{n_0}{2}}\exp\Big(-\pi(|Z^\perp|^2+|Z'^\perp|^2)\Big),\end{aligned} \tag{0.7}$$

*then there exists $C''>0$ such that for any $k,m,m',m''\in\mathbb{N}$, there exists $C>0$ such that for $x_0\in X_G$, $Z,Z'\in T_{x_0}B$, $|Z|,|Z'|\le\varepsilon_0$,*

$$\begin{aligned}&(1+\sqrt{p}|Z^\perp|+\sqrt{p}|Z'^\perp|)^{m''}\sup_{|\alpha|+|\alpha'|\le m}\Bigg|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\\ &\quad\Bigg(p^{-n+\frac{n_0}{2}}(h\kappa^{\frac12})(Z)(h\kappa^{\frac12})(Z')P^G_p\circ\Psi(Z,Z')-\sum_{r=0}^k P^{(r)}_{x_0}(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac r2}\Bigg)\Bigg|_{\mathscr{C}^{m'}(X_G)}\\ &\le Cp^{-(k+1-m)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^{2(n+k+2)+m}\exp(-\sqrt{C''}\sqrt{p}|Z-Z'|)+\mathscr{O}(p^{-\infty}).\end{aligned} \tag{0.8}$$

Let $\widetilde{h}$ denote the restriction to $X_G$ of the function $h$. Let $\mathscr{I}_p$ be a section of $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B$ on $X_G$ defined by

$$\mathscr{I}_p(x_0)=\int_{Z\in N_G,|Z|\le\varepsilon_0}h^2(x_0,Z)P^G_p\circ\Psi((x_0,Z),(x_0,Z))\kappa(x_0,Z)dv_{N_G}(Z). \tag{0.9}$$

By Theorem 0.1, modulo $\mathscr{O}(p^{-\infty})$, $\mathscr{I}_p(x_0)$ does not depend on $\varepsilon_0$, and

$$\dim(\mathrm{Ker}\,D_p)^G=\int_{X_G}\mathrm{Tr}[\mathscr{I}_p(x_0)]dv_{X_G}(x_0)+\mathscr{O}(p^{-\infty}). \tag{0.10}$$

**Theorem 0.3.** *If $(X,\omega)$ is a compact Kähler manifold and $L, E$ are holomorphic vector bundles with holomorphic Hermitian connections $\nabla^L,\nabla^E$, $\mathbf{J}=J$, and $G$ acts freely on $\mu^{-1}(0)$, then for $p$ large enough, $\mathscr{I}_p(x_0)\in\mathrm{End}(E_G)_{x_0}$, and there exist $\Phi_r(x_0)\in\mathrm{End}(E_G)_{x_0}$ and $\Phi_0=\mathrm{Id}_{E_G}$ such that*

$$\Big|p^{-n+n_0}\mathscr{I}_p(x_0)-\sum_{r=0}^k\Phi_r(x_0)p^{-r}\Big|_{\mathscr{C}^{m'}}\le C_{k,m'}p^{-k-1}. \tag{0.11}$$

*Moreover*

$$\Phi_1(x_0) = \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{3}{4\pi}\Delta_{X_G} \log \widetilde{h} + \frac{1}{2\pi} R^{E_G}_{x_0}(w^0_j, \overline{w}^0_j). \tag{0.12}$$

*Here $r^{X_G}$ is the Riemannian scalar curvature of $(TX_G, g^{TX_G})$, $\Delta_{X_G}$ is the Bochner-Laplacian on $X_G$, and $\{w^0_j\}$ is an orthonormal basis of $T^{(1,0)}X_G$.*

Let $i : P \hookrightarrow X$ be the natural injection. Let $\pi_G : \mathscr{C}^\infty(P, L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$ be the natural identification. Then by a result of Zhang, for $p$ large enough, the map

$$\pi_G \circ i^* : \mathscr{C}^\infty(X, L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$$

induces a natural isomorphism

$$\sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \to H^0(X_G, L^p_G \otimes E_G). \tag{0.13}$$

(When $E = \mathbb{C}$, this result was first proved by Guillemin-Sternberg.)

Let $dv_{X_G}$ be the Riemannian volume form on $(X_G, g^{TX_G})$. Let $\langle\, ,\, \rangle_{L^p_G \otimes E_G}$ be the metric on $L^p_G \otimes E_G$ induced by $h^{L_G}$ and $h^{E_G}$. In view of the analytic approach to the geometric quantization conjecture of Guillemin-Sternberg given in [9], the natural Hermitian product on $\mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$ is the following weighted Hermitian product $\langle\, ,\, \rangle_{\widetilde{h}}$:

$$\langle s_1, s_2 \rangle_{\widetilde{h}} = \int_{X_G} \langle s_1, s_2 \rangle_{L^p_G \otimes E_G}(x_0) \widetilde{h}^2(x_0)\, dv_{X_G}(x_0). \tag{0.14}$$

**Theorem 0.4.** *The isomorphism $(2p)^{-\frac{n_0}{4}}\sigma_p$ is an asymptotic isometry from $(H^0(X, L^p \otimes E)^G, \langle\, ,\, \rangle)$ onto $(H^0(X_G, L^p_G \otimes E_G), \langle\, ,\, \rangle_{\widetilde{h}})$: i.e. if $\{s^p_i\}_{i=1}^{d_p}$ is an orthonormal basis of $(H^0(X, L^p \otimes E)^G, \langle\, ,\, \rangle)$, then*

$$(2p)^{-\frac{n_0}{2}} \langle \sigma_p s^p_i, \sigma_p s^p_j \rangle_{\widetilde{h}} = \delta_{ij} + \mathscr{O}\left(\frac{1}{p}\right). \tag{0.15}$$

The basic philosophy developed in [3], [5], [6] is that the spectral gap properties for the operators proved in [2], [4] implies the existence of the asymptotic expansion for the corresponding Bergman kernels, by using the analytic localization technique inspired by [1, §11]. The key observation here is that the $G$-invariant Bergman kernel is exactly smooth kernel of the orthogonal projection onto the zero space of a deformation of $D^2_p$ by the Casimir operator (i.e., to consider $D^2_p - p\mathrm{Cas}$) which has a spectral gap. Thus the above philosophy applies to the proof of Theorems 0.1, 0.2.

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# Holomorphic Functions of Slow Growth on Coverings of Strongly Pseudoconvex Manifolds

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## 1. Introduction

In my talk I consider certain problems for holomorphic functions of slow growth defined on coverings of strongly pseudoconvex manifolds. The subject was originally motivated by the paper of Gromov, Henkin and Shubin [GHS] on holomorphic $L^2$ functions on coverings of pseudoconvex manifolds. In turn, in the latter paper the authors were trying to find a new approach to a problem of Shafarevich on the holomorphic convexity of the universal covering of a complex projective manifold. Indeed, according to a Grauert theorem, any complex projective manifold $M$ of dimension $n$ admits a holomorphic embedding into a strongly pseudoconvex manifold $L$ of dimension $n+1$, with the same fundamental group. Thus the main idea of [GHS] was to try to develop the complex analysis on coverings $L'$ of $L$ and then, taking restrictions of holomorphic functions on $L'$ to the corresponding coverings $M'(\subset L')$, to study holomorphic functions on $M'$.

In [GHS] the von Neumann dimension was used to measure the space of holomorphic $L^2$-functions on *regular* coverings of a strongly pseudoconvex manifold $M$. In particular, it was shown that the space of such functions is infinite-dimensional. It was also asked whether the regularity of the covering is relevant for the existence of many holomorphic $L^2$-functions on $M'$ or it is just an artifact of the chosen methods which requires a use of von Neumann algebras.

In my talk I will show that actually the regularity of $M'$ is irrelevant for the existence of many holomorphic functions on $M'$. Moreover, I will also present a substantial extension of main results of [GHS]. My method of the proof is completely different and much easier from that used in [GHS] and is based on the $L^2$-cohomology techniques, as well as, on the geometric properties of $M$.

Also in the talk, I will formulate some results related to several interesting problems posed in the paper [GHS]: theorems on peak points for holomorphic $L^2$-functions on $M'$, Hartogs type theorems for holomorphic functions of exponential (:= *slow*) growth defined in certain infinite domains on $M'$, some interpolation theorems for holomorphic functions of slow growth on $M'$ etc.

Concerning the Shafarevich problem, the results of [Br1]-[Br5] and [GHS] don't imply directly any new results in this area. However, one obtains a rich complex function theory on coverings of strongly pseudoconvex manifolds $L'$ (as above). Thus there is a hope that together with some additional ideas and methods it could give an information about holomorphic functions on $M'(\subset L')$. For now, the strongest result in this area is due to Eyssidieux [E]. It states that the regular covering of a complex projective manifold $M$ corresponding to the kernel of all representations $\pi_1(M) \to GL_n(\mathbb{C})$, with a fixed $n$, is holomorphically convex. I also mention another interesting result in this area proved independently by Campana [Ca] and by myself [Br6] which states that the universal covering of a complex projective manifold with a residually solvable fundamental group is holomorphically convex.

## 2. Formulation of Main Results

**2.1.** In order to formulate our main results we first introduce some notation and basic definitions.

Let $M \subset\subset N$ be a domain with smooth boundary $bM$ in an $n$-dimensional complex manifold $N$, specifically,

$$M = \{z \in N \ : \ \rho(z) < 0\} \tag{2.1}$$

where $\rho$ is a real-valued function of class $C^2(\Omega)$ in a neighbourhood $\Omega$ of the compact set $\overline{M} := M \cup bM$ such that

$$d\rho(z) \neq 0 \quad \text{for all} \quad z \in bM\ . \tag{2.2}$$

Let $z_1, \dots, z_n$ be complex local coordinates in $N$ near $z \in bM$. Then the tangent space $T_zN$ at $z$ is identified with $\mathbb{C}^n$. By $T_z^c(bM) \subset T_zN$ we denote the complex tangent space to $bM$ at $z$, i.e.,

$$T_z^c(bM) = \{w = (w_1, \dots, w_n) \in T_z(N) \ : \ \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j = 0\}\ . \tag{2.3}$$

The *Levi form* of $\rho$ at $z \in bM$ is a hermitian form on $T_z^c(bM)$ defined in local coordinates by the formula

$$L_z(w, \overline{w}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k\ . \tag{2.4}$$

The manifold $M$ is called *pseudoconvex* if $L_z(w, \overline{w}) \geq 0$ for all $z \in bM$ and $w \in T_z^c(bM)$. It is called *strongly pseudoconvex* if $L_z(w, \overline{w}) > 0$ for all $z \in bM$ and all $w \neq 0$, $w \in T_z^c(bM)$.

Equivalently, strongly pseudoconvex manifolds can be described as the ones which locally, in a neighbourhood of any boundary point, can be presented as strictly convex domains in $\mathbb{C}^n$. It is also known (see [C], [R]) that any strongly pseudoconvex manifold admits a proper holomorphic map with connected fibres onto a normal Stein space. In particular, if $M$ is a strongly pseudoconvex non-Stein manifold of complex dimension $n \geq 2$, then the union $C_M$ of all compact complex subvarieties of $M$ of complex dimension $\geq 1$ is a compact complex subvariety of $M$.

Without loss of generality we may and will assume that $\pi_1(M) = \pi_1(N)$ for $M$ as above. Let $r : N' \to N$ be an unbranched covering of $N$. By $M' := r^{-1}(M)$ we denote the corresponding covering of $M$. Also, by $bM' := r^{-1}(bM)$ and $\overline{M'} := M' \cup bM'$ we denote the boundary and the closure of $M'$ in $N'$.

Let $dV_{M'}$ be the Riemannian volume form on $M'$ obtained by a Riemannian metric pulled back from $N$. Let $\psi : N' \to \mathbb{R}_+$ be such that $\log \psi$ is uniformly continuous with respect to the path metric induced by this Riemannian metric. By $H^2_\psi(M')$ we denote the Hilbert space of holomorphic functions $g$ on $M'$ with norm

$$\left( \int_{z \in M'} |g(z)|^2 \psi(z) dV_{M'}(z) \right)^{1/2} . \tag{2.5}$$

For $\psi \equiv 1$, we write $H^2(M')$ instead of $H^2_1(M')$.

Let $X$ be a subspace of the space $\mathcal{O}(M')$ of all holomorphic functions on $M'$.

A point $z \in bM'$ is called a *peak point* for $X$ if there exists a function $f \in X$ such that $f$ is unbounded on $M'$ but bounded outside $U \cap M'$ for any neighbourhood $U$ of $z$ in $N'$.

The Oka-Grauert theorem [G] implies that if $M$ is strongly pseudoconvex and $bM$ is not empty then every $z \in bM$ is a peak point for $H^2(M)$. In general it is not known whether a similar statement is true for boundary points of an infinite covering $M'$ of $M$.

Let us introduce the Hilbert space $l_{2,\psi,x}(M')$ of functions $g$ on $x' := r^{-1}(x)$, $x \in M$, with norm

$$|g|_{2,\psi,x} := \left( \sum_{y \in x'} |g(y)|^2 \psi(y) \right)^{1/2} . \tag{2.6}$$

Let $z_i$, $1 \le i \le m$, be distinct points in $M \setminus C_M$.

**Theorem 2.1** *If $M$ is strongly pseudoconvex, then*

*(a) For any $f_i \in l_{2,\psi,z_i}$, $1 \le i \le m$, there exists $F \in H^2_\psi(M')$ such that $F|_{z_i'} = f_i$, $1 \le i \le m$;*

*(b) If $\psi$ is such that $\log \psi$ is bounded from below on $N'$, then each point in $bM'$ is a peak point for $H^2_\psi(M')$.*

**Example 2.2** Let $d$ be the path metric on $M'$ induced by a Riemannian metric pulled back from $N$. For a point $o \in M'$ we set $d_o(x) := d(o, x)$, $x \in M'$. Then as a function $\psi$ one can take, e.g., $e^{cd_o}$ with $c \in \mathbb{R}$ in Theorem 2.1 (a) and with $c \ge 0$ in Theorem 2.1 (b).

Theorem 2.1 gives a substantial extension of one of the main results of [GHS] (see [GHS, Theorem 02]). Similar results are valid for certain weighted $L_p$ spaces of holomorphic functions on $M'$. It is worth noting that results much stronger than Theorem 2.1 can be obtained if $M$ is a strongly pseudoconvex Stein manifold, see [Br1], [Br2] for an exposition.

**2.2.** The Hartogs type theorem presented in this section is related to one of the problems formulated in [GHS].

We retain the notation of the previous section. Consider a domain $D \subset M'$ with a connected piecewise smooth boundary $bD$ such that

$$r(D) \subset\subset M. \tag{2.7}$$

Next, for a fixed $o \in D$ we set $d_o(z) := d(o, z)$, $z \in M'$. Also, by $\overline{D} \subset M'$ we denote the closure of $D$ and by $\mathcal{O}(D)$ the space of holomorphic functions on $D$. Now, recall that a continuous function $f$ on $bD$ is called $CR$ if for every smooth $(n, n-2)$-form $\omega$ on $M'$ with compact support one has

$$\int_{bD} f \cdot \overline{\partial}\omega = 0\,.$$

If $f$ and $bD$ are smooth this is equivalent to $f$ being a solution of the tangential $CR$-equations: $\overline{\partial}_b f = 0$ (see, e.g., [KR]).

Suppose that $f \in C(bD)$ is a $CR$-function satisfying for some positive numbers $c, \delta$ the following conditions

(1)
$$|f(z)| \le e^{cd_o(z)} \quad \text{for all} \quad z \in bD\,;$$

(2) for any $z_1, z_2 \in bD$ with $d(z_1, z_2) \le \delta$
$$|f(z_1) - f(z_2)| \le e^{c\max\{d_o(z_1), d_o(z_2)\}} d(z_1, z_2)\,.$$

**Theorem 2.3** *There is a constant $\tilde{c} = \tilde{c}(c, \delta) > 0$ such that for any $CR$-function $f$ on $bD$ satisfying conditions (1) and (2) there exists $\hat{f} \in \mathcal{O}(D) \cap C(\overline{D})$ such that*

$$\hat{f}|_{bD} = f \quad \text{and} \quad |\hat{f}(z)| \le e^{\tilde{c}d_o(z)} \quad \text{for all} \quad z \in D.$$

**Remark 2.4** (A) If, in addition, $bD$ is smooth of class $C^k$, $1 \le k \le \infty$, and $f \in C^s(bD)$, $1 \le s \le k$, then the extension $\hat{f}$ belongs to $\mathcal{O}(D) \cap C^s(\overline{D})$. This follows from [HL, Theorem 5.1].
(B) Condition (2) means that $f$ is locally Lipschitz with local Lipschitz constants growing exponentially. For instance, from the Cauchy integral formula it follows that this is true if $f$ is the restriction to $bD$ of a holomorphic function of exponential growth defined in a neighbourhood of $bD$ whose width decreases exponentially.
(C) From Theorem 2.1 it follows that holomorphic functions of exponential growth on $M'$ separate points on $M' \setminus C'_M$ where $C'_M := r^{-1}(C_M)$. Thus there are sufficiently many $CR$-functions $f$ on $bD$ satisfying conditions (1) and (2).
(D) Results much stronger than Theorem 2.3 can be obtained if $M$ is a strongly pseudoconvex Stein manifold, see [Br2], [Br3].

**2.3.** In this section we present one of the interpolation theorems for holomorphic functions on coverings of strongly pseudoconvex manifolds.

Let $Y$ be a closed complex submanifold of some neighbourhood of $\overline{M}$. We set $X := Y \cap M$ and assume that

$$X \cap C_M = \emptyset. \tag{2.8}$$

For a covering $r : M' \to M$ of $M$ as above we set $X' := r^{-1}(X)$. Next for a function $\psi : N' \to \mathbb{R}_+$ such that $\log \psi$ is uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$ we define the Banach space $\mathcal{H}_{2,\psi}(X')$ of functions $f$ holomorphic on $X'$ with norm

$$|f|^X_{2,\psi} := \sup_{x \in X} |f|_{2,\psi,x}$$

where $|\cdot|_{2,\psi,x}$ is defined by (2.6). Similarly one defines $\mathcal{H}_{2,\psi}(M')$.

**Theorem 2.5** *For every function $f \in \mathcal{H}_{2,\psi}(X')$, there exists a function $F \in \mathcal{H}_{2,\psi}(M')$ such that $F = f$ on $X'$.*

Analogous results hold for spaces of holomorphic functions $\mathcal{H}_{p,\psi}(X')$ and $\mathcal{H}_{p,\psi}(M')$, $1 \le p \le \infty$, defined similarly in case $r : M' \to M$ is a regular covering of $M$.

# 3. Method of the Proof

**3.1.** The case of coverings of pseudoconvex domains in Stein manifolds is considered in [Br1]-[Br3]. Our method of the proof is based on the theory of coherent Banach sheaves together with Cartan's vanishing cohomology theorems, see, e.g., [Le] for an exposition.

**3.2.** In the case of coverings of strongly pseudoconvex (non-Stein) manifolds $M$ we proceed as follows. First, we construct a complete Kähler metric on $M' \setminus C'_M$ for a covering $r : M' \to M$. Then we define a specific Hermitian vector bundle on $M'$ whose space of holomorphic $L_2$ sections can be identified with the required space of holomorphic $L_2$ functions on $M'$. Finally, we apply standard Complex Analysis techniques based on the $L_2$ Kodaira-Nakano vanishing theorem for cohomology groups, see [D], [O], to get the desired results.

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# REMOVABLE SINGULARITY THEOREM FOR PSEUDO-HOLOMORPHIC MAPS

JAE-CHEON JOO

The primary goal concerns a removable singularity theorem for pseudo-holomorphic mappings between manifolds with non-integrable almost complex structures:

**Theorem 1** ([1]). *Let $A$ be a thin subset of an almost complex manifold $X$ and let $M$ be a compact Kobayashi hyperbolic almost complex manifold. Then every holomorphic map $f : X \setminus A \to M$ extends a holomorphic map on $X$ into $M$.*

Let $D$ be the unit disc in the complex plane $\mathbf{C}$. We call a smooth map $g$ from $D^{d-1} \times D$ into an almost complex manifold $X$ of dimension $2d$ a *local foliation of $X$ by pseudo-holomorphic discs around $p$*, if

(i) $g$ is a diffeomorphism onto a neighborhood $V$ of $p$,
(ii) $g(0,0) = p$, and
(iii) $g(z', \cdot) : D \to X$ is a pseudo-holomorphic embedding for every $z' \in D^{d-1}$.

A closed subset $A$ of $X$ is called a *thin subset* if there exists a local foliation $g = g_p$ of $X$ by pseudo-holomorphic discs around $p$, for every $p \in A$ which satisfies the following properties:

(A) There is a positive constant $r < 1$ such that $A_{z'} = \{w \in D : g(z', w) \in A\}$ is a finite point set contained in the $r$-disc $D_r = \{w \in \mathbf{C} : |w| < r\}$ for every $z' \in D^{d-1}$.
(B) There exist sequences $\{r_j\}$ and $\{s_j\}$ of real numbers less than 1 such that $r_j \to 0$ and the cylinder $\{(z', w) : |w| = r_j,\ |z'| < s_j\}$ does not intersect $g^{-1}(A)$ for every $j = 1, 2, \dots$ .

As an example, every analytic subvariety of a complex manifold is a thin subset.

The study pertaining to removable singularity theorems shows an impressive history. Among all the significant contributions, we have been influenced by [2], [3], [4], [5], [6].

Our starting point is to describe a pseudo-holomorphic curve $f$ from a Riemann surface $S$ to an almost complex manifold $M$ as a harmonic map. This can be achieved if one chooses an appropriate affine connection for the tangent bundle of the target manifold:

**Proposition 1.** *Let $(M, J)$ be an almost complex manifold and let $S$ be a Riemann surface with a conformal metric. Suppose that $\nabla$ is an affine connection of $M$ which satisfies the following conditions:*

(C1) *$\nabla$ is $J$-linear, i.e. $\nabla J = 0$.*

(C2) *For every $p \in M$ and for every $\Xi \in T_pM$, $\mathcal{T}(\Xi, J\Xi) = 0$ where $\mathcal{T}$ is the torsion tensor of $\nabla$.*

*Then a pseudo-holomorphic mapping $f : S \to M$ satisfies the harmonic map equation with respect to the connection $\nabla$, regardless of the choice of the conformal structure on $S$.*

A connection on $M$ which satisfies the conditions (C1) and (C2) is said to be *compatible with the almost complex structure $J$ of $M$*. If $z = x^1 + \sqrt{-1}x^2$ is a local complex coordinate of $S$ and if $(y^1, ..., y^n)$ is a smooth local coordinate system of $M$, then the harmonic map equation has a following local expressions:

$$\Delta f^i + \sum_{\alpha,j,k} \Gamma^i_{jk}(f) \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\alpha} = 0$$

for $i = 1, ..., n$. Here, $\Delta = \left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2$ and $\Gamma^i_{jk}$ is the Christoffel symbol of the connection that we choose.

To prove Theorem 1, we first consider pseudo-holomorphic curves with discrete singularities, that is, we consider a pseudo-holomorphic map $f$ defined on $D^* = D\backslash\{0\}$. The existence of continuous extensions can be proved by the method used in [6]. One of the crucial part is an area estimate from below for pseudo-holomorphic curves, which is already known as Gromov's monotonicity lemma. (See [7] for instance.) The conformal property of $f$ implies that $f$ is indeed a weak harmonic map on $D$ if

$$L(f(\sigma_r)) \to 0$$

as $r \to 0$ where $L$ is the length function of curves induced by a hermitian metric on $M$ and

$$\sigma_r = \{z \in \mathbf{C} : |z| = r\}.$$

By the hyperbolic property, there is a constant $C$ such that

$$L(s) \leq C\, L^{Kob}(s)$$

for every piecewise smooth curve $s$ in $M$, where $L^{Kob}$ represents the length function induced by the Kobayashi metric. By the decreasing property of the Kobayashi metric,

$$L(f(\sigma_r)) \le C\, L^{Kob}(f(\sigma_r)) \le C L^{Kob}(\sigma_r) = \frac{C}{|\log r|} \to 0$$

as $r \to 0$. Therefore, $f$ is a weak harmonic map defined on the entire disc $D$ and the regularity theorem for continuous weak harmonic maps ensures the smoothness of the pseudo-holomorphic curve.

Next, we consider the higher dimensional cases. The condition (A) for thin subsets and the 1-dimensional extension theorem enable us to use the normal family theorem for pseudo-holomorphic curves (See [7].) to prove the continuity of the pseudo-holomorphic mappings with singularities contained in a thin subset. Since the Riemann extension theorem and the Cauchy integral formula are not available for the pseudo-holomorphic maps between non-integrable almost complex manifolds, we exploit a scheme of the Implicit Function Theorem to prove the smoothness:

For $p \in A$, choose a local foliation $g : D^{d-1} \times D \to X$ around $p$ which satisfies the conditions (A) and (B). By the continuity of $f$ up to $A$, we may assume that $f \circ g$ maps $D^d$ into a single coordinate neighborhood $U$ of $f(p)$ with coordinates $(y^1, ..., y^n)$. Let $f_z = f \circ g(z, \cdot) : D \to U$ and let $v_z = f_z|_{\partial D}$. We define a non-linear functional $\mathcal{G} : C^{2,\lambda}(D, \mathbf{R}^n) \times C^{2,\lambda}(\partial D, \mathbf{R}^n) \to C^{0,\lambda}(D, \mathbf{R}^n) \times C^{2,\lambda}(\partial D, \mathbf{R}^n)$ by

$$\mathcal{G}(h, v) = \left( \left( \Delta h^i + \sum_{\alpha,j,k} \Gamma^i_{jk}(h) \frac{\partial h^j}{\partial x^\alpha} \frac{\partial h^k}{\partial x^\alpha} \right)_{i=1,\dots,n}, \ h|_{\partial D} - v \right),$$

where $C^{k,\lambda}(D, \mathbf{R}^n)$ and $C^{k,\lambda}(\partial D, \mathbf{R}^n)$ are spaces of $\mathbf{R}^n$-valued $(k, \lambda)$-Hölder functions on $D$ and $\partial D$, respectively. (In fact, $\mathcal{G}$ is defined only on a neighborhood of $(f_0, v_0)$.) By Proposition 1, $\mathcal{G}(f_z, v_z) = 0$ for every $z \in D^{d-1}$. We denote by $\mathcal{L} = \partial\mathcal{G}/\partial h$, the linearization of $\mathcal{G}$ in direction of $h$ at $(f_0, v_0)$. If we reparametrize $g$ by the dilation $(z, w) \to (tz, tw)$ for $t > 0$, it can be shown that $\mathcal{L}$ becomes invertible if $t$ is sufficiently small. It is achieved by the unique solvability and the Hölder estimates of the Dirichlet problem of the Poisson equation. Applying the Implicit Function Theorem, we can prove that $f$ is smooth near $p$ if we choose the reparametrizing constant $t$ by $r_j$ in the condition (B).

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## STRONGLY PSEUDOCONVEX HOMOGENEOUS DOMAINS IN ALMOST COMPLEX MANIFOLDS

KANG-HYURK LEE

ABSTRACT. Main result of author's Ph.D. thesis [6] is the classification of strongly pseudoconvex homogeneous domains in almost complex manifolds. The origin of this work is in the Wong-Rosay theorem. In this article, we introduce the Wong-Rosay theorem in almost complex manifolds and we give a brief process of the classification.

### 1. THE WONG-ROSAY THEOREM IN ALMOST COMPLEX MANIFOLDS

A strongly pseudoconvex homogeneous domain in a complex manifold has to be biholomorphic to the unit ball in the complex Euclidean space. It can be obtained by the Wong-Rosay theorem (see [2, 8, 9]) which says that

*A domain in a complex manifold which admits an automorphism orbit accumulating at a strongly pseudoconvex boundary point has to be biholomorphic to the unit ball.*

The most general version of the Wong-Rosay theorem for complex manifolds (see [2]) has been obtained by the scaling method which was initialized by Pinchuk [7]. In [3], Gaussier and Sukhov modified the scaling method to generalize the Wong-Rosay theorem to the real 4-dimensional almost complex manifolds. But for higher dimensional cases, it turns out that there are infinitely many domains in almost complex manifolds which satisfy the condition of the Wong-Rosay theorem but whose structures are non-integrable.

**Definition 1.** Let $\mathsf{Q} = (\mathsf{Q}_{j,\bar{k}})$ be a $N \times N$ positive definite hermitian matrix and $\mathsf{B} = (\mathsf{B}_{j,k})$ be a $N \times N$ skew-symmetric complex matrix. We call a pair $(G_{\mathsf{Q}}, J_{\mathsf{B}})$ a *model domain* where

(1) $G_{\mathsf{Q}}$ is the domain in $\mathbb{C}^{N+1}$ defined by $G_{\mathsf{Q}} = \{z \in \mathbb{C}^{N+1} : \operatorname{Re} z_{N+1} + \mathsf{Q}(z', z') < 0\}$ where $z' = (z_1, \ldots, z_N)$ and $\mathsf{Q}(z', w') = \sum_{j,k=1}^{N} \mathsf{Q}_{j,\bar{k}} z_j \bar{w}_k$ is the hermitian inner product on $\mathbb{C}^N$,
(2) $J_{\mathsf{B}}$ is the almost complex structure of $\mathbb{C}^{N+1}$ defined by

$$J_{\mathsf{B}}\left(\frac{\partial}{\partial z_j}\right) = i\frac{\partial}{\partial z_j} + \sum_{k=1}^{N} \mathsf{B}_{j,k} z_k \frac{\partial}{\partial \bar{z}_{N+1}} \quad \text{and} \quad J_{\mathsf{B}}\left(\frac{\partial}{\partial \bar{z}_j}\right) = -i\frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{N} \bar{\mathsf{B}}_{j,k} \bar{z}_k \frac{\partial}{\partial z_{N+1}}$$

(we call this structure a *model structure*). Here, we let $\mathsf{B}_{j,k} = 0$ for $j = N+1$ or $k = N+1$ for convenience.

It can be easily verified that the domain $G_{\mathsf{Q}}$ is strongly $J_{\mathsf{B}}$-pseudoconvex at the origin and the dilation $\Lambda_\tau$ which is defined by $\Lambda_\tau(z) = (\sqrt{\tau} z_1, \ldots, \sqrt{\tau} z_N, \tau z_{N+1})$, is an automorphism of $(G_{\mathsf{Q}}, J_{\mathsf{B}})$ for $\tau > 0$. The point $\mathbf{-1} = (0, \ldots, 0, -1)$ always belongs to $G_{\mathsf{Q}}$ and the automorphism orbit $\Lambda_\tau(\mathbf{-1})$ tends to the origin as $\tau \to 0$. Since the matrix $\mathsf{B}$ represents the Nijenhuis tensor of $J_{\mathsf{B}}$ (a torsion for an integrability of an almost complex structure), the model structure $J_{\mathsf{B}}$ is non-integrable if $\mathsf{B} \neq 0$. Now we have infinitely many examples which show that the classical Wong-Rosay theorem does not hold in the almost complex case.

## 2. Scaling method

We introduce the scaling method for the Wong-Rosay theorem and the modification of this method by Gaussier and Sukhov [3, 4]

2.1. **The complex case.** Let $\Omega$ be a domain of a complex manifold $M$ which admits an automorphism orbit $\varphi^\nu(p)$ accumulating at a strongly pseudoconvex boundary point $q$. The scaling method is to follow the next steps:

(1) For some coordinate neighborhood $U$ of $q$, one may assume that $\Omega \cap U$ is a domain in $\mathbb{C}^{N+1}$ defined by
$$\Omega \cap U = \left\{ z \in U : \operatorname{Re} z_{N+1} + \mathsf{Q}(z', z') + o\left( |z_{N+1}| + |z'|^2 \right) < 0 \right\}$$
for some $\mathsf{Q} > 0$. Choose a point $p_\nu^* \in \partial\Omega \cap U$ such that
$$\operatorname{dist}(p_\nu, p_\nu^*) = \operatorname{dist}(p_\nu, \partial\Omega) = \tau_\nu$$
where $p_\nu = \varphi^\nu(p)$.

(2) Choose a complex rigid motion $L^\nu$ so that
$$L^\nu(p_\nu^*) = 0\,, \quad L^\nu(p_\nu) = (0, \ldots, 0, -\tau_\nu) \quad \text{and} \quad T_0 L^\nu(\partial\Omega \cap U) = \{\operatorname{Re} z_{N+1} = 0\}\,.$$

(3) For each $\tau_\nu > 0$, let $\Lambda^\nu(z) = \Lambda_{\tau_\nu^{-1}}$ for each $\nu = 1, 2, \ldots$.

Now we consider the *scaling sequence* defined by
$$F^\nu = \Lambda^\nu \circ L^\nu \circ \varphi^\nu\,.$$

Analyzing the boundary of $\Lambda^\nu \circ L^\nu(\Omega \cap U)$, we have that the sequence of sets $\Lambda^\nu \circ L^\nu(\Omega \cap U)$ converges to $G_{\mathsf{Q}}$ in the sense of local Hausdorff set convergence. Moreover $F^\nu$ is a normal family and its subsequential limit is a biholomorphism $F : \Omega \to G_{\mathsf{Q}}$. By the Cayley transform, we have $G_{\mathsf{Q}} \simeq \mathbb{B}_{N+1}$. Therefore the Wong-Rosay theorem is obtained.

2.2. **The almost complex case.** Let $\Omega$ be a domain in an almost complex manifold $(M, J)$ and $\varphi^\nu$ be a sequence of automorpisms of $(\Omega, J)$ which generates an automorphism orbit accumulating at a strongly $J$-pseudoconvex boundary point $q$. Then we can also construct a scaling sequence $F^\nu$ as we introduced. In the limiting process, we must consider not only a sequence of domains $\Lambda^\nu \circ L^\nu(\Omega \cap U)$ but also a sequence of almost complex structures which is generated by $F^\nu$ in the sense of
$$J^\nu = dF^\nu \circ J \circ (dF^\nu)^{-1}\,.$$

It can be verified that the sequence $J^\nu$ converges to a model structure $J_{\mathsf{B}}$ for some $\mathsf{B}$. Consequently, we can obtain a subsequence of $F^\nu$ which converges to a $(J, J_{\mathsf{B}})$-biholomorpism $F : \Omega \to G_{\mathsf{Q}}$.

**Theorem 2** ([4, 5]). *Let $(M, J)$ be an almost complex manifold equipped with the almost complex structure $J$ of Hölder class $C^{1,\alpha}$. Suppose that a domain $\Omega$ in $M$ admits an automorphism orbit accumulating at a strongly $J$-pseudoconvex boundary point. Then $(\Omega, J)$ is biholomorphic to a model domain $(G_{\mathsf{Q}}, J_{\mathsf{B}})$ for some $\mathsf{Q}$ and $\mathsf{B}$.*

As a result, all model domains are stongly pseudoconvex and homogeneous. Hence the classification of model domains is the same as the classification of strongly pseudoconvex homogeneous domains in almost complex manifolds.

## 3. Basic properties of model domains

In this section, we introduce an automorphism group and a biholomorphis equivalence of model domains

**3.1. The automorphism groups.** Let $(G_{\mathsf{Q}}, J_{\mathsf{B}})$ be a model domain. For any point $\zeta = (\zeta', \zeta_{N+1})$, $\xi = (\xi', \xi_{N+1}) \in \mathbb{C}^{N+1}$, we define a binary operation $*_{(\mathsf{Q},\mathsf{B})}$ by

$$\zeta *_{(\mathsf{Q},\mathsf{B})} \xi = \left(\zeta' + \xi', \zeta_{N+1} + \xi_{N+1} - 2\mathsf{Q}(\xi', \zeta') + \frac{i}{2}\mathrm{Re}\,\mathsf{B}(\xi', \zeta')\right) .$$

Here, $\mathsf{B}(\xi', \zeta') = \sum_{j,k=1}^{N} \mathsf{B}_{j,k}\zeta_j\xi_k$. Then the boundary $\partial G_{\mathsf{Q}}$ is closed under this operation so that $H_{(\mathsf{Q},\mathsf{B})} = (\partial G_{\mathsf{Q}}, *_{(\mathsf{Q},\mathsf{B})})$ is a Lie group. Note that $H_{(\mathsf{Q},0)}$ is the usual Heisenberg group.

It can be verified that for each $\zeta \in \partial G_{\mathsf{Q}}$, the mapping

$$\Psi^{\zeta}_{(\mathsf{Q},\mathsf{B})} = \zeta *_{(\mathsf{Q},\mathsf{B})} z$$

belongs to $\mathrm{Aut}(G_{\mathsf{Q}}, J_{\mathsf{B}})$. Since $\Psi^{\zeta}_{(\mathsf{Q},\mathsf{B})} \circ \Psi^{\xi}_{(\mathsf{Q},\mathsf{B})} = \Psi^{\zeta *_{(\mathsf{Q},\mathsf{B})} \xi}_{(\mathsf{Q},\mathsf{B})}$, the group $H_{(\mathsf{Q},\mathsf{B})}$ can be identified as a subgroup of $\mathrm{Aut}(G_{\mathsf{Q}}, J_{\mathsf{B}})$.

**Theorem 3.** *The automorphism group of model domain can be decomposed by*

$$\mathrm{Aut}(G_{\mathsf{Q}}, J_{\mathsf{B}}) = \mathrm{Aut}_{-\mathbf{1}}(G_{\mathsf{Q}}, J_{\mathsf{B}}) \circ \mathcal{D} \circ H_{(\mathsf{Q},\mathsf{B})}$$

*where*

(1) $\mathrm{Aut}_{-\mathbf{1}}(G_{\mathsf{Q}}, J_{\mathsf{B}}) = \{\Phi \in \mathrm{Aut}(G_{\mathsf{Q}}, J_{\mathsf{B}}) : \Phi(-\mathbf{1}) = -\mathbf{1}\}$ *an isotropy subgroup,*
(2) $\mathcal{D} = \{\Lambda_\tau : \tau > 0\}$.

It is easily verified that the action by $\mathcal{D} \circ H_{(\mathsf{Q},\mathsf{B})}$ is transitive.

**Corollary 4.** *Model domains are homogeneous.*

**3.2. Biholomorphic equivalence.** Since model domains are homogeneous, the equivalence problem of model domains is the same as the existence of a biholomorphism which leaves the common point $-\mathbf{1}$ fixed. Concerning possible differentials of biholomorphisms at $-\mathbf{1}$, it is successful to obtain

**Lemma 5.** *Two pseudo-Siegel domains* $(G_{\mathsf{Q}}, J_{\mathsf{B}})$ *and* $(G_{\tilde{\mathsf{Q}}}, J_{\tilde{\mathsf{B}}})$ *are biholomorphically equivalent if and only if*

$$A^t\mathsf{Q}\overline{A} = \tilde{\mathsf{Q}} \quad \textit{and} \quad A^t\mathsf{B}A = \tilde{\mathsf{B}}$$

*for some* $A \in \mathrm{GL}(N, \mathbb{C})$.

If $(G_{\mathsf{Q}}, J_{\mathsf{B}})$ and $(G_{\tilde{\mathsf{Q}}}, J_{\tilde{\mathsf{B}}})$ are biholomorphic, then the biholomorphism can be realized by $\Phi(z) = (A(z'), z_{N+1})$ for $A$ as in the lemma.

When $\mathsf{Q} = I$, the domain $G_{\mathsf{Q}}$ is the Siegel half plane $\mathbf{H} = \{z \in \mathbb{C}^{N+1} : \mathrm{Re}\, z_{N+1} + |z'|^2 < 0\}$. For each $\mathsf{Q}$, it is possible to choose $A \in \mathrm{GL}(N, \mathbb{C})$ with $A^t\mathsf{Q}\overline{A} = I$. Therefore the set of model domains can be reduced by the set of model domains whose underlying domain is $\mathbf{H}$.

**Corollary 6.** *Two model domain* $(\mathbf{H}, J_{\mathsf{B}})$ *and* $(\mathbf{H}, J_{\tilde{\mathsf{B}}})$ *are biholomorically equivalent if and only if* $\mathsf{B} = A^t\tilde{\mathsf{B}}A$ *for some* $\mathrm{U}(N)$.

In Theorem 3, there is no information for the isotropy subgroup of the model domain. Using Cartan's uniqueness theorem and Lemma 5, we have

**Corollary 7.** *For any model domain* $(\mathbf{H}, J_{\mathsf{B}})$, $\mathrm{Aut}_{-1}(\mathbf{H}, J_{\mathsf{B}}) \simeq \{A \in \mathrm{U}(N) : A^t\mathsf{B}A = \mathsf{B}\}$.

## 4. Classification

Let $\mathrm{Sk}(N)$ be the space of $N \times N$ complex skew-symmetric matrices. Due to Corollary 6 and 7, it is natural to consider the following unitary action $\mathcal{U}$ on $\mathrm{Sk}(N)$:

$$\begin{aligned} \mathcal{U} : \mathrm{U}(N) \times \mathrm{Sk}(N) &\longrightarrow \mathrm{Sk}(N) \\ (A, \mathsf{B}) &\longmapsto A^t\mathsf{B}A\,. \end{aligned}$$

From now on, we concentrate on

> **Modified Problems:** *Compute the quotient space* $\mathrm{Sk}(N)/\mathcal{U}$, *and the isotropy subgroup* $\mathcal{U}_{\mathsf{B}} = \{A \in \mathrm{U}(N) : \mathcal{U}(A, \mathsf{B}) = \mathsf{B}\}$.

Given $\mathsf{B} \in \mathrm{Sk}(N)$, we denote by $\mathrm{Ann}(\mathsf{B})$ the *annihilator* of $\mathsf{B}$, defined by

$$\mathrm{Ann}(\mathsf{B}) = \{v \in \mathbb{C}^N : \mathsf{B}(v, w) = 0 \text{ for any } w \in \mathbb{C}^N\}\,.$$

The the orthogonal decomposition $\mathbb{C}^N = \mathrm{Ann}(\mathsf{B}) \oplus \mathrm{Ann}(\mathsf{B})^\perp$ is an invariant under $\mathcal{U}$ in the sense that

**Proposition 8.** *Let* $\mathsf{B}, \mathsf{B}' \in \mathrm{Sk}(N)$. *If* $A^t\mathsf{B}A = \mathsf{B}'$ *for some* $A \in \mathrm{GL}(N, \mathbb{C})$, *then* $A(\mathrm{Ann}(\mathsf{B}')) = \mathrm{Ann}(\mathsf{B})$ *and* $A(\mathrm{Ann}(\mathsf{B}')^\perp) = \mathrm{Ann}(\mathsf{B})^\perp$.

Since $\mathsf{B} = 0$ on $\mathrm{Ann}(\mathsf{B})$, it suffices to consider the restriction of $\mathsf{B}$ on $\mathrm{Ann}(\mathsf{B})^\perp$. On $\mathrm{Ann}(\mathsf{B})^\perp$ (of complex even dimension), $\mathsf{B}$ defines a non-degenerate skew-symmetric bilinear form (usually call *complex symplectic form*). Hence we first consider the non-degenerate case.

4.1. **The non-degenerate case.** Let $\mathsf{B}$ be a complex symplectic form on $\mathbb{C}^{2n}$ and denote by $\mathrm{Sp}(\mathsf{B}) = \{A \in \mathrm{GL}(2n,\mathbb{C}) : A^t\mathsf{B}A = \mathsf{B}\}$. Then the isotropy subgroup of $\mathsf{B}$ is

$$\mathcal{U}_{\mathsf{B}} = \mathrm{U}(2n) \cap \mathrm{Sp}(\mathsf{B}) \ .$$

Our problems are closely related with the dimension of the intersection of $\mathrm{U}(2n)$ and $\mathrm{Sp}(\mathsf{B})$. The maximal case is well-known. Let $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ be the standard symplectic form.

**Proposition 9.** *The symplectic group* $\mathrm{Sp}(n,\mathbb{C}) = \mathrm{Sp}(\Omega)$ *is a non-compact Lie group and the symplectic compact group* $\mathrm{Sp}(n) = \mathcal{U}_\Omega$ *is a maximal compact subgroup of* $\mathrm{Sp}(n,\mathbb{C})$.

Since $\mathrm{Sp}(\mathsf{B}) \simeq \mathrm{Sp}(n,\mathbb{C})$ for any complex symplectic form $\mathsf{B}$, an isotropy subgroup $\mathcal{U}_{\mathsf{B}} = \mathrm{Sp}(\mathsf{B}) \cap \mathrm{U}(2n)$ is also isomorphic to a compact subgroup of $\mathrm{Sp}(n,\mathbb{C})$. Therefore it follows that

$$\dim_{\mathbb{R}} \mathcal{U}_{\mathsf{B}} \leq \dim_{\mathbb{R}} \mathrm{Sp}(n) \ .$$

The maximal case can be characterized by the compatibility of a symplectic form and the standard hermitian inner product $h$.

**Definition 10.** We call a real linear transformation $\mathbf{J} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ a *quaternion structure* if $\mathbf{J}$ is anti-complex linear and defines another complex structure on $\mathbb{C}^{2n}$, i.e. $\mathbf{J}i = -i\mathbf{J}$ and $\mathbf{J}^2 = -\mathrm{Id}$. A pair $(\mathsf{B}, h)$ is called *compatible* if there is a quaternion structure $\mathbf{J}$ such that

$$\mathsf{B}(\mathbf{J}v, \mathbf{J}w) = \overline{\mathsf{B}(v,w)} \ , \quad h(\mathbf{J}v, \mathbf{J}w) = \overline{h(v,w)} \quad \text{and} \quad h(v, \mathbf{J}w) = -\mathsf{B}(v,w)$$

for any $v, w \in V$.

Then we have

**Proposition 11.** *A symplectic form* $\mathsf{B}$ *is compatible with* $h$ *if and only if* $\mathsf{B} \in \mathcal{U}(\Omega)$.

In an analogy way to find a complex structure which is compatible with a real symplectic form (see [1]), we have

**Lemma 12.** *For any complex symplectic form* $\mathsf{B}$, *there exist a complex symplectic form* $\mathsf{B}'$, *positive real numbers* $\lambda_1 > \cdots > \lambda_\mu > 0$, *positive interges* $k_1, \ldots, k_\mu$ *and an orthogonal decomposition* $\mathbb{C}^{2n} = V_1 \oplus \cdots \oplus V_\mu$, *uniquely so that*

(1) $\mathsf{B}'$ *is compatible with* $h$,
(2) $\dim_{\mathbb{C}} V_\nu = k_\nu$,
(3) $\mathsf{B} = \lambda_\nu \mathsf{B}'$ *on* $V_\nu$.

For $\mathsf{B}'$ as in Lemma 12, there is $A \in \mathrm{U}(2n)$ such that $A^t\mathsf{B}A = \Omega$ by Proposition 11. By the condition (2) and (3), we can choose a suitable $A$ so that

$$A^t\mathsf{B}A = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_\mu \end{pmatrix}$$

where each $\lambda_\nu$ appears in $k_\nu$ times. Moreover

$$\mathcal{U}_{\mathsf{B}} \simeq \mathrm{Sp}(k_1) \oplus \cdots \oplus \mathrm{Sp}(k_\mu)\,.$$

4.2. **The general case.** For any $\mathsf{B} \in \mathrm{Sk}(N)$, let

$$\mathbb{C}^N = \underbrace{\mathrm{Ann}(\mathsf{B})}_{k} \oplus \underbrace{\mathrm{Ann}(\mathsf{B})^\perp}_{2n}\,.$$

Since $\mathsf{B}$ is a complex symplectic form on $\mathrm{Ann}(\mathsf{B})^\perp$, we can apply Lemma 12. Then

**Theorem 13.** *There exist positive real numbers $\lambda_1 > \cdots > \lambda_\mu$ and positive integers $k_1 + \cdots + k_\mu = 2n$ uniquely such that for some unitary matrix $A \in \mathrm{U}(N)$,*

$$A^t \mathsf{B} A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & -D & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_\mu \end{pmatrix}$$

*where each $\lambda_\nu$ appears in $k_\nu$ times. Moreover*

$$\mathcal{U}_{\mathsf{B}} \simeq \mathrm{U}(k) \oplus \mathrm{Sp}(k_1) \oplus \cdots \oplus \mathrm{Sp}(k_\mu)\,.$$

This solves our problems in the front of this section. So it is successful to classify strongly pseudoconvex homogeneous domains in almost complex manifolds.

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# Recent Progress in the Theory of Holomorphic Curves

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We will discuss some new results in the Nevanlinna theory of holomorphic curves into algebraic varieties. The central problem is the following conjecture, strengthened from the original of Griffiths (1972):

*Griffiths Conjecture 1* (1972). Let $f : \mathbf{C} \to M$ be an algebraically non-degenerate holomorphic curve into a complex projective manifold $M$. Let $D$ be an effective reduced divisor of simple normal crossings. Then we have

$$T_f(r; L(D)) + T_f(r; K_M) \leq N_1(r; f^*D) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \tag{0.1}$$

Here $N_1(r; f^*D)$ stands for the counting function truncated to level one.

Vojta formulated an analogue of this conjecture in Diophantine approximation theory with the non-truncated counting function $N(r; f^*D)$ and proposed Vojta's dictionary, which has brought interesting observations and motivations in the both theories.

Griffiths Conjecture 1 implies

*Griffiths Conjecture 2* (1972). Let $X$ be a (complex) algebraic variety of log general type. Then every holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate.

## 1 Order function.

We need to define the order function of $f$ in a more general form than those already known (cf., e.g., Stoll [21], Noguchi-Ochiai [8]).

In what follows $X$ is a compact complex reduced space and a subspace is a closed one. Let $\mathcal{O}_X$ denote the structure sheaf of local holomorphic functions over $X$. Let $Y$ be a subspace of $X$, not necessarily reduced, and let $\mathcal{I} \subset \mathcal{O}_X$ be the defining coherent ideal sheaf of $Y$. Here one may begin with taking a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ and take a subspace $Y$ defined by $\mathcal{I}$. In any case, there are a finite open covering $X = \bigcup U_\lambda$ of $X$ and holomorphic functions $\sigma_{\lambda 1}, \ldots, \sigma_{\lambda l_\lambda}$ on $U_\lambda$ such that at every point $x \in U_\lambda$ their germs $\underline{\sigma_{\lambda 1}}_x, \ldots, \underline{\sigma_{\lambda l_\lambda}}_x$ generate the fiber $\mathcal{I}_x$ of $\mathcal{I}$ at $x$. Take relatively compact open covering $V_\lambda \Subset U_\lambda, X = \bigcup V_\lambda$. We take $\rho_\lambda \in C_0^\infty(U_\lambda)$ with $\rho_\lambda|_{V_\lambda} \equiv 1$ and set

$$d_Y(x) = d_{\mathcal{I}}(x) = \sum_\lambda \rho_\lambda(x) \left( \sum_{j=1}^{l_\lambda} |\sigma_{\lambda j(x)}|^2 \right)^{1/2}, \qquad x \in N \tag{1.1}$$

(cf. [11] Chap. 2 §3, [25] §2, [16]). Another finite open covering and another local generators of $\mathcal{I}_Y$ yield a function $d'_Y$ by the same construction as above. Then there is a constant $C > 0$ such that

$$|\log d_Y(x) - \log d'_Y(x)| \leq C, \qquad x \in X. \tag{1.2}$$

The function $d_Y(x)$ stands for "a sort of the distance" between $x$ and the subspace $Y$. We call

$$\phi_Y(x) = \phi_{\mathcal{I}}(x) = -\log d_Y(x), \quad x \in X$$

the *Weil function* or the *proximity (approximation) potential* of $Y$.

For a holomorphic curve $f : \mathbf{C} \to X$ with $f(\mathbf{C}) \not\subset \operatorname{Supp} Y$ we define

$$\begin{aligned} \omega_{Y,f} = \omega_{\mathcal{I},f} &= -dd^c \phi_Y(z) = -\frac{i}{2\pi} \partial\bar{\partial} \phi_Y(z) \\ &= dd^c \log \frac{1}{d_Y \circ f(z)}, \end{aligned} \tag{1.3}$$

which is a smooth (1,1)-form on $\mathbf{C}$. The order function of $f$ for $Y$ or $\mathcal{I}$ is defined by

$$(1.4) \qquad T(r;\omega_{Y,f}) = T(r;\omega_{\mathcal{I},f}) = \int_1^r \frac{dt}{t}\int_{|z|<t}\omega_{Y,f}.$$

When $\mathcal{I}$ defines a Cartier divisor $D$ on $M$, we see that

$$T(r;\omega_{\mathcal{I},f}) = T_f(r;L(D)) + O(1),$$

where $T_f(r;L(D))$ is the order function defined by the Chern class of $L$ (cf. [8]).

Similarly taking a hermitian metric form $\omega$ on $X_{\mathrm{red}}$, we define an order function of $f$ with respect to $\omega$ by

$$T_f(r) = T(r;f^*\omega) = \int_1^r \frac{dt}{t}\int_{|z|<t} f^*\omega.$$

Then in general we have

$$T(r;\omega_{\mathcal{I},f}) = O(T_f(r)).$$

The *proximity function* (or approximation function) of $f$ for $Y$ is defined by

$$(1.5) \qquad m_f(r,Y) = m_f(r,\mathcal{I}) = \int_{|z|=r}\phi_Y\circ f(z)\frac{d\theta}{2\pi}.$$

It follows from (1.1) that the integral is finite, and from (1.2) that $m_f(r,Y)$ is well-defined up to $O(1)$-term.

Let $Y$, $X=\bigcup U_\lambda$ and $\sigma_{\lambda 1},\dots,\sigma_{\lambda l_\lambda}$ be as above. Suppose that $f(\zeta)\in U_\lambda$. Then $\sigma_{\lambda j}\circ f(z)$ are local holomorphic functions in a neighborhood of $\zeta$ vanishing at $\zeta$ with multiplicity $\mathrm{mult}_\zeta\,\sigma_{\lambda j}\circ f$. We define the intersection multiplicity of $f$ with $Y$ by

$$\mathrm{mult}_\zeta f^*Y = \min\{\mathrm{mult}_\zeta\sigma_{\lambda j}\circ f; 1\le j\le l_\lambda\},$$

which is independent of the choice of local generators $\sigma_{\lambda j}$. The counting function with truncation level $k\le\infty$ is defined by

$$N_k(r;f^*Y) = N_k(r;f^*\mathcal{I}) = \int_1^r\frac{dt}{t}\sum_{|\zeta|<t}\min\{\mathrm{mult}_\zeta f^*Y,k\}.$$

We set $N(r;f^*Y)=N(r;f^*\mathcal{I})=N_\infty(r;f^*Y)$.

**Theorem 1.6** ([25], [17]) *Let $f:\mathbf{C}\to X$ and $\mathcal{I}$ be as above. Then we have the following:*

(i) (First Main Theorem) $T(r;\omega_{\mathcal{I},f}) = N(r;f^*\mathcal{I}) + m_f(r;\mathcal{I}) - m_f(1;\mathcal{I})$.

(ii) *Let $\mathcal{I}_i$ $(i=1,2)$ be coherent ideal sheaves of $\mathcal{O}_X$ and let $Y_i$ be the subspace defined by $\mathcal{I}_i$. If $\mathcal{I}_1\subset\mathcal{I}_2$ or equivalently $Y_1\supset Y_2$, then*

$$m_f(r;Y_2)\le m_f(r;Y_1)+O(1).$$

(iii) *Let $\phi:X_1\to X_2$ be a holomorphic mappings between compact complex manifolds. Let $\mathcal{I}_2\subset\mathcal{O}_{X_2}$ be a coherent ideal sheaf and let $\mathcal{I}_1\subset\mathcal{O}_{X_1}$ be the coherent ideal sheaf generated by $\phi^*\mathcal{I}_2$. Then*

$$m_f(r;\mathcal{I}_1) = m_{\phi\circ f}(r;\mathcal{I}_2)+O(1).$$

(iv) *Let $\mathcal{I}_i$, $i=1,2$, be two coherent ideal sheaves of $\mathcal{O}_X$. Suppose that $f(\mathbf{C})\not\subset\mathrm{Supp}\,(\mathcal{O}_X/\mathcal{I}_1\otimes\mathcal{I}_2)$. Then we have*

$$T(r;\omega_{\mathcal{I}_1\otimes\mathcal{I}_2,f}) = T(r;\omega_{\mathcal{I}_1,f}) + T(r;\omega_{\mathcal{I}_2,f}) + O(1).$$

(v) *A holomorphic curve $f:\mathbf{C}\to X$ is a rational curve if and only if $T_f(r)=O(\log r)$, provided that $X$ is algebraic.*

Here we recall the classical result for a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ into the complex projective space of dimension $n$. We set $T_f(r) = T(r; \Omega)$ with Fubini-Study metric form $\Omega$.

**Theorem 1.7** (Nevanlinna-Cartan) *Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate holomorphic curve, i.e., $f(\mathbf{C})$ is not contained in a hyperplane. Let $\{H_j\}_{j=1}^q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Then*

$$(q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_n(r, f^*H_j) + O(\log r) + O(\log T_f(r))||, \tag{1.8}$$

*where the symbol "$||$" stands for the estimate to hold for $r > 0$ outside a Borel subset of finite total Lebesgue measure.*

# 2 Min Ru's result.

In the Diophantine approximation theory, P. Corvaja and U. Zannier [1] generalized Schmidt's Subspace Theorem to the case of hypersurfaces in the projective space $\mathbf{P}^n$, and then J.-H. Evertse and R.G. Feretti [3], [4] generalized it to the case of subspace $M \subset \mathbf{P}^n$.

Min Ru [18], [19] found their analogue to be valid in the theory of holomorphic curves and proved the following:

**Theorem 2.1** *Let $M \subset \mathbf{P}^N(\mathbf{C})$ be a smooth subvariety of dimension $n$. Let $D_i, 1 \leq i \leq q$ be hypersurfaces of degree $d_i$ in $\mathbf{P}^N(\mathbf{C})$ which are in general position in $M$; i.e.,*

$$M \cap D_{i_1} \cap \cdots \cap D_{i_{n+1}} = \emptyset$$

*for all $1 \leq i_1 < \cdots < i_{n+1} \leq q$. Let $f : \mathbf{C} \to M$ be an algebraically non-degenerate holomorphic curve. Then*

$$(q-n-1-\epsilon)T_f(r; O(1)) \leq \sum_{i=1}^{q} \frac{1}{d_i} N(r; f^*D_i)||_\epsilon, \quad \forall \epsilon > 0.$$

In the proof the following approximation theorem due to H. Cartan is one key:

**Theorem 2.2** *Let $L_j, j \in Q = \{1, \ldots, q\}$ be linear forms on $\mathbf{P}^n(\mathbf{C})$ in general position. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate holomorphic curve. Then*

$$\int_{|\zeta|=r} \max_K \sum_{j \in K} \log \frac{\|f(\zeta)\| \|L_j\|}{|L_j(f(\zeta))|} \frac{d\theta}{2\pi} \leq (n+1+\epsilon)T_f(r; O(1))||_\epsilon,$$

*where $K \subset Q$ runs with $|K| = n+1$.*

This is showing the limit how much $f(\zeta)$ can approximate the divisor $\prod_{j \in Q} L_j = 0$ on $\mathbf{P}^n(\mathbf{C})$. They apply a very elaborate combinatorial argument for Veronese embeddings of degree $m$ as $m \to \infty$ $(\epsilon \to 0)$.

# 3 Dethloff-Lu's result.

**Theorem 3.1** (Log Bloch-Ochiai (N. '77–'81, N.-Winkelmann [12])) *Let $X$ be a Zariski open subset of a compact Kähler manifold $\bar{X}$ such that the log irregularity $\bar{q}(X) > \dim_{\mathbf{C}} X$. Then no holomorphic curve $f : \mathbf{C} \to X$ has a Zariski dense image in $\bar{X}$.*

*Problem.* What happens in the case of $\bar{q}(X) = \dim_{\mathbf{C}} X$?

A holomorphic curve $f : \mathbf{C} \to M$ into a compact hermitian manifold $M$ is called a Brody curve if the norm $\|f'(z)\|$ of the differential of $f$ is bounded on $\mathbf{C}$.

As for Griffiths Conjecture 2 G. Dethloff and S. Lu [2] dealt with Brody curves into algebraic surfaces.

**Theorem 3.2** *Let $X$ be a smooth algebraic surface of log general type with log irregularity $\bar{q}(X) = 2$, and let $\bar{X}$ be a smooth compactification with s.n.c. $\partial X = \bar{X} \setminus X$. Then every Brody curve $f : \mathbf{C} \to X \subset \bar{X}$ is algebraically degenerate.*

**Proposition 3.3** *Let $X$ be an algebraic surface with $\bar{\kappa}(X) = 1$ and $\bar{q}(X) = 2$. Assume that the quasi-Albanese map $\alpha_X : X \to A_X$ is proper (a bit more general assumption works). Then every holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate.*

By Kawamata's theorem this is easily reduced to the case of $\dim X = \bar{q}(X) = \bar{\kappa}(X) = 1$, and then little Picard's theorem is applied.

They gave an interesting example.

**Remark 3.4** *There is an algebraic surface $X$ with $\bar{\kappa}(X) = 1$ and $\bar{q}(X) = 2$ which admits an algebraically non-degenerate $f : \mathbf{C} \to X$.*

On the other hand, J. Winkelmann gave another interesting example:

**Remark 3.5** *There is a compact projective threefold $X$ such that*

(i) *$\kappa(X) = 0$ and $q(X) = 3$,*

(ii) *the Kobayashi hyperbolic pseudodistance $d_X \equiv 0$,*

(iii) *there is a holomorphic curve $f : \mathbf{C} \to X$ with the dense image in the sense of the differential topology,*

(iv) *there is a proper subvariety $Z \subset X$ satisfying that for every Brody $g : \mathbf{C} \to X$, $g(\mathbf{C}) \subset Z$.*

# 4 Semi-abelian varieties.

Let $f : \mathbf{C} \to A$ be a holomorphic curve and let $J_k(f) : \mathbf{C} \to J_k(A)$ denote the $k$-jet lift of $f$ into the $k$-jet space $J_k(A)$ over $A$. Let $X_k(f)$ denote the Zariski closure of the image of $J_k(f)$.

**Theorem 4.1** (N.-Winkelmann-Yamanoi [16]) *Let $A$ be a semi-abelian variety. Let $f : \mathbf{C} \to A$ be a holomorphic curve with Zariski dense image.*

(i) *Let $Z$ be an algebraic reduced subvariety of $X_k(f)$ ($k \geq 0$). Then there exists a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that*

$$(4.2) \qquad T(r; \omega_{\bar{Z}, J_k(f)}) \leq N_1(r; J_k(f)^* Z) + \epsilon T_f(r) ||_\epsilon, \quad \forall \epsilon > 0,$$

*where $\bar{Z}$ is the closure of $Z$ in $\bar{X}_k(f)$.*

(ii) *Moreover, if $\operatorname{codim}_{X_k(f)} Z \geq 2$, then*

$$(4.3) \qquad T(r; \omega_{\bar{Z}, J_k(f)}) \leq \epsilon T_f(r) ||_\epsilon, \quad \forall \epsilon > 0.$$

(iii) *In the case when $k = 0$ and $Z$ is an effective divisor $D$ on $A$, the compactification $\bar{A}$ of $A$ can be chosen as smooth, equivariant with respect to the $A$-action, and independent of $f$; furthermore, (4.2) takes the form*

$$(4.4) \qquad T_f(r; L(\bar{D})) \leq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D})) ||_\epsilon, \quad \forall \epsilon > 0.$$

Note that in the above estimate (4.2), (4.3) or (4.4) the error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log r) + O(\log T_f(r))$" (see [15] Example (5.36)).

**Remark 4.5** (i) In N.-Winkelmann-Yamanoi [15] we proved (4.4) with a higher level truncated counting function $N_l(r; f^* D)$. In the case of abelian $A$ (4.4) with truncation level one was obtained by Yamanoi [26].

(ii) Theorem 4.1 is considered as the analogue of abc-Conjecture over semi-abelian varieties. Cf. Vojta [24] for a result without order truncation.

# 5 Application and conjecture.

As applications for Griffiths Conjecture 2 we have the following (see [17]).

**Theorem 5.1** *Let $X$ be a complex algebraic variety and let $\pi : X \to A$ be a finite morphism onto a semi-abelian variety $A$. Let $f : \mathbf{C} \to X$ be an arbitrary entire holomorphic curve. If $\bar{\kappa}(X) > 0$, then $f$ is algebraically degenerate.*

*Moreover, the normalization of the Zariski closure of $f(\mathbf{C})$ is a semi-abelian variety which is a finite étale cover of a translate of a proper semi-abelian subvariety of $A$.*

**Corollary 5.2** *Let $X$ be a complex algebraic variety whose quasi-Albanese map is a proper map. Assume that $\bar{\kappa}(X) > 0$ and $\bar{q}(X) \geq \dim X$. Then every entire holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate.*

**Theorem 5.3** *Let $E_i, 1 \leq i \leq q$, be smooth hypersurfaces of the complex projective space $\mathbf{P}^n(\mathbf{C})$ of dimension $n$ such that $E = \sum E_i$ is a divisor of simple normal crossings. Assume that*

(i) $q \geq n + 1$.

(ii) $\deg E \geq n + 2$.

*Then every holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus E$ is algebraically degenerate.*

**Remark 5.4** In Theorem 5.3 the case when $n = 2$, $E_i, i = 1, 2$, are lines and $E_3$ is a quadric was a conjecture of M. Green [5].

Let $A$ be a semi-abelian variety and let $D$ be an effective reduced divisor on $A$. Assume that the stabilizer $\{a \in Aa + D = D\}^0 = \{0\}$. Then there is an equivariant compactification $\bar{A}$ of $A$ such that the closure $\bar{D}$ of $D$ in $\bar{A}$ contains no $A$-orbit ([24], [16]). Let $\partial A = \bar{A} \setminus A$ denote the boundary divisor, which has only simple normal crossings.

*Conjecture.* Let $f : \mathbf{C} \to \bar{A}$ be an algebraically non-degenerate holomorphic curve. Then we have

$$m_f(r; \bar{D}) + m_f(r; \partial A) \leq T_f(r; L(\partial D)) + O(\log r) + O(\log T_f(r))||. \tag{5.5}$$

When $f(\mathbf{C}) \cap \partial A = \emptyset$, (5.5) was proved in [15].

# 6 Analogue in Diophantine approximation.

We first recall

*Abc-Conjecture.* Let $a, b, c \in \mathbf{Z}$ be co-prime numbers satisfying

$$a + b = c. \tag{6.1}$$

Then for an arbitrary $\epsilon > 0$ there is a number $C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{\text{prime } p|(abc)} p^{1+\epsilon}.$$

Notice that the order of *abc* at every prime $p$ is counted only by "$1+\epsilon$" (truncation) when it is positive.

As in §1 we put $x = [a, b] \in \mathbf{P}^1(\mathbf{Q})$. After Vojta's notational dictionary ([22]), this is equivalent to

$$(1 - \epsilon)h(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon \tag{6.2}$$

for $x \in \mathbf{P}^1(\mathbf{Q})$ (cf. [7], [23] for $\mathbf{P}^n$). This is quite analogous to (1.8). Here we follow the notation in Vojta [22] for number theory and Noguchi-Ochiai [8] for the Nevanlinna theory in particular,

$h(x)$ = the height of $x$.

$N_1(x; *)$ = the counting function at $*$ truncated to level 1 (see below).

Motivated by the results in sections 3 and 4, we formulate an analogue of abc-Conjecture over semi-abelian varieties. Let $k$ be an algebraic number field and let $S \subset M_k$ be an arbitrarily fixed finite subset of places of $k$ containing all infinite places. Let $A$ be a semi-abelian variety over $k$, let $D$ be a reduced divisor on $A$, let $\bar{A}$ be an equivariant compactification of $A$ such that $\bar{D}(\subset \bar{A})$ contains no $A$-orbit, and let $\sigma_{\bar{D}}$ be a regular section of the line bundle $L(\bar{D})$ defining the divisor $\bar{D}$.

*Abc-Conjecture over semi-abelian variety.* For an arbitrary $\epsilon > 0$, there exits a constant $C_\epsilon > 0$ such that for all $x \in A(k) \setminus D$

$$(1-\epsilon)h_{L(\bar{D})}(x) \leq N_1(x; S, \bar{D}) + C_\epsilon. \tag{6.3}$$

Here $h_{L(\bar{D})}(x)$ denotes the height function with respect to $L(\bar{D})$ and $N_1(x, \bar{D}; S)$ denotes the $S$-counting function truncated to level one:

$$N_1(x; S, \bar{D}) = \frac{1}{[k:\mathbf{Q}]} \sum_{\substack{v \in M_k \setminus S \\ \operatorname{ord}_{\mathfrak{p}_v} \sigma_{\bar{D}}(x) \geq 1}} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v).$$

*Remark.* Cf. [14] for the analogue over algebraic function fields.

It may be interesting to specialize the above conjecture in dimension one.

*Abc-Conjecture for S-units.* We assume that $a$ and $b$ are $S$-units in (6.1); that is, $x$ in (6.2) is an $S$-unit. Then for arbitrary $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$(1-\epsilon)h(x) \leq N_1(x; S, 1) + C_\epsilon. \tag{6.4}$$

*Abc-Conjecture for elliptic curves.* Let $C$ be an elliptic curve defined as a closure of an affine curve,

$$y^2 = x^3 + c_1 x + c_0, \quad c_i \in k^*.$$

In a neighborhood of $\infty \in C$, $\sigma_\infty = x/y$ gives an affine parameter with $\sigma_\infty(\infty) = 0$. Then for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that for $w \in C(k)$

$$\begin{aligned}(1-\epsilon)h(w) &\leq N_1(w; S, \infty) + C_\epsilon \\ &= \frac{1}{[k:\mathbf{Q}]} \sum_{\substack{v \in M_k \setminus S \\ \operatorname{ord}_{\mathfrak{p}_v} \sigma_\infty(w) \geq 1}} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v) \quad + C_\epsilon.\end{aligned}$$

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# EXTENSION OF CR STRUCTURES ON THREE DIMENSIONAL COMPACT PSEUDOCONVEX CR MANIFOLDS

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Let $(M, \mathcal{S})$ be an abstract smoothly bounded orientable $CR$-manifold of dimension $2n-1$ with $CR$-dimension equal to $n-1$ (i.e., $dim_{\mathbb{C}}\mathcal{S} = n-1$). Since $M$ is orientable, there exist a smooth real nonvanishing 1–form $\eta$ and a smooth real vector field $X_0$ on $M$ so that $\eta(X) = 0$ for all $X \in \mathcal{S}$ and $\eta(X_0) = 1$. Define the Levi-form of $\mathcal{S}$ on $M$ by

$$i\eta([L', \overline{L}'']), \quad L', L'' \in \mathcal{S}. \tag{1}$$

**Definition 1.** *We say $(M, \mathcal{S})$ is strictly pseudoconvex (resp. pseudoconvex) if the Levi-form defined in (1) is strictly positive definite (resp. non-negative definite).*

Then we have the following celebrating local embedding theorem by Kuranishi.

**Theorem K.** *(Kuranishi 81') If $(M, \mathcal{S})$ is strictly pseudoconvex and $dim_{\mathbb{R}} M = 2n-1 \geq 9$, then $(M, \mathcal{S})$ can be locally embedded as a real hypersurface in $\mathbb{C}^n$.*

Theorem (K) has been improved by Akahori and Webster(Theorem A and W) in 1985. They showed that $(M, \mathcal{S})$ can be locally embedded as a real hypersurface in $\mathbb{C}^n$ provided $(M, \mathcal{S})$ is strictly pseudoconvex and $dim_{\mathbb{R}} M \geq 7$. In Theorem (K,A,W), they used solvability and estimates of $\bar{\partial}_b$ equation, the tangential Cauchy Riemann equation.

In 1994, Catlin proposed another Approach : Extend the given $CR$ structure to an integrable almost complex structure by deforming the given almost complex structure [1,2].

**Theorem C.** *(Catlin 94') : If $(M, \mathcal{S})$ has either 3 positive eigenvalues or (n-1)-negative eigenvalues, then there is a tubular neighborhood $\Omega$ so that $M \subset b\Omega$ and $\Omega$ is an integrable complex manifold.*

**Corollary.** *If $(M,\mathcal{S})$ is pseudoconvex near $z_0$ and has 3 + eigenvalues, then $M = b\Omega$, where $\Omega$ is a complex manifold*

**Corollary.** *: If $(M,\mathcal{S})$ is strongly pseudoconvex and $dim_{\mathbb{R}} M \geq 7$, then $M$ can be embedded in $\mathbb{C}^n$ as a real hypersurface type. Same conclution holds if $M$ has three positive eigenvalues and three negative eigenvalues.*

For weakly pseudoconvex CR manifold $(M,\mathcal{S})$ of finite type, the second author proved some series of extension problems [3,4].

**Theorem Ch1.** *(Cho, S. 97') Let $(M,\mathcal{S})$ be a smooth compact pseudoconvex CR manifold of finite type with $dim_R M = 3$. Then there exists a tubular neighborhood $\Omega$ on the concave side of $M$ so that $M \subset b\Omega$, and $\Omega$ is an integrable complex manifold. That is, there exists an integrable almost complex structure $\mathcal{L}$ on $\Omega$ such that for all $x \in M$, $\mathcal{L}_{(x,0)} \cap \mathbb{C}TM = \mathcal{S}_x$.*

**Corollary.** *If $M$ is the boundary (or a portion of the boundary) of a complex manifold $D$ with $dim_{\mathbb{C}} D = 2$ and assume that $M$ is of finite type. Then the complex structure of $D$ extends smoothly beyond $M$. That is, there is a complex manifold $\Omega$, $dim_{\mathbb{C}}\Omega = 2$, such that $D \cup M \subset \Omega$.*

**Theorem Ch2.** *(Cho, S.,2003 ) Let $(M,\mathcal{S})$ be a smooth pseudoconvex CR manifold of finite type with $dim_R M = 2n-1$, and the Levi-form of $M$ has at least $(n-2)$-positive eigenvalues. Then there exists a tubular neighborhood $\Omega$ on the concave side of $M$ so that $M \subset b\Omega$, and $\Omega$ is an integrable complex manifold. That is, there exists an integrable almost complex structure $\mathcal{L}$ on $\Omega$ such that for all $x \in M$, $\mathcal{L}_{(x,0)} \cap \mathbb{C}TM = \mathcal{S}_x$.*

**Theorem Ch3.** *(Cho, S.,2002 ) Let $(M,\mathcal{S})$ be a smooth pseudoconvex CR manifold of finite type with $dim_R M = 2n-1$, and the Levi-form of $M$ has at least $(n-2)$-positive eigenvalues. Then there exists a tubular neighborhood $\Omega$ on the convex side of $M$ so that $M \subset b\Omega$, and $\Omega$ is an integrable complex manifold. That is, there exists an integrable almost complex structure $\mathcal{L}$ on $\Omega$ such that for all $x \in M$, $\mathcal{L}_{(x,0)} \cap \mathbb{C}TM = \mathcal{S}_x$.*

**Corollary.** *Let $M$ be as in Theorem Ch2. Then $M$ can be locally embedded as a real hypersurface in $\mathbb{C}^n$.*

We note that $dim_{\mathbb{R}} M = 5$ case is still open.

Let us study the extension problem of compact pseudoconvex $CR$ manifold of $dim_{\mathbb{R}} M = 3$. Let $(M,\mathcal{S})$ be a smooth compact orientable pseudoconvex $CR$ manifold. Using the vector field $X_0$, we can define a projection $\Pi^{0,1}$ of $\mathbb{C}TM$ onto $\overline{\mathcal{S}}$. If $Y \in \mathbb{C}TM$, we can uniquely write $Y = Y' + Y'' + cX_0$, where $Y' \in \mathcal{S}$, $Y'' \in \overline{\mathcal{S}}$. Define $\Pi^{0,1} Y = Y''$. In terms of this projection, we can define the Hessian $H_\lambda$ of a smooth function $\lambda$. If $L_1, L_2$ are sections of $\mathcal{S}$, define

$$H_\lambda(L_1,\overline{L}_2) = L_1\overline{L}_2\lambda - [L_1,\overline{L}_2]''\lambda. \tag{2}$$

In general, this Hessian depends on the choice of $X_0$. However, note that if $L_1$ and $L_2$ are both in the null space of the Levi-form at a point $p \in M$, then $[L_1, \overline{L}_2]$ is in the span of $\mathcal{S}$ and $\overline{\mathcal{S}}$ at $p$ so that $H_\lambda(L_1, \overline{L}_2)$ is independent of the choice of $X_0$.

**Definition 2.** *We say $\lambda$ is strictly subharmonic near the set $W$ of weakly pseudoconvex points if $H_\lambda(L, L) > 0$, $L \in \mathcal{S}_p$ where $L$ is a nonzero vector in $\mathcal{S}$ and $p$ is any point in $W$.*

Set

$$\Omega = \{(x, r)\ ;\ x \in M,\ -1 < r < 1\},$$
$$\Omega^- = \{(x, r) \in \Omega\ ;\ -1 < r \leq 0\},$$

and for a sufficiently small $\epsilon > 0$, let

$$\Omega_\epsilon = \{(x, r) \in M \times (-1, 1)\ ;\ -\epsilon < r < \epsilon\}$$

and set

$$\Omega_\epsilon^+ = \{(x, r)\ ;\ x \in M,\ 0 \leq r < \epsilon\}.$$

Then it follows that

$$\partial\bar{\partial}r(L, \overline{L}) = \frac{i}{2}\eta([L, \overline{L}]).$$

Assume that if $\lambda$ is strictly subharmonic near the set $W \subset M$ of weakly pseudoconvex points. Then we can arrange so that $\lambda$ is strictly plurisubharmonic for some $\epsilon_0 > 0$ on $\Omega_{\epsilon_0}$.

Let $P_+ : \Omega_\epsilon^+ \longrightarrow M$ be the projection map. Then our main results are as follows (Catlin, Cho).

**Theorem (C,Ch).** *Let $M$ be a smooth compact orientable pseudoconvex $CR$ manifold of real dimension three with a given $CR$ structure $\mathcal{S}$ and assume that there is a smooth function $\lambda$ which is strictly subharmonic near the set $W$ of weakly pseudoconvex points. Then there exist a small positive number $\epsilon > 0$ and a smooth integrable almost complex structure $\mathcal{L}$ on $\Omega_\epsilon^+$ such that for all $x \in M$, $\mathcal{L}_{(x,0)} \cap \mathbb{C}TM = \mathcal{S}_x$. Furthermore, if $\mathcal{J}_\mathcal{L} : T\Omega_\epsilon \rightarrow T\Omega_\epsilon$ is the map associated with the complex structure $\mathcal{L}$, then $dr(\mathcal{J}_\mathcal{L}(X_0)) < 0$ at all points of $M_0 = \{(x, 0); x \in M\}$.*

Note that we are extending the given $CR$ structure on $M$ to the concave side (instead of convex side) of $M$. Theorem (C,Ch), in general, does not imply that the given $CR$ structure can be locally embedded in $\mathbb{C}^2$ [6](cf. the example of non-solvable elliptic PDE of Nirenberg). When $M$ is compact strictly pseudoconvex of real dimension three and has the property that the range of $\overline{\partial}_b$ is closed, then the results of Kohn and Burns imply that $M$ can be embedded in $\mathbb{C}^N$, for some $N$. The finite type analog of this result is due to Christ. If the $CR$ structure already extends to the pseudoconvex side, then one can embed $M$ in a manifold of the same dimension:

**Corollary 3.** *Let $D$ be a compact pseudoconvex complex manifold with smooth boundary and $\dim_{\mathbb{C}} D = 2$. Suppose that the complex structure on $D$ extends smoothly up to the boundary $bD$ of $D$, and that there is a smooth function $\lambda$ which is strictly subharmonic near the set $W$ of weakly pseudoconvex points of $bD$. Then there exists a complex manifold $\tilde{D}$, $\dim_{\mathbb{C}} \tilde{D} = 2$, such that $\overline{D}$ can be holomorphically embedded into $\tilde{D}$.*

Also, as an application of Theorem (C,Ch), we have the following local extension theorem.

**Theorem 4.** *Let $M$ be a $CR$ manifold of real dimension three and assume that $M$ is pseudoconvex in a neighborhood $U$ of $z_0 \in M$. Then there exist $\epsilon > 0$ and a neighborhood $V \subset U$ of $z_0 \in M$ such that the given $CR$ structure $S|_V$ on $M$ can be extended to an integrable almost complex structure $\mathcal{L}$ on $V_\epsilon^+ = P_+^{-1}(V)$.*

**Corollary 5.** *Let $D$ be a complex manifold with smooth boundary and $\dim_{\mathbb{C}} D = 2$. Suppose that the associated almost complex structure on $D$ extends smoothly up the boundary $bD$ of $D$, and $bD$ is pseudoconvex near $z_0 \in bD$. Then there is a neighborhood $V$ of $z_0$ such that $D$ can be embedded in a larger complex manifold $\tilde{D}$ so that $\overline{V} \cap bD$ lies in the interior of $\tilde{D}$ as a real hypersurface.*

**Sketch of the Proof**

Set $\Omega = M \times (-1, 1)$. Extend $L_1, \dots, L_{n-1} \in E_x$ so that it does not depend on $t$. For a real vector field $X_0$ with $\eta(X_0) = 1$, we set

$$L_n = \partial/\partial t - iX_0,$$

and set $\mathcal{L} = span\{L_1, \dots, L_n\}$. Then $(\Omega, \mathcal{L})$ is an almost complex manifold. By recursive process, we first prove that there exists $(\Omega, \mathcal{L}_0)$ such that if $L_1, L_2 \in \mathcal{L}_0$, and $w \in \Lambda^{1,0}$, then $w([\overline{L}_1, \overline{L}_2]) = 0$ to infinite order along $M$. That is, $\mathcal{L}_0$ is integrable up to infinite order along $M$.

For $q = 0, 1, \dots, n$, set

$$\Gamma^q = \Lambda^{(0,q)} \otimes \mathcal{L}.$$

If $\{L_1, \dots, L_n\}$ is a local frame and $\{\omega^1, \dots, \omega^n\}$ is its dual frame, then each $A \in \Gamma^1$ can be written as

$$A = \sum_{j,k=1}^{n} A_{j,k}(x)\overline{\omega}^k \cdot L_j,$$

where $A_{j,k}$ are smooth functions. Similarly, every $B \in \Gamma^q$ can be written as:

$$B = \sum_{l=1}^{n} \sum_{|J|=q} B_J^l(x)\overline{\omega}^J \cdot L_l.$$

Therefore, we can define norms for $B \in \Gamma^q$ ($C^k$-norms, Sobolev-norms, etc.) on $\Omega$ by using component functions of $B$ and by using a partition of unity on $\Omega$.

Note that if $A \in \Gamma^1$, then $A$ is a $\mathbb{C}$-linear bundle homomorphism from $\overline{\mathcal{L}}$ to $\mathcal{L}$. That is if $\overline{L} = \sum_{k=1}^{n} b_k \overline{L}_k \in \overline{\mathcal{L}}$, then

$$A(\overline{L}) = \sum_{j,k=1}^{n} A_{j,k} b_k L_j \in \mathcal{L}.$$

If $A_{j,k}$ is sufficiently small, we set

$$P_A(\overline{L}) = \overline{L} + A(\overline{L}) = \overline{L}_A,$$

and set

$$\overline{\mathcal{L}}_A = \{P_A(\overline{L}) : \overline{L} \in \overline{\mathcal{L}}\}.$$

Then $\overline{\mathcal{L}}_A$ is a deformation (or a perturbation) of $\mathcal{L}$. Set $\omega_A = \omega - A^*\omega$, where $A^* : \Lambda^{1,0} \longrightarrow \Lambda^{0,1}$ is given by

$$(A^*\omega)(\overline{L}) = \omega(A(\overline{L})), \quad \overline{L} \in \overline{\mathcal{L}}, \quad \omega \in \Lambda^{1,0}.$$

Then $\omega_A$ is a dual of $\mathcal{L}_A$. That is,

$$\omega_A(\overline{L}_A) = 0, \quad \overline{L}_A \in \overline{\mathcal{L}}_A.$$

**Q1 : Find $A$ (i.e., a deformation of $\mathcal{L}$) so that $(\Omega_\epsilon^+, \mathcal{L}_A)$ is integrable.**

That is, find $A$ so that for each $\omega_A \in \Lambda_A^{1,0}$, and for all $L_1, L_2 \in \mathcal{L}$,

(3)
$$\begin{aligned} &\omega_A([P_A(\overline{L}_1), P_A(\overline{L}_2)]) \\ &\equiv (\omega - A^*\omega)([\overline{L}_1 + A(\overline{L}_1), \overline{L}_2 + A(\overline{L}_2)]) \equiv 0. \end{aligned}$$

Define a non-linear operator $\Phi : \Gamma^1 \longrightarrow \Gamma^2$ by :

$$\Phi(A)(\overline{L}', \overline{L}'', \omega) = \omega_A([P_A(\overline{L}_1), P_A(\overline{L}_2)).$$

So our question is reduced to :

**(Q2) Find $A \in \Lambda^1$ so that $\Phi(A) = 0$.**

Note that (3) and hence (**Q2**) are non-linear problems. We use the Nash-Moser implicit function theorem. Up to second order term of $A$ (i.e., by linearizing), (3) becomes :

$$\omega([\overline{L}_1, A(\overline{L}_2)]) + \omega([A(\overline{L}_1), \overline{L}_2) - A^*\omega([\overline{L}_1, \overline{L}_2]) = -\omega([\overline{L}_1, \overline{L}_2]). \tag{4}$$

For $L \in \mathbb{C}T_z$, let $L = L' + L''$, where $L' \in \mathcal{L}_z$ and $L'' \in \overline{\mathcal{L}}_z$. Define

$$(D_2A)(\overline{L}_1, \overline{L}_2) = [\overline{L}_1, A(\overline{L}_2)]' - [\overline{L}_2, A(\overline{L}_1)]' - A([\overline{L}_1, \overline{L}_2]'').$$

Then $D_2A \in \Gamma^2$, and (4) is equivalent to

$$D_2A = -F,$$

where $F \in \Gamma^2$ satisfies $F(\overline{L}_1, \overline{L}_2) = [\overline{L}_1, \overline{L}_2]'$, which measures the extend to which fails to be integrable.

Let $\mathcal{P}_A^q : \Gamma_A^q \to \Gamma^q$ be the bundle isomorphism defined by

$$\mathcal{P}_A^q G(\overline{L}_1, \ldots, \overline{L}_q, \Omega) = G(P_A(\overline{L}_1), \ldots, P_A(\overline{L}_q), Q_A(\Omega)).$$

Then it turns out that

$$\Phi'(A) = \mathcal{P}_A^2 D_2^A (\mathcal{P}_A^1)^{-1} + \mathcal{O}(\Phi(A)A). \tag{5}$$

Set $A_0 = 0$. By induction, we set

$$d_{A_k} = \mathcal{P}_{A_k}^1 \circ (D_2^t)^* \circ N_{A_k}^t \circ (\mathcal{P}_{A_k}^2)^{-1}(-\Phi(A_k)) \tag{6}$$

where $N_{A_k}^t$ is the Neumann operator with respect to the structure $\mathcal{L}^{A_k}$ with weight $e^{-t\lambda}$, $t \geq T_k$, for some $T_k$ depending on $k$.

Using weighted $D_2^t$ estimates and the careful inspection of the parameters $\epsilon$ and $t$, we see that

$$\begin{aligned} \|d_A^k\|_{k,t} &\lesssim \|\Phi(A)\|_{k,t} + \epsilon^{-k^2}|A|_{k+1}\|\Phi(A)\|_{0,t} \\ &\quad + \epsilon^{-k^2} t^k \|\Phi(A)\|_{0,t}, \end{aligned} \tag{7}$$

and

$$\|\Phi(A) + \Phi'(A)(d_A^k)\|_k \lesssim \epsilon^{-k^2-k-10} t^{k+3} e^{\frac{t}{8}} \cdot \cdot \|\Phi(A)\|_{k+3} \|\Phi(A)\|_3 \left(1 + |A|_{k+1}\right). \tag{8}$$

If we set $A_{k+1} = A_k + S_{\theta_k} d_{A_k}$, where $S_{\theta_k} d_{A_k}$ is a smoothing operator, then by combining (5)–(8), we see that

$$\Phi(A_{k+1}) = \Phi(A_k) + \Phi'(A_k) d_{A_k} + \mathcal{O}(|d_{A_k}|^2) = \mathcal{O}(|d_{A_k}|^2),$$

that is, $\Phi(A_{k+1})$ vanishes second order in $d_{A_k}$, or in second order in the right hand side of (7).

(7) and (8) are called the **Tame estimates**, which are necessary in the variant of Nash-Moser iteration process. Using the variant of Nash-Moser theorem [5], we obtain our main theorem.

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# THE DIMENSION OF THE AUTOMORPHIC FORMS OF N-BALL

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ABSTRACT. In this paper, we get the classification of conjugacy classes in the group of automorphisms on 3-dimensional ball. The contributions from the conjugacy classes of the regualr elliptic elements and some hyperelliptic elements in the dimension formula are obtained.

## 1. AUTOMORPHIC FORMS ON $n$-DIMENSIONAL BALL.

Let $B_n$ be $n$-dimensional ball: $B_n = \{z = (z_1, ..., z_n) \in \mathbf{C}^n | 1 - z^t \bar{z} > 0\}$.

Let $I_n$ be the identity matrix of degree $n$, $I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$ and

$$SU(n,1) = \{g \in SL(n+1, \mathbf{C}) | {}^t\bar{g} I_{n,1} g = I_{n,1}\}.$$

As a well-known result, $B_n$ is a classical domain of type I and $SU(n,1)$ is group of automorphisms which act transitively on $B_n$(cf.[2],[3],and[4]).

Let $\Gamma$ be an arithmetic subgroup of $SU(n,1)$. We say a holomorphic function $f$ on $B_n$ an **automorphic form** of weight $m$ for $\Gamma$ if $j(\gamma, z)^m f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$ and $z \in B_n$. Here $j(\gamma, z)$ denotes the jacobian of the mapping $\gamma \in SU(n,1)$ at $z \in B_n$ which is given by $j(\gamma, z) = (c^t z + d)^{-n-1}$ for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ where $d \in \mathbf{R}$. Let

$$k(z, w) = (1 - z^t \bar{w})^{-n-1} \tag{1.1}$$

which is the Bergaman kernel function of $B_n$(cf.[2],[3],and[4]). Then the volume element $dV(w) = k(w, w)dw$(where by $dw$ we denote the Euclidean volume element) is invariant under $SU(n,1)$. A fundamental domain of $\Gamma$ acting on $B_n$(i.e. a measurable set of orbit representations) is denoted by $F$. For the automorphic forms $f_1$ and $f_2$ of weight $m$ for $\Gamma$, we set

$$(f_1, f_2) = \int_F f_1(z) \overline{f_2(z)} k(z, z)^{-m} dV(z), \tag{1.2}$$

which is called the **Petersson inner product**. It is easy to prove that the integral in (1.2) is independent of the selection of the fundamental domain.

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Let $A_m(\Gamma)$ denote the space of automorphic forms of weight $m$ for $\Gamma$ which have the property that $k(z,z)^{-\frac{m}{2}}f(z)$ is bounded on a fundamental domain $F$(i.e. a measurable set of orbit representations)of $\Gamma$ acting on $B_n$, we also call them **cusp forms.** It is obvious that any cusp form $f$ is square integral with respect to the Petersson inner product, i.e.$(f,f)<\infty$.

We set a map $A$: $z_1=\dfrac{w-i}{w+i}$, $z_2=\dfrac{\sqrt{2}u_1}{w+i},\ldots,z_n=\dfrac{\sqrt{2}u_{n-1}}{w+i}$. It is easy to show that $A$ is a bijective biholomorphic transformation of a Siegel domain of type 2:

$$\mathcal{D}_n=\{(w,u)\in \mathbf{C}\times\mathbf{C^{n-1}}|2\mathrm{Im}w-u^t\bar{u}>0\}.$$

onto the $n$-dimensional ball $B_n$.

Let $f$ be an automorphic form of weight $m$ for $\Gamma$ on $B_n$. It is easy to prove that

$$F(Z)=f(AZ)j(A,Z)^m \qquad Z\in\mathcal{D}_n,\ \ j(A,Z)=\frac{2^{\frac{n+1}{2}}}{(w+i)^{n+1}}i \tag{1.3}$$

is an automorphic form of weight $m$ for $A^{-1}\Gamma A$ on $\mathcal{D}_n$, i.e., $F$ is analytic in $\mathcal{D}_n$ and satisfies the functional equation $F(\mu Z)j(\mu,Z)^m=F(Z)$ for any $\mu\in A^{-1}\Gamma A$ and $Z\in\mathcal{D}_n$. As shown in[5], $F$ has a Fourier-Jacobi expansion of the form:

$$F(w,u)=\sum_{r=0}^{\infty}g_r(u)\exp(2\pi irw), \tag{1.4}$$

Which is called the **Fourier-Jacobi expansion** of $f$, where $g_r$ are theta functions on $\mathbf{C^n}$

**Theorem 1.1.** An automorphic form $f(z)$ on $B_n$ is a cusp form if and only if the first coefficient $g_0$ of its Fourier-Jacobi expansion (1.4) is 0.

## 2. Kernel function.

**Proposition 2.1.** Let $m\geq 2$. For $(z,w)\in B_n\times B_n$ define

$$K(z,w)=a(m)\sum_{\gamma\in\Gamma}k(\gamma z,w)^m j(\gamma,z)^m, \tag{2.1}$$

where

$$a(m)=\pi^{-n}\frac{\Gamma((n+1)(m-1)+n)}{\Gamma((n+1)(m-1))},$$

and $k(*,*)$ is the Bergman kernel functional defined by (1.1). Then one has:

(a)For any $w\in B_n$ the series on the right hand side of (2.1)—considered as a series of functions in $z$—is normally convergent on every compact subset of $\mathbf{H}\times\mathbf{C}$.

(b)$K(z,w)$ is the reproducing kernel function for $A_m(\Gamma)$ with repect to the Petersson inner product, i.e.:

(i)$K(z,w)=\overline{K(w,z)}$;

(ii)for any $w\in B_n$, the function $K(*,w)$ is a cusp form in $A_m(\Gamma)$;

(iii)for any $w \in B_n$ and any cusp form $f \in A_m(\Gamma)$, one has

$$< f, K(*, w >= f(w).$$

**Theorem 2.2.** For $m \geq 2$, one has

$$\dim A_m(\Gamma) = a(m) \int_F \sum_{\gamma \in \Gamma} k(\gamma z, z)^m j(\gamma, z)^m k(z, z)^{-m} dV(z). \quad (2.2)$$

## 3. Contribution from the conjugacy class of a regular elliptic element

The matrix $I_{n,1}$ determines a Hermitian form ( , ) on $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$ by the $(x,y) =^t \bar{y} I_{n,1} x$. Our group $SU(n,1)$ can then characterized as the group of matrices $g \in SL(n+1, \mathbf{C})$ such that $(gx, gy) = (x, y)$ for all $x, y \in \mathbf{C}^{n+1}$. We say that the nonzero vector $x \in \mathbf{C}^{n+1}$ is positive, isotropic or negative according as $(x,x)$ is positive, zero or negative.

**Definition 3.1.** An element $g \in SU(n,1)$ with $g \neq 1$ is **regular elliptic** if $g$ has a positive eigenvector and has no isotropic eigenvector.

**Proposition 3.2.** For $m \geq 2$ and a regular elliptic element $g$, we have:

(i) $\int_{B_n} k(gz,z)^m j(g,z)^m k(z,z)^{-m} dV(z)$ is convergent;

(ii)the contribution from elements in $\Gamma$ which are conjugate in $\Gamma$ to $g$ in the dimension formula (2.2) is given by $N(g) = \frac{a(m)}{|C(g)|} \int_{B_n} k(gz,z)^m j(g,z)^m k(z,z)^{-m} dV(z)$.

To calculate the integral, we obtain:

**Theorem 3.3.** For $m \geq 2$,

$$N(g) = \frac{\lambda^{n-(n+1)m}}{|C(g)|(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)}.$$

**Remark 3.4.** If $n = 1$, one has $N(g) = \frac{\lambda^{1-2m}}{|C(g)|(\lambda-\lambda_1)}$, which is well known in the theory of trace formula for modular forms of one variable. If $n = 2$, $N(g) = \frac{\lambda^{2-3m}}{|C(g)|(\lambda-\lambda_1)(\lambda-\lambda_2)}$ which has been calculated by Cohn[1].

In addition to elliptic elements, we will generalize Cohn's(cf.[1]) method and calculate contributions from other conjuacy classes in $SU(3,1)$. Since the bijective biholomorphic transformation $A^{-1}$ map $B_n$ to $\mathcal{D}_n$, it follows that we discuss the Siegel domain $\mathcal{D}_3$.

## 4. The group of automorphisms and its subgroups on $\mathcal{D}_3$

We can easily get the group of automorphisms which act transitively on $\mathcal{D}_3$:

$$G = t^{-1} SU(3,1) t = \{g \in SL(4, C) | {}^t\bar{g} H_{2,2} g = H_{2,2}\},$$

where $t \in SL(4, Z[i])$ is the transformation matrix of the bijective biholomorphic transformation $A : \mathcal{D}_3 \to B_3$, and $H_{2,2} = \begin{pmatrix} & & i \\ & -I_2 & \\ -i & & \end{pmatrix}$.

We take our discrete group $\widetilde{\Gamma} = G \cap SL(4, Z[i])$.

we now define the (parabolic) subgroup $P$ of $G$ to be the group of upper triangular matrices belongs to $G$. Then we get that unipotent radical $P_u$ of $P$ is the set of matrices of the form

$$\begin{pmatrix} 1 & \gamma & \frac{|\gamma|^2}{2}i + r \\ & I_2 & i^t\overline{\gamma} \\ & & 1 \end{pmatrix} = [\gamma, r] \quad (\gamma = (\gamma_1, \gamma_2) \in C^2; r \in R; |\gamma|^2 = |\gamma_1|^2 + |\gamma_2|^2).$$

We have the multiplication rule $[\alpha, j_1][\beta, j_2] = [\alpha+\beta, j_1+j_2-\mathrm{Im}a_1\bar{b}_1-\mathrm{Im}a_2\bar{b}_2]$, where $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$. We also define the subgroups $\Gamma_\infty^{(1)} = \widetilde{\Gamma} \cap P$ and $\Gamma_\infty = \widetilde{\Gamma} \cap P_u$. It is easy to show that $\Gamma_\infty = \{[\gamma, r] \in P | \gamma_1, \gamma_2 \in (1+i)Z[i]$ or $\gamma_1, \gamma_2 \in (1+i)Z[i] + 1, r \in Z\}$ and $\Gamma_\infty^{(1)} = \{i^\varepsilon g_0^\eta [\gamma, r] | [\gamma, r] \in \Gamma_\infty, \varepsilon \in Z, \eta \in Z\} \cong \Gamma_\infty \times Z/(4) \times Z/(4)$, where $g_0 = diag\{i, -1, 1, i\} \in \widetilde{\Gamma}$.

Then we get the following conclusions which are generalizations of Cohn[1](pp13~27).

**Conclusion 4.1.** If $\Gamma = t\widetilde{\Gamma}t^{-1}$, and $G_{Q[i]} = \widetilde{\Gamma} P_{Q[i]}$, and if $4m \in Z$ and $m \geq 2$, then

$$\begin{aligned} &a(m)^{-1} dim A_m(\Gamma) = c(m, \widetilde{\Gamma}) vol(\tilde{F}) \\ &+ \sum_{\gamma \in \widetilde{\Gamma} - (\Gamma_\infty^{(1)} \cup Z_{\widetilde{\Gamma}})} \int_{\tilde{F}} \frac{\tilde{k}(p, \gamma p)^m}{\tilde{k}(p, p)^m} \overline{j(\gamma, p)}^m dp + \lim_{s \to +0} \sum_{\gamma \in (\Gamma_\infty^{(1)} - Z_{\widetilde{\Gamma}})} \int_{\tilde{F}} \frac{\tilde{k}(p, \gamma p)^m}{\tilde{k}(p, p)^{m(1-s)}} \overline{j(\gamma, p)}^m dp \\ &= c(m, \widetilde{\Gamma}) vol(\tilde{F}) + \sum_{[\gamma]_{\widetilde{\Gamma}} : [\gamma]_{\widetilde{\Gamma}} \cap \Gamma_\infty^{(1)} = \phi} \int_{\tilde{F}_\gamma} \frac{\tilde{k}(p, \gamma p)^m}{\tilde{k}(p, p)^m} \overline{j(\gamma, p)}^m dp \\ &+ \lim_{s \to +0} \sum_{[\gamma]_{\widetilde{\Gamma}} : [\gamma]_{\widetilde{\Gamma}} \cap (\Gamma_\infty^{(1)} - Z_{\widetilde{\Gamma}}) \neq \phi} [\int_{\substack{\tilde{F}_\gamma - \cup \delta^{-1} \tilde{F} \\ \delta\gamma\delta^{-1} \in (\Gamma_\infty^{(1)} - Z_{\widetilde{\Gamma}})}} \frac{\tilde{k}(p, \gamma p)^m}{\tilde{k}(p, p)^m} \overline{j(\gamma, p)}^m dp \\ &+ \sum_{\delta\gamma\delta^{-1} \in (\Gamma_\infty^{(1)} - Z_{\widetilde{\Gamma}})} \int_{\delta^{-1}\tilde{F}} \frac{\tilde{k}(p, \gamma p)^m}{\tilde{k}(p, p)^{m(1-s)}} \frac{\overline{j(\gamma, p)}^m}{|j(\delta, p)|^{2ms}} dp. \end{aligned} \tag{4.1}$$

Here $Z_{\widetilde{\Gamma}}$ is the center of $\widetilde{\Gamma}$, and $c(m, \widetilde{\Gamma}) = \sum_{\gamma \in Z_{\widetilde{\Gamma}}} \overline{j(\gamma, p_0)}^m$, where $p_0 = (i, 0, 0) \in \mathcal{D}_3$, and $\tilde{k}(p, q) = (i(\bar{x} - w) - u^t\bar{v})^{-4}$ for any $p = (w, u), q = (x, v) \in \mathcal{D}_3$; $dp = \tilde{k}(p, p) dw du_1 du_2$; $\widetilde{\Gamma}$ is a discrete group of $G$, and $\tilde{F}$ is a fundamental domain in $\mathcal{D}_3$ for $\widetilde{\Gamma}$; $G_\gamma$ is the centralizer of $\gamma$ in $G$; $\widetilde{\Gamma}_\gamma = G_\gamma \cap \widetilde{\Gamma}$ is the centralizer of $\gamma$ in $\widetilde{\Gamma}$; and $\tilde{F}_\gamma = \cup_{\delta \in \widetilde{\Gamma}/\widetilde{\Gamma}_\gamma} \delta^{-1} \tilde{F}$ is a fundamental domain in $\mathcal{D}_3$ for $\widetilde{\Gamma}_\gamma$.

**Remark 4.2.** We shall denote by $I_m(\gamma; s)$ the integral $\int_{\tilde{F}_\gamma} \frac{\tilde{k}(p, \gamma p)^m}{\tilde{k}(p, p)^m} \overline{j(\gamma, p)}^m dp$ $(\gamma \in \widetilde{\Gamma}, s \geq 0)$. We write $I_m(\gamma)$ for $I_m(\gamma; 0)$.

**Conclusion 4.3.** Every primitive isotropic vector $v \in \widetilde{\Gamma}$ can be embedded in a basis $\{v, y_1, y_2, v'\}$ of $\widetilde{\Gamma}$ with $v'$ isotropic, $(v', v) = i$, $(y_j, y_j) = -1$, $(y_1, y_2) = 0$ and $y_j \perp v, v'$.

**Remark 4.4.** Depending on conclusion 4.3 and Lemma 6.2, we easily get $G_{Q[i]} = \widetilde{\Gamma} P_{Q[i]}$.

## 5. Classification of conjugacy classes in the group of automorphisms $G$

In addition to the regular elliptic element, when $n = 3$, we obtain the following definition of classification of the other conjugacy classes in $G$ referring to [1].

**Definition 5.1.** If $g \in G$ and $g$ is not in the center of $G$, we say $g$ is hyperelliptic if there exists a hypebolic plane $W \subset C^4$(i.e. a two-dimensional non-degenerate subspace containing an isotropic vector) such that $g|W$ is multiplication by a scalar (of absolute value 1); hyperbolic if $g$ is not hyperelliptic and has linearly independent isotropic vector $x_1$ and $x_2$ in $C^4$ such that $g(x_i) = \gamma x_i (i = 1, 2)$ with $\lambda_i \in C$,$\lambda_1 \neq \lambda_2$; or parabolic if $g$ has an isotropic eigenvector and is neither hyperelliptic nor hyperbolic.

**Theorem 5.2.** For any $g \in G$ and $g$ is not in the center of $G$, then $g$ belongs to one of the regular elliptic elements of conjugacy classes or the above types of conjugacy classes.

**Remark 5.3.** From the above theorem, we get the following results(Let $g(v_j) = \lambda_j v_j (j \leq 4)$):

(i). The element $g$ is regular elliptic, if $C^4 = \bigoplus_{j=1}^{4} Cv_j$, with $v_1$ positive,$v_j (j = 2, 3, 4)$ negative, $\lambda_1 \neq \lambda_j (j = 2, 3, 4)$.

(ii). The element $g$ is hyperelliptic, if $C^4 = \bigoplus_{j=1}^{4} Cv_j$ and $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_3$ or $\lambda_1 = \lambda_4$, with $v_1$ positive, $v_j (j = 2, 3, 4)$ negative.

(iii). The element $g$ is hyperbolic, if $C^4 = Cv_1 \oplus Cv_2 \oplus (Cv_3 + Cv_4)$, with $v_1$ and $v_2$ negative, $v_3$ and $v_4$ isotropic,$\lambda_3 \neq \lambda_4$. If $g$ has only three eigenspaces $V_{\lambda_1} = Cv_1, V_{\lambda_2} = Cv_2$ and $V_{\lambda_3} = Cv_3$, with $v_1$ negative, $v_2$ and $v_3$ isotropic, $v_1 \perp w (= Cv_2 + Cv_3)$, $\lambda_2 \neq \lambda_3$, then $g$ is also hyperbolic.

(iv). The element $g$ is parapolic, if $g$ has only three eigenspaces $V_{\lambda_1} (= Cv_j, j = 1, 2, 3)$, with $v_1$ and $v_2$ negative, $v_3$ isotropic, $v_1, v_2, v_3$ pairwise orthogonal. If $g$ has only two eigenspaces $V_{\lambda_1} = Cv_1$ and $V_{\lambda_2} = Cv_2$, with $v_1$ negative, $v_2$ isotropic, $v_1 \perp v_2$, then $g$ is also parapolic. In addition, if $g$ has no positive or negative eigenvectors, then $g$ is still parapolic as it must have at least one eigenvector which must be isotropic.

## 6. Hyperelliptic conjugacy classes

Suppose that $\gamma \in \widetilde{\Gamma}$ is hyperelliptic and that $\gamma(v_j) = \lambda_j v_j (j = 1, 2, 3, 4)$ with $v_1$ positive, $v_2$,$v_3$ and $v_4$ negative, $v_j \perp v_k (j \neq k)$,$|\lambda_j|^2 = 1$,$\prod_{j=1}^{4} \lambda_j = 1$, $\lambda_1 = \lambda_4$.

**Lemma 6.1.** $\gamma$ belongs to one of the following three types:

(i). $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = i^{\varepsilon}(1, -1, -1, 1)$; (ii).$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = i^{\varepsilon}(i, -1, 1, i)$; (iii).$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = i^{\varepsilon}(1, \rho, \rho^2, 1)$. Here ($\varepsilon = 0, 1, 2$ or 3; $\rho = \exp(2\pi i/3)$).

Now, we will devote to computing the contributions from the conjugacy classes of the hyperelliptic elements of type (ii). By means of the conclusion 4.3 in section 4, we get the following lemmas.

**Lemma 6.2.** If $\gamma \in \widetilde{\Gamma}$ is hyperelliptic of type (i) and type (ii), then $[\gamma]_{\widetilde{\Gamma}} \cap P \neq \phi$.

**Remark 6.3.** By the above Lemma 6.2 and Cohn[1], every hyperelliptic $\widetilde{\Gamma}$-conjugacy class $[\gamma]_{\widetilde{\Gamma}}$ of the type ii) has a representative in $\Gamma_{\infty}^{(1)} = \widetilde{\Gamma} \cap P$. But if $\gamma, \gamma' \in \Gamma_{\infty}^{(1)}$ are hyperelliptic and conjugate in $\widetilde{\Gamma}$, they are conjugate in $\Gamma_{\infty}^{(1)}$.

**Remark 6.4.** By Lemma 6.1 and Lemma 6.3, it is easy to show that there are the following $\widetilde{\Gamma}$-conjugacy classes of hyperelliptic elements of type ii): $i^{\varepsilon}(g_0)^{\eta}$, $i^{\varepsilon}\delta_1(g_0)^{\eta}(\delta_1)^{-1}$, $i^{\varepsilon}\delta_2(g_0)^{\eta}(\delta_2)^{-1}$, $i^{\varepsilon}\tau_1 g_0(\tau_1)^{-1}$, $i^{\varepsilon}\tau_2(g_0)^3(\tau_2)^{-1}$, $i^{\varepsilon}\tau_3 g_0(\tau_3)^{-1}$, $i^{\varepsilon}\tau_4(g_0)^3(\tau_4)^{-1}$, $i^{\varepsilon}\tau_5 g_0(\tau_5)^{-1}$, $i^{\varepsilon}\tau_6(g_0)^3(\tau_6)^{-1}$. Here $\eta = 1$, or 3, $\delta_1 = [(1,0),0]$, $\delta_2 = [(0,-1),0]$, $\tau_1 = [(-\frac{1+i}{2}, \frac{i-1}{2}), 0]$, $\tau_2 = [(\frac{i-1}{2}, -\frac{1+i}{2}), 0]$, $\tau_3 = [(-\frac{1+i}{2}, \frac{i-3}{2}), 0]$, $\tau_4 = [(\frac{i-1}{2}, -\frac{1+3i}{2}), 0]$, $\tau_5 = [(-\frac{1+3i}{2}, \frac{i-1}{2}), 0]$, $\tau_6 = [(\frac{i-3}{2}, -\frac{1+i}{2}), 0]$.

By means of Cohn[1](p31), let $r_1 = [\widetilde{\Gamma}_{(g_0)^{\eta}} : \widetilde{\Gamma}_{(g_0)^{\eta}} \cap \rho^{-1}\widetilde{\Gamma}\rho], r_2 = [G_{(g_0)^{\eta}} \cap \rho^{-1}\widetilde{\Gamma}\rho] : \widetilde{\Gamma}_{(g_0)^{\eta}} \cap \rho^{-1}\widetilde{\Gamma}\rho]$ for$\eta = 1, 3$ $\rho = \delta_j (j = 1, 2)$, or $\eta = 1, \rho = \tau_{j'} (j' = 1, 3, 5)$, or $\eta = 3, \rho = \tau_{j'} (j' = 2, 4, 6)$, we have that the contribution of the conjugacy classes of $\rho(g_0)^{\eta}\rho^{-1}$ is given by $I_m(\rho(g_0)^{\eta}\rho^{-1}) = \frac{r_1}{r_2} I_m((g_0)^{\eta})$. we also have that $I_m(i^{\varepsilon}(g_0)^{\eta}) = i^{4\varepsilon m} I_m((g_0)^{\eta})$.

**Theorem 6.5.** $I_m((g_0)^{\eta}) = \frac{\pi^3}{3\times 16} \frac{i^{(4m-2)\eta}}{(4m-1)(4m-3)(i^{-\eta}-1)(i^{-\eta}-(-1)^{\eta})}$ i.e.

$$N((g_0)^{\eta}) = \frac{1}{3 \times 16} \frac{4m-2}{(i^{-\eta} - 1)(i^{-\eta} - (-1)^{\eta})} i^{(4m-2)\eta} (\eta = 1, 2 \text{ or } 3).$$

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# HARMONIC AND PLURIHARMONIC BEREZIN TRANSFORMS

MIROSLAV ENGLIŠ

ABSTRACT. We show that, perhaps surprisingly, the asymptotic behaviour of the Berezin transform as well as some properties of Toeplitz operators on a variety of weighted harmonic and pluriharmonic Bergman spaces seem to be the same as in the holomorphic case.

Let $\Omega$ be a bounded domain in $\mathbf{C}^n$, $L^2_{\text{hol}}(\Omega) \subset L^2(\Omega)$ the Bergman space of all square-integrable holomorphic functions on $\Omega$, and $K(x,y)$ its reproducing kernel, i.e. the Bergman kernel. Thus

$$f(x) = \int_\Omega f(y)\, K(x,y)\, dy = \langle f, K_x \rangle, \qquad K_x := K(\cdot, x),$$

for all $f \in L^2_{\text{hol}}$ and $x \in \Omega$. Recall that for $\phi \in L^\infty(\Omega)$, the Toeplitz operator $T_\phi$ with symbol $\phi$ is defined by

$$T_\phi : L^2_{\text{hol}} \to L^2_{\text{hol}}, \qquad T_\phi f := P(\phi f),$$

where $P : L^2 \to L^2_{\text{hol}}$ is the orthogonal projection (the Bergman projection). The Berezin symbol of a (bounded linear) operator $T$ on $L^2_{\text{hol}}$ is, by definition, the function $\widetilde{T}$ on $\Omega$ defined by

$$\widetilde{T}(x) := \frac{\langle TK_x, K_x \rangle}{K(x,x)} = \Big\langle T \frac{K_x}{\|K_x\|}, \frac{K_x}{\|K_x\|} \Big\rangle.$$

Finally, the Berezin transform of $f \in L^\infty$ is, by definition, the Berezin symbol of the Toeplitz operator $T_f$:

$$Bf(x) = \widetilde{T_f}(x) = K(x,x)^{-1} \int_\Omega f(y)\, |K(x,y)|^2\, dy.$$

It is immediate that the mapping $T \mapsto \widetilde{T}$ is linear, $\widetilde{I} = \mathbf{1}$, $(T^*)^\sim = \overline{\widetilde{T}}$, $\|\widetilde{T}\|_\infty \leq \|T\|$, and $\widetilde{T}$ is a real-analytic function on $\Omega$; similarly for $f \mapsto Bf$. Since the function $\langle TK_y, K_x \rangle$, being holomorphic in $x$ and $\overline{y}$, is uniquely determined by its restriction to the diagonal $x = y$, it also follows that both mappings $T \mapsto \widetilde{T}$ and $f \mapsto Bf$ are one-to-one — a fact which is of crucial importance for some applications.

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There are also the weighted analogues of all the above objects: namely, for any continuous, positive weight function $w$ on $\Omega$, the subspace $L^2_{\text{hol}}(\Omega, w)$ of all holomorphic functions in $L^2(\Omega, w)$ is closed and possesses a reproducing kernel $K_w(x, y)$ — the weighted Bergman kernel; and one may define the Toeplitz operators $T_f^{(w)}$, Berezin symbols $\widetilde{T}^{(w)}$ and Berezin transform $B^{(w)}$ in the same way as before.

Consider now a strictly plurisubharmonic function $\Phi$ on $\Omega$. Then $g_{i\bar{j}} = \dfrac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j}$ defines a Kähler metric on $\Omega$, with the associated volume element $d\mu(z) = \det[g_{i\bar{j}}]\, dz$ ($dz$ being the Lebesgue measure). For any $h > 0$, we then have, in particular, the weighted Bergman spaces $L^2_{\text{hol}}(\Omega, e^{-\Phi/h}\, d\mu) =: L^2_{\text{hol},h}$, and the corresponding reproducing kernels $K_h(x, y)$, Toeplitz operators $T_f^{(h)}$, and Berezin transforms $B_h f$. It turns out that the following theorem holds.

**Theorem.** ([E1],[BMS]) *Assume that $\Omega$ is smoothly bounded and strictly pseudoconvex, and $e^{-\Phi}$ is a defining function for $\Omega$. Then as $h \searrow 0$,*

$$K_h(x, x) \approx e^{\Phi(x)/h}\, h^{-n} \sum_{j=0}^{\infty} h^j\, b_j(x); \tag{1}$$

$$B_h f \approx \sum_{j=0}^{\infty} h^j\, Q_j f; \qquad \text{and} \tag{2}$$

$$T_f^{(h)} T_g^{(h)} \approx \sum_{j=0}^{\infty} h^j\, T^{(h)}_{C_j(f,g)} \qquad \text{(in operator norm)}, \tag{3}$$

*for some functions $b_j \in C^\infty(\Omega)$, some differential operators $Q_j$, with $Q_0 = I$ and $Q_1 = g^{\bar{j}i} \partial_i \bar{\partial}_j$, the Laplace-Beltrami operator with respect to the metric $g_{i\bar{j}}$; and some bidifferential operators $C_j$, where $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \frac{i}{2\pi}\{f, g\}$ (the Poisson bracket of $f$ and $g$).*

The proof of the theorem makes use of the domain

$$\widetilde{\Omega} := \{(x, t) \in \Omega \times \mathbf{C} : \ |t|^2 < e^{-\Phi(x)}\}$$

which by the hypotheses is smoothly bounded and strictly pseudoconvex, and admits $r(x, t) := |t|^2 - e^{-\Phi(x)}$ as a defining function. Its boundary $\mathcal{X} = \partial\widetilde{\Omega}$ is a compact manifold, and $\alpha = \operatorname{Im} \partial r$ is a contact form on $\mathcal{X}$ (i.e. $\alpha \wedge (d\alpha)^{n-1}$ is a non-vanishing volume element). Let $H^2(\mathcal{X})$ be the Hardy subspace of all functions in $L^2(\mathcal{X})$ that extend holomorphically to $\widetilde{\Omega}$. According to a formula of Forelli, Rudin and Ligocka, the reproducing kernel $K_{\mathcal{X}}$ of $H^2(\mathcal{X})$ — the Szegö kernel — satisfies

$$K_{\mathcal{X}}((x, t), (y, s)) = \frac{1}{2\pi n!} \sum_{k=0}^{\infty} (t\bar{s})^k\, K_{1/(k+n+1)}(x, y).$$

On the other hand, by results of Fefferman, Boutet de Monvel and Sjöstrand,

$$K_{\mathcal{X}}\,|_{\text{diagonal}} = \frac{a}{r^{n+1}} + b \log r, \qquad a, b \in C^\infty(\overline{\widetilde{\Omega}}).$$

Employing the usual Cauchy estimates for the function $f_x(t\overline{s}) := K_{\mathcal{X}}((x,t),(x,s))$ of one complex variable on the disc $|t\overline{s}| < e^{-\Phi(x)}$, the expansion (1) follows (where $h = 1/(k+n+1)$, $k \to \infty$). In fact, this even gives a similar expansion for $K_h(x,y)$ for $(x,y) \in \Omega \times \Omega$ close to the diagonal, and (2) then follows by an application of the stationary phase method. Finally, (3) can be proved using the Boutet de Monvel-Guillemin theory of generalized Toeplitz operators (with pseudodifferential symbols).

A completely analogous result also holds for an arbitrary Kähler manifold $\Omega$ such that the second cohomology class $[\omega]$ of the Kähler form $\omega$ is integral: namely, there exists then an Hermitian line bundle $\mathcal{L}$ over $\Omega$ with compatible connection $\nabla$ such that curv $\nabla = \omega$. For $k = 1, 2, \dots$, consider, instead of the spaces $L^2_{\text{hol}}(\Omega, e^{-k\Phi}\, d\mu)$, the subspaces of all holomorphic square-integrable sections of the $k$-th power $\mathcal{L}^{*\otimes k}$ of the dual bundle $\mathcal{L}^*$. Taking the unit disc bundle $\widetilde{\Omega} \subset \mathcal{L}^*$ in $\mathcal{L}^*$ in the place of the domain $\widetilde{\Omega}$ from the preceding paragraph, a totally parallel argument again shows that (1) and (2) hold, and the Guillemin-Boutet de Monvel theory of generalized Toeplitz operators again yields also (3) (cf. [BMS],[Zel]).

The last theorem has an elegant application to *quantization on Kähler manifolds.* Recall that the traditional problem of quantization consists in looking for a map $f \mapsto Q_f$ from $C^\infty(\Omega)$ into operators on some (fixed) Hilbert space which is linear, conjugation-preserving, $Q_1 = I$, and as the Planck constant $h \searrow 0$,

$$[Q_f, Q_g] \approx \frac{ih}{2\pi} Q_{\{f,g\}}. \tag{4}$$

(The spectrum of $Q_f$ is then interpreted as the possible outcomes of measuring the observable $f$ in an experiment; and (4) amounts to a correct semiclassical limit.) Our last theorem implies that (4) holds for $Q_f = T_f^{(h)}$, the Toeplitz operators on the Bergman spaces $L^2_{\text{hol},h}$ (or on the spaces of holomorphic $L^2$-sections of the bundles $\mathcal{L}^{*\otimes 1/h}$). This is the so-called *Berezin-Toeplitz quantization.*

There is also another approach to quantization, discarding the operators $Q_f$ but rather looking for a noncommutative associative product $*$ on $C^\infty(\Omega)$, depending on $h$, such that as $h \searrow 0$,

$$f * g \to fg, \qquad \frac{f*g - g*f}{h} \to \frac{i}{2\pi}\{f,g\}.$$

Such products are called a <u>star-products</u>, and are the subject of *deformation quantization.* The relationship to Bergman spaces is the following: in view of the injectivity of the map $T \mapsto \widetilde{T}$ from operators to their Berezin symbols, we can define for two bounded operators $T, U$ on $L^2_{\text{hol},h}$ a "product" of their symbols by

$$\widetilde{T} * \widetilde{U} := \widetilde{TU}.$$

This gives a noncommutative associative product on

$$\{\widetilde{T} : T \text{ a bounded operator on } L^2_{\text{hol},h}\} \subset C^\omega(\Omega).$$

It can be shown from part (2) of the last theorem (i.e. from the asymptotics of $B_h$) that if $h$ is made to vary, these products can be glued into a star-product on $C^\infty(\Omega)$. This is the so-called *Berezin quantization.*

From the point of view of these applications, the weighted Bergman spaces $L^2_{\text{hol},h}$ (or their analogues $L^2_{\text{hol}}(\mathcal{L}^{*\otimes k})$ for manifolds) have an obvious disadvantage in that their very definition requires a holomorphic structure (hence, in particular, they can make sense only on Kähler manifolds). On the other hand, the other ingredients — the operator symbols, the Toeplitz operators and the Berezin transform — make sense not only for $L^2_{\text{hol}}$, but for any subspace of $L^2$ with a reproducing kernel. Hence it seems very natural to investigate whether any such spaces other than weighted Bergman spaces can be used for quantization.

For instance, one such candidate might be the harmonic Bergman spaces $L^2_{\text{harm}}$ of all harmonic functions in $L^2$. As in the holomorphic case, these possess a reproducing kernel, the *harmonic Bergman kernel* $H(x,y)$; in contrast to the usual Bergman kernel, $H(x,y)$ is real-valued and symmetric, $H(x,y) = H(y,x) \in \mathbf{R}$. Similarly, one has *pluriharmonic Bergman spaces* $L^2_{\text{ph}}$ (and pluriharmonic Bergman kernels).

Still another candidate are *Sobolev spaces of holomorphic functions* (Sobolev-Bergman spaces), i.e. the subspaces $W^s_{\text{hol}}$ of all holomorphic functions in the (possibly weighted) Sobolev spaces $W^s$, $s \in \mathbf{R}$. In fact, one can show that in the situation from the last theorem (i.e. when $e^{-\Phi}$ is a defining function), the weighted Bergman spaces $L^2_{\text{hol},h}$, for $h = 1/m$, coincide (as sets) with $W^s_{\text{hol}}(\Omega)$ where $s = \frac{n+1-m}{2} \leq 0$.

It is also possible to combine these two approaches and look at Sobolev spaces of (pluri)harmonic functions.

In this talk, we discuss in more detail the situation for the harmonic and pluriharmonic Bergman spaces.

Unfortunately, it turns out that — from the point of view of the quantization applications at least — bad things happen. First of all, recall that for the Berezin-Toeplitz quantization we needed that the Toeplitz operators satisfy

$$\frac{1}{h}[T^{(h)}_f, T^{(h)}_g] \approx \frac{i}{2\pi} T^{(h)}_{\{f,g\}} \qquad \text{as } h \searrow 0.$$

However, for Toeplitz operators on $L^2_{\text{harm}}$, this fails even on $\Omega = \mathbf{D}$, the unit disc in $\mathbf{C}$, with the hyperbolic metric (given by Kähler potential $\Phi(z) = \log \frac{1}{1-|z|^2}$) and $f(z) = z$, $g(z) = \overline{z}$. Second, recall that the Berezin quantization (the star-products) was based on the fact that the correspondence $T \mapsto \widetilde{T}$ between operators and their symbols was one-to-one. However, this fails on any harmonic Bergman space: if $f, g$ are any two linearly independent elements in $L^2_{\text{harm}}$, then the operator $T = \langle \cdot, \overline{f} \rangle g - \langle \cdot, \overline{g} \rangle f$ is easily seen to satisfy $\langle TH_x, H_x \rangle = f(x)g(x) - g(x)f(x) = 0 \; \forall x$; hence $\widetilde{T} \equiv 0$, while apparently $T \neq 0$. Thus, there is no hope to perform the quantization. (See [E2] for the details.)

In view of these failures, it would be only natural to expect that also the other assertions of our theorem (e.g. the asymptotics of the Berezin transform, or the injectivity of the map $f \mapsto Bf$) break down. The following results therefore came as some surprise for the author.

Recall that a domain $\Omega \subset \mathbf{C}^n$ is called complete Reinhardt if $x \in \Omega$ and $|y_j| \leq |x_j|\ \forall j$ imply $y \in \Omega$. In particular, such domains are invariant under the rotations

$$z \mapsto (z_1 e^{i\theta_1}, z_2 e^{i\theta_2}, \dots, z_n e^{i\theta_n}), \qquad \forall \theta_1, \dots, \theta_n \in \mathbf{R}. \tag{5}$$

**Theorem 1.** *Let $\Omega \subset \mathbf{C}^n$ be complete Reinhardt and let $\nu$ be any finite measure on $\Omega$ invariant under the rotations* (5). *Then on $L^2_{\text{ph}}(\Omega, d\nu)$,*

$$\widetilde{T}_f = 0 \implies T_f = 0 \qquad (\text{i.e. } \widetilde{B}f = 0 \implies f = 0).$$

Thus, although the Berezin symbol map $T \mapsto \widetilde{T}$ is not injective on all operators, it is injective on Toeplitz operators.

**Theorem 2.** *Consider the following situations,*

$$L^2_{\text{harm}}(\mathbf{D}, \tfrac{1+h}{\pi h}(1-|z|^2)^{1/h}),$$
$$L^2_{\text{ph}}(\mathbf{C}^n, h^{-n} e^{-|z|^2/h})$$

*(i.e. the harmonic Bergman spaces on the disc with respect to the usual weights and the pluriharmonic Fock spaces on $\mathbf{C}^n$), and also the pluriharmonic analogues of the standard weighted Bergman spaces on bounded symmetric domains in $\mathbf{C}^n$. Then the associated Berezin transforms possess the asymptotic expansion* (2), *i.e. there exist differential operators $Q_j$ such that $\forall f \in C^\infty \cap L^\infty$,*

$$B_h f(x) = \sum_{j=0}^{\infty} h^j Q_j f(x) \qquad \text{as } h \searrow 0.$$

*In fact, these are the same $Q_j$ as in the holomorphic case.*

**Theorem 3.** *The assertion of the last theorem also holds for*

$$L^2_{\text{harm}}(\mathbf{R}^n, h^{-n/2} e^{-|x|^2/h})$$

*(the harmonic Fock space on $\mathbf{R}^n$), with $Q_j = (\Delta/4)^j$.*

The proofs of these theorems go by explicit calculations of the reproducing kernels in question (which are possible owing to the rotational symmetry of the domains and measures) and the method of stationary phase; see [E3]. (For Theorem 3, one also needs the properties of certain spherical harmonics [ABR], and an interesting special function — one of the hypergeometric functions of Horn — plays a role.)

In a way, these theorems raise more questions than they answer. First of all, it is not clear whether the results are anomalies whose validity stems from the abundant symmetries of the domains, or whether they hold in more general settings. For instance, does Theorem 1 hold for the Toeplitz operators on the pluriharmonic Bergman space on a general smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$? Or does Theorem 2 hold for the pluriharmonic analogues of the spaces $L^2_{\text{hol}}(\Omega, e^{-\Phi/h} d\mu)$ from the traditional Berezin and Berezin-Toeplitz quantizations? For Theorem 3, it even makes sense to study the problem not only for pseudoconvex domains in $\mathbf{C}^n$, which are the natural arena for holomorphic functions, but for any open set in $\mathbf{R}^n$. (Currently, it is even unknown whether an analogue of Theorem 3 holds for the unit ball of $\mathbf{R}^n$.)

We remark that in the holomorphic case, the asymptotics of the weighted Bergman kernels, of the Berezin transform and of the Toeplitz operators were derived from the boundary behaviour of the Szegö kernel of the "inflated" domain $\widetilde{\Omega} = \{(x,t) \in \Omega \times \mathbf{C} : |t|^2 < e^{-\Phi}\}$, using the formula of Forelli-Rudin-Ligocka and the Fefferman-Boutet de Monvel-Sjöstrand theorem. It should be noted that the Forelli-Rudin-Ligocka formula holds also in the pluriharmonic case: if we denote by $H^2_{\text{ph}}(\mathcal{X})$, $\mathcal{X} = \partial\widetilde{\Omega}$, the subspace in $L^2(\mathcal{X})$ of all functions that have a pluriharmonic extension inside $\widetilde{\Omega}$, then the reproducing kernel of $H^2_{\text{ph}}(\mathcal{X})$ is given by

$$K^{\text{ph}}_{\mathcal{X}}((x,t),(y,s)) = \frac{1}{2\pi n!} \sum_{j=-\infty}^{\infty} (s\bar{t})^{[j]} \, K^{\text{ph}}_{1/(|j|+n+1)}(x,y),$$

where $z^{[j]} = z^j$ or $\bar{z}^{-j}$ according as $j \geq 0$ or $< 0$, and $K^{\text{ph}}_{1/m}(x,y)$ is the reproducing kernel of $L^2_{\text{ph}}(\Omega, e^{-m\Phi}\, d\mu)$. Thus in principle we can again get the asymptotics of $K^{\text{ph}}_{1/m}$, and of the pluriharmonic Berezin transform, from the boundary singularity of $K^{\text{ph}}_{\mathcal{X}}$. Unfortunately, what is missing is the pluriharmonic analogue of the Fefferman-Boutet de Monvel-Sjöstrand theorem, i.e. the description of the boundary singularity of the pluriharmonic Szegö or Bergman kernels.

Similarly, it seems unknown what is the boundary singularity of the harmonic Bergman (or Szegö) kernel of a domain in $\mathbf{R}^n$. (There exist optimal estimates for the boundary growth, though; see [KK].) However, in this case there is no analogue of the Forelli-Rudin-Ligocka formula.

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# The Logarithmic Singularities of the Bergman Kernels for model domains

HANJIN LEE

## 1. Statement of theorem

A domain $\Omega \in \mathcal{M}$ if and only if

$$\Omega = \{(z_0, z) \in \mathbb{C} \times \mathbb{C}^n : \Im(z_0) > F(z)\}$$

$F$ : real analytic strictly plurisubharmonic function on $\mathbb{C}^n$ such that

1. $F(0) = \nabla F(0) = 0$
2. $F(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) = F(z_1, \cdots, z_n)$ for any $\theta_j \in \mathbb{R}$
3. There are small positive numbers $c$ and $\epsilon$ such that $F(z) \geq c|z|^\epsilon$ for sufficiently large $|z| := (\sum_{j=1}^n |z_j|^2)^{1/2}$.

**Theorem 1.** *Suppose $n = 2$. Let $\Omega$ be a domain that belongs to the class $\mathcal{M}$. Then $\Omega$ is biholomorphic to the ball if, and only if, its Bergman kernel function does not have logarithmic singularity at the boundary.*

## 2. Background of theorem

**Theorem 2 (Fefferman).** *Let $G \subset \mathbb{C}^n$, bounded strictly pseudoconvex domain, $\partial G$ is smooth and $G = \{r > 0\}$ for smooth defining function $r$ then,*

$$B_G = \frac{\varphi}{r^{n+1}} + \psi \log r$$

*where $\varphi$, $\psi \in C^\infty(\overline{G})$*

Expansions of $\varphi, \psi$ :

$$\varphi = \sum_{k=0}^{n} \varphi_k r^k \bmod O(r^{n+1}), \qquad \psi \sim \sum_{k=0}^{\infty} \psi_k r^k$$

where $\phi_k, \psi_k \in C^\infty(\overline{G})$

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## Fefferman's program

Choose $r = r^F$ which satisfies certain transformation rule under biholomorphism. Then $\varphi_k, \psi_k$ are CR invariants, that is, polynomials in Moser's normal form coefficients satisfying certain transformation rule with weight $k$ and $n+1+k$. By Chern-Moser theory, Moser's normal form coefficients are expressed in terms of CR curvature tensors. It implies that certain conditions on singularities $\varphi_k, \psi_k$ decide the geometry of domains.

**Theorem 3 (Burns / Graham).** *Let $G \subset \mathbb{C}^2$. The boundary of $G$ is locally CR equivalent to the sphere if $\psi = O(r^2)$.*

Our theorem is an attempt to generalize Burns-Graham's theorem for higher dimension. For 2 dimensional case, with additional assumption of complete Reinhardtness it is known that the vanishing of log term implies that G is equivalent to the ball (Boichu and Coeuré, Nakazawa). For general dimension, if the domains are ellipsoids close to the ball,then vanishing of log term implies that the domain is the ball (Hirachi).

## 3. Proof of theorem

We use basically ideas and methods in Kamimoto's work (2004).

### Part 1. Formula of log singularities

### Haslinger's formula

$$H_\tau(\mathbb{C}^n) = \{g \in \mathcal{O}(\mathbb{C}^n) : \int_{\mathbb{C}^n} |g|^2 e^{-2\tau F} dV < \infty\}$$

$K(\cdot;\tau)$ : Bergman kernel for $H_\tau(\mathbb{C}^n)$

$$B_\Omega(z_0, z) = \frac{1}{2\pi} \int_0^\infty e^{-2\Im(z_0)\tau} K(z;\tau)\tau d\tau$$

$$K(z;\tau) = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{|z|^{2\alpha}}{c_\alpha(\tau)^2}$$

where $|z|^{2\alpha} = |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n}$, and

$$c_\alpha(\tau)^2 = \int_{\mathbb{C}^n} |z|^{2\alpha} e^{-2\tau F(z)} dV(z)$$

**Singularity formula**

$$B_\Omega(z_0, z) = \frac{1}{8\pi}\sum_{j=0}^{n+1} \varphi_j(z)(\Im z_0)^{-j-1} + \frac{1}{8\pi}\sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!}\psi_p(z)(\Im z_0)^p \log(\Im z_0)$$

where

$$\varphi_j(z) = \sum_{\alpha\in\mathbb{Z}_+^n} e_{\alpha,|\alpha|+n+1-j}\ |z|^{2\alpha}$$

$$\psi_p(z) = \sum_{\alpha\in\mathbb{Z}_+^n} e_{\alpha,|\alpha|+n+2+p}\ |z|^{2\alpha}$$

$$F(z) = \sum_{j=1}^{n} |z_j|^2 + \sum_{l\geq 2} P_l(|z_1|^2, \ldots, |z_n|^2)$$

where

$$P_l(y_1, \ldots, y_n) = \sum_{|\beta|=l} C_\beta^{(l)} y^\beta$$

Set $S_+ = \{y \in \mathbb{R}_+^n : y_1 + \cdots + y_n = 1\}$. $d\mu$ is surface measure on $S_+$ and $d\mu_\alpha = y^\alpha d\mu$

$$\begin{aligned}
e_{\alpha,|\alpha|+p+n+2} &= \int_{S_+} P_{|\alpha|+p+n+3}\, d\mu_\alpha \\
&+ \int_{S_+} P_{|\alpha|+p+n+2}\, P_2\, d\mu_\alpha + \int_{S_+} P_{|\alpha|+p+n+2}\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \\
&+ \int_{S_+} P_{|\alpha|+p+n+1}\, P_3\, d\mu_\alpha + \int_{S_+} P_{|\alpha|+p+n+1}\, d\mu_\alpha \int_{S_+} P_3\, d\mu_\alpha \\
&+ \int_{S_+} P_{|\alpha|+p+n+1}\, P_2^2\, d\mu_\alpha + \int_{S_+} P_{|\alpha|+p+n+1}\, P_2\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \\
&+ \int_{S_+} P_{|\alpha|+p+n+1}\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \\
&+ \cdots \\
&+ \sum_{k=1}^{|\alpha|+p+n+2} \sum_{\substack{l_1+\ldots+l_k \\ =|\alpha|+p+n+2}} \int_{S_+} P_2^{l_1}\, d\mu_\alpha \cdots \int_{S_+} P_2^{l_k}\, d\mu_\alpha
\end{aligned}$$

where each term has proper constants, but we didn't consider them here.

**Part 2**

We consider the case $n = 2$. ( Most of key lemmas still hold for general dimension, but some arguments depend on the partition of $n$ )

**Theorem 4.** *$\Psi = \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!} \psi_p(z)(\Im z_0)^p = O((\Im z_0)^{43})$ implies $P_k = 0$ for all $k \geq 2$.*

From the observation

$$e_{\alpha,(l)} = \sum_{|\beta|=l+n+3} C_{\beta}^{(l+n+3)} B(\beta_1, \dots, \beta_n) + \text{polynomial}\,(C_{\beta}^{(l+n+2)}, \dots, C_{\beta}^{(2)})$$

we can consider

$$e_{\alpha,(|\alpha|+p=l)} = e_{\alpha,|\alpha|+p+n+2=l+n+2} = 0, \qquad |\alpha| \leq l \qquad l = 0, 1, \dots$$

as system of polynomial equations

$$E_j^{(k)}(P_k, \dots, P_2) = 0, \qquad j = 1, \dots, n_k \qquad k = 2, 3 \dots$$

We may assume $P_2 \neq 0$ or $P_3 \neq 0$ because if $P_2 = P_3 = 0$, we can show that $P_k = 0$ for all $k$.

## (A) We assume $P_2 \neq 0$.

**Lemma 1.** *The whole system ( in fact, finite subsystem ) of equations is reduced to*

$$E_j^{(2)}(P_2) = 0, \qquad j = 1, 2, 3$$

$$E_j^{(3)}(P_3, P_2) = 0, \qquad j = 1, 2, 3, 4$$

*and in particular $P_2 = P_3 = 0$.*

Key idea for lemma is to find smallest $L$ such that

$$n_2 + \dots + n_L \geq (2+1) + (3+1) + \dots + (L+1).$$

For $k > 5$

$$n_k = \begin{cases} 3/2\,k - 5 & k \text{ is even} \\ 3/2\,(k+1) - 6 & k \text{ is odd.} \end{cases}$$

Since we have redundant terms when $k \geq 17$, we replace $n_k$ with $\tilde{n}_k = (k+1) + 2$ for $k \geq 17$, then $L = 26$ is smallest number such that

$$n_2 + \dots + n_{16} + \tilde{n}_{17} + \dots + \tilde{n}_L \geq (2+1) + (3+1) + \dots + (L+1).$$

$\Rightarrow \int P_{26} P_2^{21}\, d\mu$ comes from

$$e_{\alpha,(|\alpha|+p=42)} = \int P_{47}+P_{46}P_2+\ldots+P_{26}(P_2^{21}+\ldots+P_{22})+(\text{l.o.t})\, d\mu+(P_r : r \leq 46) = 0$$

$\Rightarrow$ Condition for vanishing order of log term

## Reduction

$$\int P_k\, y_1^{j_p}\, d\mu = \int P_{k-1}\, y_1^{j_p+2}\, d\mu + \text{l.o.t}$$

for $j_1 = 0, \ldots, j_{k-4} = k-5, \ldots, j_{n_k} = 2(k-4)$

Suppose $n_k > k+1$. LHS can be considered linear equations in $C_0^{(k)}, \ldots C_k^{(k)}$. Solve equations for $C_0^{(k)}, \ldots C_k^{(k)}$

$$\begin{cases} \int P_k\, y_1^{j_1}\, d\mu = \int P_{k-1}\, y_1^{j_1+2}\, d\mu + \text{l.o.t} \\ \vdots \\ \int P_k\, y_1^{j_{k+1}}\, d\mu = \int P_{k-1}\, y_1^{j_{k+1}+2}\, d\mu + \text{l.o.t} \end{cases}$$

Then

$$(*) \quad C_j^{(k)} = \sum_{p=1}^{k+1} \left( \int P_{k-1}\, y_1^{2+j_p}\, d\mu \right) + \text{l.o.t}$$

$$\int P_k\, y^{j_p}\, d\mu|_{(*)} = \int P_{k-1}\, y_1^{2+j_p}\, d\mu + \text{l.o.t}$$

for $p = k+2, \ldots, n_k$

$$LHS = \sum_{j=0}^{k} B(j_p + k - j, j) C_j^{(k)} | \, (*)$$

$$\int P_{k-1}\, y_1^{2+j_p}\, d\mu = \sum_{p \leq k+1} \int P_{k-1}\, y_1^{2+j_p}\, d\mu + \text{l.o.t}$$

for $p = k+2, \ldots, n_k$

## Equations for $P_2$

$(k, m)$ represents $\int P_k P_2^m\, d\mu$, and $\rightarrow$ means reduction.
$(26, 21) \rightarrow (25, 22) \rightarrow (24, 23) \rightarrow (23, 24) \rightarrow \ldots$
$(25, 21) \rightarrow (24, 22) \rightarrow (23, 23) \rightarrow \ldots$
$(25, 20) \rightarrow (24, 21) \rightarrow (23, 22) \rightarrow \ldots$
$(24, 20) \rightarrow (23, 21) \rightarrow \ldots$

$(25,20),(25,21),(26,21)$ forms a tail in the whole system. Full reduction $\Rightarrow$ $Q_d(P_2)=0,\ d=44,45,46\ \Rightarrow P_2=0$.

**(B) Case $P_2=0, P_3\neq 0$**

$F(z)=\sum_{k\geq 3}P_k(|z_1|^2,\dots,|z_n|^2)$ and $e_{\alpha,(|\alpha|+p)}(P_2=0)=0$ for all $\alpha\in\mathbb{Z}_+^n, p\in\mathbb{Z}_+$. Apply previous arguments to

$$F^\epsilon(z)=\epsilon P_2(|z_1|^2,\dots,|z_n|^2)+F(z)$$

where $P_2|_{S_+}=1$

$$\begin{aligned} e_{\alpha,(|\alpha|+p)}(F^\epsilon) &= \epsilon^{|\alpha|+p+4}+\dots+\epsilon(P_{|\alpha|+p+4}+\dots+(P_k:k\geq 3))+\text{ terms without } P_2 \\ &= \epsilon(P_{|\alpha|+p+4}+\dots+(P_k:k\geq 3)+O(\epsilon)) \end{aligned}$$

$$\begin{cases} \gamma_0=e_{\alpha,(|\alpha|+p=0)}=(P_4)+O(\epsilon) \\ \gamma_{1,|\alpha|}=e_{\alpha,(|\alpha|+p=1)}=(P_5)+(P_3^2)+O(\epsilon), \quad |\alpha|=0,1 \\ \gamma_{2,|\alpha|}=e_{\alpha,(|\alpha|+p=2)}=(P_6)+(P_4P_3)+O(\epsilon), \quad |\alpha|=0,1,2 \\ \gamma_{3,|\alpha|}=e_{\alpha,(|\alpha|+p=3)}=(P_7)+(P_5P_3)+(P_4^2)+O(\epsilon), \quad |\alpha|=0,1,2,3 \\ \dots \end{cases}$$

$$\Rightarrow \begin{cases} (P_3^2)=\gamma_1'+O(\epsilon) \\ (P_4P_3)=\gamma_{2,j}'+O(\epsilon) \quad j=1,2 \\ (P_5P_3)=(P_4^2)+\gamma_{3,j}'+O(\epsilon) \quad j=1,2,3 \\ \dots \end{cases}$$

$$\Rightarrow \begin{cases} n_3-(3+1)=-3, \\ n_4-(4+1)=-2, \\ n_5-(5+1)=-1, \\ \dots \end{cases}$$

$\Rightarrow$ Finite system with $|\alpha|+p\leq 42$
$\Rightarrow$ (Reduction) Overdetermined system for $P_3$
$\Rightarrow P_3=0$

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# HEARING THE TYPE OF A DOMAIN IN $\mathbb{C}^2$ WITH THE $\bar{\partial}$-NEUMANN LAPLACIAN

SIQI FU

## 1. Introduction

Motivated by Mark Kac's famous question [Kac66] "Can one hear the shape of a drum?", we study the interplays between the geometry of a bounded domain in $\mathbb{C}^n$ and the spectrum of the $\bar{\partial}$-Neumann Laplacian. Since the work of Kohn [Ko63], it has been discovered that regularity of the $\bar{\partial}$-Neumann Laplacian is intimately connected to the boundary geometry. (See, for example, the surveys [BSt99, Ch99, DK99, FS01].) It is then natural to expect that one can "hear" more about the geometry of a bounded domain in $\mathbb{C}^n$ with the $\bar{\partial}$-Neumann Laplacian than with the usual Dirichlet or Neumann Laplacian.

For bounded domains in $\mathbb{C}^n$, it follows from Hörmander's $L^2$-estimates of the $\bar{\partial}$-operator [H65] that pseudoconvexity implies positivity of the spectrum of the $\bar{\partial}$-Neumann Laplacian on all $(0,q)$-forms, $1 \leq q \leq n-1$. The converse is also true (under the assumption that the interior of the closure of the domain is the domain itself). This is a consequence of the sheaf cohomology theory dated back to Oka and H. Cartan (see [Se53, L66, O88]). (See [Fu05] for a discussion and proofs of this and the analogous result for the Kohn Laplacian without the sheaf cohomology theory.) Therefore, in Kac's language, we can "hear" pseudoconvexity via the $\bar{\partial}$-Neumann Laplacian.

Regularity and spectral theories of the $\bar{\partial}$-Neumann Laplacian closely intertwine. For example, on the one hand, by a classical theorem of Hilbert in general operator theory, compactness of the $\bar{\partial}$-Neumann operator is equivalent to emptiness of the essential spectrum of the $\bar{\partial}$-Neumann Laplacian. On the other hand, by a result of Kohn and Nirenberg [KN65], compactness of the $\bar{\partial}$-Neumann operator implies exact global regularity of the $\bar{\partial}$-Neumann Laplacian on $L^2$-Sobolev spaces. It was shown in [FS98] that for a bounded convex domain in $\mathbb{C}^n$, the $\bar{\partial}$-Neumann operator on $(0,q)$-forms is compact if and only if the boundary contains no $q$-dimensional complex varieties. (It is noteworthy that the proof of the necessity of this result is based on the Ohsawa-Takegoshi extension theorem [OT87].) However, such characterization does not hold even for complete pseudoconvex Hartogs domains in $\mathbb{C}^2$ ([Ma97], see also [FS01]). It was observed in [FS02] that compactness of the $\bar{\partial}$-Neumann operator on complete Hartogs domains in $\mathbb{C}^2$ is intimately related to diamagnetism and paramagnetism for certain Schrödinger operators with infinitely degenerating magnetic fields. The desired paramagnetic property (in semi-classical limits) was finally established in [CF05]. As a consequence, for smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$, compactness of the $\bar{\partial}$-Neumann operator on $(0,1)$-forms implies that the boundary contains no pluripotentials (more precisely, it satisfies property $(P)$ in the sense of Catlin [Ca84b] or equivalently is $B$-regular in the sense of Sibony [Si87]). This, together with an earlier result of Catlin [Ca84b] (compare [St97]), shows that one can determine whether or

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not the boundary of a Hartogs domain in $\mathbb{C}^2$ contains pluripotentials via the spectrum of the $\bar{\partial}$-Neumann Laplacian.

The main purpose of this note is to sketch the proof of the following theorem. We refer the reader to [Fu05a] for the detail.

**Theorem 1.1.** *Let $\Omega$ be a smooth bounded domain in $\mathbb{C}^2$. Let $\mathcal{N}(\lambda)$ be the number of eigenvalues of the $\bar{\partial}$-Neumann Laplacian that are less than or equal to $\lambda$. Then $b\Omega$ is pseudoconvex of finite type if and only if $\mathcal{N}(\lambda)$ has at most polynomial growth.*

Recall that the type of a smooth boundary $b\Omega \subset \mathbb{C}^2$ (in the sense of Kohn [Ko72]) is the maximal order of contact of a (regular) complex variety with $b\Omega$. (See [Ko79, D82, Ca84a, D93] for more information on this and other notions of finite type.)

We divide the proof of Theorem 1.1 into two parts. For the sufficiency, we establish the following result.

**Theorem 1.2.** *Let $\Omega \subset \mathbb{C}^2$ be smooth bounded pseudoconvex domain of finite type $2m$. Then $\mathcal{N}(\lambda) \lesssim \lambda^{m+1}$.*

The Weyl type asymptotic formula for $\mathcal{N}(\lambda)$ for strictly pseudoconvex domains in $\mathbb{C}^n$ was established in [Me81] by Metivier via an analysis of the spectral kernel of the $\bar{\partial}$-Neumann Laplacian. The heat kernel of the $\bar{\partial}$-Neumann Laplacian on strictly pseudoconvex domains, as well as that of the Kohn Laplacian on the boundary, were studied extensively in a series of papers by Stanton, Beals-Greiner-Stanton, Stanton-Tartakoff, Beals-Stanton, and others (see [S84, BGS84, ST84, BeS87, BeS88]). Metivier's formula was recovered as a consequence. Recently, the heat kernel of the Kohn Laplacian on finite type boundaries in $\mathbb{C}^2$ was studied by Nagel and Stein [NS01], from which one could also deduce a result similar to Theorem 1.2 for the Kohn Laplacian on the boundary.

We follow Metivier's approach in proving Theorem 1.2 by studying the spectral kernel. We are also motivated by the work on the Bergman kernel by Catlin [Ca89], Nagel et al [NRSW89] and McNeal [Mc89] as well as related work of Christ [Ch88] and Fefferman and Kohn [FeK88]. Since the spectral kernel does not transform well under biholomorphic mappings, instead of (locally) rescaling the domain to unit scale and studying the $\bar{\partial}$-Neumann Laplacian on the rescaled domain as in the Bergman kernel case, we rescale both the domain and the $\bar{\partial}$-Neumann Laplacian as in [Me81]. In doing so, we are led to study anisotropic bidiscs that have larger radii in the complex normal direction. Roughly speaking, at a boundary point of type $2m$, the quotient of the radii in the complex tangential and normal directions for the bidiscs used here is $\tau : \tau^m$ while in the Bergman kernel case it is $\tau : \tau^{2m}$ ($\tau > 0$ is small). To establish desirable properties, such as doubling and engulfing properties, for these anisotropic bidiscs, we employ both pseudoconvexity and the finite type condition. Note that only the finite type condition was used in establishing these properties for the smaller bidiscs used in the Bergman kernel case. Here in our analysis of these bidiscs, we make essential use of an observation by Fornæss and Sibony [FoS89]. Also crucial to our analysis is a uniform Kohn type Gårding's inequality on the rescaled $\bar{\partial}$-Neumann Laplacian.

By carefully flattening the boundary, we then reduce the problem to estimating eigenvalues of auxiliary operators on the half-space, which ultimately boils down to estimating eigenvalues of certain Schrödinger operators with finitely degenerating magnetic fields.

For the necessity, we prove the following slightly more general result.

**Theorem 1.3.** *Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $\mathcal{N}_q(\lambda)$ be the number of eigenvalues of the $\bar{\partial}$-Neumann Laplacian on $(0,q)$-forms that are less than or* *equal to* $\lambda$. *If* $\mathcal{N}_q(\lambda)$ *has at most polynomial growth for some* $q$, $1 \le q \le n-1$, *then* $b\Omega$ *is of finite* $D_{n-1}$*-type.*

Recall that the $D_{n-1}$-type of $b\Omega$ is the maximal order of contact of $(n-1)$-dimensional (regular) complex varieties with $b\Omega$. It was observed by D'Angelo [D87] that the $D_{n-1}$-type is identical to the second entry in Catlin's multitype. An ingredient in the proof of Theorem 1.3 is a wavelet construction of Lemarié and Meyer [LM86]. A result similar to Theorem 1.3 for the Kohn Laplacian on the boundaries in $\mathbb{C}^2$ is also known to M. Christ [Ch].

## 2. Proof of Necessity

In this section, we sketch the proof of Theorem 1.3.

I. Let $Q(u,u) = (\bar\partial u, \bar\partial u) + (\bar\partial^* u, \bar\partial^* u)$ be the quadratic form associated with the $\bar\partial$-Neumann Laplacian $\Box_q$ on $(0,q)$ forms. By the min-max principle, $\lambda_j \gtrsim j^\varepsilon$ implies that there exist at most $\lesssim j$ many orthogonal $u_k \in \mathrm{Dom}(Q)$ such that

$$Q(u_k, u_k) \lesssim j^\varepsilon \|u_k\|^2.$$

Therefore, it suffices to prove that if $b\Omega$ is of infinite $D_{n-1}$-type, then there exist $\gg$ j many such $u_k$.

II. It is not possible to construct many orthogonal $u_k$'s without further symmetry assumptions on $\Omega$. To overcome this difficulty, we use the following variation of the min-max principle.

**Lemma 2.1** (Min-Max). *Suppose that* $u_k \in \mathrm{Dom}(Q)$, $1 \le k \le j$, *satisfies the following Riesz type condition*

$$\|\sum_{k=1}^{j} c_k u_k\| \gtrsim (\sum_{k=1}^{j} |c_k|^2)^{1/2}. \qquad \textit{(Riesz)}$$

*Then*

$$\lambda_j \gtrsim j^\varepsilon \Rightarrow \max_{1\le k\le j} Q(u_k, u_k) \gtrsim j^\varepsilon$$

III. To construct the $u_k$ in the above lemma, we use the following wavelet lemma due to Lemarié and Meyer.

**Lemma 2.2** (Wavelet). *Let* $b(t)$ *be a smooth cut-off function supported in [-1/2, 1],* $\equiv 1$ *on [0, 1/2], and* $b^2(t)+b^2(t-1) \equiv 1$ *on [1/2, 1]. Then* $\{b(t)e^{2\pi k t\sqrt{-1}} \mid k \in \mathbb{Z}\}$ *are mutually orthogonal.*

IV. We will also need the following well-known normalized lemma:

**Lemma 2.3** (Normalization). *If the* $D_{n-1}$*-type of* $b\Omega$ *is* $\ge 2m$ *at* $z_0'$, *then after local change of coordinates,* $b\Omega$ *is defined near* $z_0' = 0$ *by*

$$r(z) = \mathrm{Re}\ z_n + f(z') + (\mathrm{Im}\ z_n) g(z') + O(|\mathrm{Im}\ z_n|^2),$$

*where* $|f(z')| \lesssim |z'|^{2m}$, $|g(z')| \lesssim |z'|^m$.

V. It follows from the Kohn-Morrey formula and the usual min-max principle that $\lambda_{q,j} \leq \lambda_{q+1,nj}$. Thus, it suffices to work on $(0, n-1)$-forms.

VI. To construct the $(0, n-1)$-forms that satisfy the condition in Lemma 2.1, we first extend the Lemarié-Meyer wavelet $b(t)$ from $\mathbb{R}$ to $\mathbb{C}$: Let

$$B(w) = (b(t) - ib'(t)s - b''(t)s^2/2)\chi(s/(1+|t|^2))$$

where $w = s + it$ and $\chi$ is a cut-off function $\equiv 1$ on [-1, 1]. Then $B(0,t) = b(t)$ and $|B_{\bar{w}}| \lesssim |s|^2$.

Now let

$$f_{j,k}(z) = k16^{(m+n-1)j}a(16^j z')B(16^{mj}z_n)e^{-2\pi k^2 16^{mj} w}$$

for any $j \in \mathbb{N}$ and $2^{mj-1} \leq k \leq 2^{mj}$, where $a(z')$ is any smooth cut-off function in $z'$. Let $\omega_1, \ldots, \omega_n$ be an orthonormal basis for $(1,0)$-forms near the origin with $\omega_n = \partial r/|\partial r|$. Let

$$u_{j,k} = f_{j,k}\bar{\omega}_1 \wedge \ldots \bar{\omega}_{n-1}.$$

Then it is not difficult to show that $u_{j,k}$ satisfies the uniform Riesz condition in Lemma 2.1:

$$\|\sum_k c_k u_{j,k}\|^2 \gtrsim \sum_k |c_k|^2$$

Furthermore, it follows from direct computations that

$$Q(u_{j,k}, u_{j,k}) \lesssim 16^{2j}.$$

Thus, by Lemma 2.1, $\exists k_0 \in [2^{mj-1}, 2^{mj}]$ such that

$$Q(u_{j,k_0}, u_{j,k_0}) \gtrsim (2^{mj})^{\varepsilon}.$$

Therefore, $2m \leq 16/\varepsilon$. We thus conclude the proof of Theorem 1.3.

## 3. Proof of Sufficiency

We sketch the proof of Theorem 1.2 in this section.

I. Let $\lambda_j$ be the eigenvalues of the $\bar{\partial}$-Neumann Laplacian $\Box$ on $(0,1)$-forms. Let $\varphi_j$ be the normalized eigenforms associated with $\lambda_j$. Throughout this section, we assume that $\Box$ has purely discrete spectrum. In this case, the spectral resolution $E(\lambda)$ of $\Box$ is given by

$$E(\lambda)f = \sum_{j:\lambda_j \leq \lambda} \langle f, \varphi_j \rangle \varphi_j.$$

Let $e(\lambda; z, z')$ be the kernel of $E(\lambda)$ (in the sense of Schwartz). Then

$$\mathcal{N}(\lambda) = \int \text{tr } e(\lambda; z, z) dV(z).$$

II. We now recall the well-known setup for finite type domains in $\mathbb{C}^2$. Let $\Omega = \{z \in \mathbb{C}^2 \mid r(z) < 0\}$ and let $z' \in b\Omega$. Let $L = r_{z_2}\partial_{z_1} - r_{z_1}\partial_{z_2}$. For $j, k \geq 1$, let

$$\mathcal{L}_{jk}\partial\bar{\partial}r(z') = \underbrace{L \ldots L}_{j-1 \text{ times}} \; \underbrace{\bar{L} \ldots \bar{L}}_{k-1 \text{ times}} \; \partial\bar{\partial}r(L, \bar{L})(z').$$

For any $2 \leq l \leq 2m$, let

$$A_l(z') = \Big( \sum_{\substack{j+k \leq l \\ j,k > 0}} |\mathcal{L}_{jk}\partial\bar{\partial}r(z')|^2 \Big)^{1/2}.$$

For any $\tau > 0$, let

$$\delta(z',\tau) = \sum_{l=2}^{2m} A_l(z')\tau^l.$$

It is evident that $b\Omega$ is of finite type $2m \iff \delta(z',\tau) \gtrsim \tau^{2m}$ uniformly for all $z' \in b\Omega$ and $\delta(z'_0,\tau) \lesssim \tau^{2m}$ for some $z'_0 \in b\Omega$.

III. We now recall the special coordinates introduced by Fornæss and Sibony: $\exists\zeta = \Phi_{z'}(z)$ such that $\Phi_{z'}(U_{z'} \cap \Omega) = \{\text{Re } \zeta_2 + h(\zeta_1, \text{Im } \zeta_2) < 0\}$, where $h(\zeta_1, \text{Im } \zeta_2)$ has the form of

$$\sum_{k=2}^{2m} P_k(\zeta_1) + (\text{Im } \zeta_2)\sum_{k=2}^{m} Q_k(\zeta_1) + O\big(|\zeta_1|^{2m+1} + |\text{ Im } \zeta_2||\zeta_1|^{m+1} + |\text{ Im } \zeta_2|^2|\zeta_1|\big).$$

It is easy to see that

$$\delta(z',\tau) \approx \sum_k \|P_k\|_\infty \tau^k,$$

where $\|P\|_\infty = \max_{|\zeta_1|=1}|P|$.

We will need the following key fact from [FoS89]: If $\Omega$ is pseudoconvex of finite type $2m$, then

$$\sum_{l=2}^{m} \|Q_l\|_\infty |\zeta_1|^l \lesssim |\zeta_1|\Big(\sum_{l=2}^{2m} \|P_l\|_\infty |\zeta_1|^l\Big)^{1/2}.$$

IV. We use the following construction of anisotropic "bidiscs":

$$R_\tau(z') = \Phi_{z'}^{-1}(|\zeta_1| < \tau, |\zeta_2| < (\delta(z',\tau))^{1/2}).$$

Notice that the bidiscs here have radii $\delta^{1/2}$ in the complex normal direction whereas those in the study of the Bergman kernel have $\delta$. For this construction of anisotropic bidiscs to be useful, we establish the following doubling and engulfing properties.

**Lemma 3.1** (Doubling/Engulfing). *If $z'' \in R_\tau(z') \cap b\Omega$, then*

$$\delta(z',\tau) \approx \delta(z'',\tau); \text{ and } R_\tau(z') \subset R_{C\tau}(z''),\ R_\tau(z'') \subset R_{C\tau}(z').$$

V. We divide $\Omega$ into two regions: the blue and red regions. On the blue region

$$\{z \in \Omega :\ d(z) \gtrsim (\delta(\pi(z), 1/\sqrt{\lambda}))^{1/2}\},$$

it follows from the interior ellipticity of $\Box$ that

$$\text{tr } e(\lambda; z, z) \lesssim \lambda(\delta(\pi(z), 1/\sqrt{\lambda}))^{1/2} \lesssim \lambda^{m+1}.$$

VI. On the red region

$$\{z \in \Omega :\ d(z) \lesssim (\delta(\pi(z), 1/\sqrt{\lambda}))^{1/2}\},$$

we shall establish

$$\int_{\text{Red Region}} \text{tr } e(\lambda; z, z) \lesssim \lambda^{m+1}.$$

By Lemma 3.1, it suffices to prove:

$$\int_{R_\tau(z')\cap\Omega} \text{tr } e(\lambda; z, z) \lesssim (\delta(z',\tau))^{-1/2}, \tag{3.1}$$

where $\tau = 1/\sqrt{\lambda}$.

VII. To prove the above estimate, we will use a rescaling method. We first flatten the boundary: Let $(\eta_1, \eta_2) = \widehat{\Phi}_{z'}(\zeta_1, \zeta_2)$:

$$(\eta_1, \eta_2) = (\zeta_1, \zeta_2 + h(\zeta_1, \text{Im } \zeta_2) - F(\zeta_1, \zeta_2))$$

where

$$\begin{aligned} F(\zeta_1, \zeta_2) &= h_2(\zeta_1)(\text{Re } \zeta_2 + h(\zeta_1, \text{Im } \zeta_2))^2/2 \\ &\quad + i(h_1(\zeta_1)(\text{Re } \zeta_2) + h_2(\zeta_1)(\text{Re } \zeta_2)(\text{Im } \zeta_2)). \end{aligned}$$

We use this choice of $F$ to make $\partial\eta_2/\partial\bar{\zeta}_2$ vanish to a desirable higher order.

VIII. The rescaling is the usual one defined by

$$(w_1, w_2) = D_{z',\tau}(\eta_1, \eta_2) = (\eta_1/\tau, \eta_2/\delta),$$

where $\delta = \delta(z', \tau)$. Let $\Omega_{z'} = \Omega \cap U_{z'}$. Let $\Psi_{z',\tau} = D_\tau \circ \widehat{\Phi} \circ \Phi$ and $\Omega_{z',\tau} = \Psi_{z',\tau}(\Omega_{z'}) = \{\text{Re } w_2 < 0 \mid \Psi^{-1}_{z',\tau}(w) \in U_{z'}\}$. Let

$$\mathcal{G}_\tau \colon L^2(\Omega_\tau) \to L^2(\Omega); \mathcal{G}(u) = |\det d\Psi_\tau|^{\frac{1}{2}} u \circ \Psi.$$

The rescaling of the $\overline{\partial}$-Neumann Laplacian is done by rescaling the quadratic form via the following formula.

$$Q_\tau(u, u) = \tau^2 Q(\mathcal{G}_\tau u, \mathcal{G}_\tau u), \quad \text{Supp}\, u \subset \Omega_{z',\tau}.$$

Let $\Box_\tau$ be the operator associated with $Q_\tau$. Roughly speaking, we have $\Box_\tau = \tau^2 \mathcal{G}_\tau^{-1} \Box \mathcal{G}_\tau$. It remains to estimate the spectral kernel of $\Box_\tau$.

IX. The estimation of the spectral kernel of $\Box_\tau$ is based on the folliwng Kohn type Gärding inequality:

**Lemma 3.2** (Kohn type Gärding inequality). *$\exists \varepsilon > 0$ such that*

$$Q_\tau(u, u) \gtrsim \| u \|_\varepsilon^2 + \tau^2 \delta^{-2} \| \frac{\partial u}{\partial \bar{w}_2} \|_{-1+\varepsilon}^2,$$

*for all $u$ supported in $\{|w_1| < 1, |w_2| < \delta^{-1/2}\}$.*

From the above lemma, we have

$$Q_\tau(u, u) \gtrsim \widetilde{Q}_\delta(u, u) \equiv \| u \|_\varepsilon^2 + \delta^{-1} \| \frac{\partial u}{\partial \bar{w}_2} \|_{-1+\varepsilon}^2$$

where $\delta = \delta(z', \tau)$. Thus, as $\delta \to 0$, $\partial u/\partial \bar{w}_2 = 0$. Using the Bergman projection and a Payley-Wiener type theorem, we obtain

$$\lambda_j(\chi_\delta \widetilde{N}_\delta) \lesssim (1 + j\delta^{1/2})^{-\varepsilon/4},$$

where $\chi_\delta(w_1, w_2) = \chi(w_1, \delta^{1/2} w_2)$ and $\widetilde{N}_\delta$ is the inverse of the operator associated with $\widetilde{Q}_\delta$. A commutator argument then yields that

$$\int_{P_\tau(z')} \text{tr } e_\tau(1; w, w) \lesssim \delta^{-1/2}$$

where $P_\tau(z') = \{|w_1| < 1, |w_2| < \delta^{-1/2}\}$. This in turn yields our goal (3.1).

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# INVARIANTS FOR HOLOMORPHIC/RATIONAL DYNAMICS: A SURVEY

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Given a dynamical system $f : X \to X$ with some (topological, measurable, differentiable, algebraic) structure, one can introduce an associated invariant. For certain holomorphic/rational dynamical systems some of these invariants can be defined, and one may ask how they are related to each other. In this talk I will survey several recent results concerning the relationship between these invariants for three classes of holomorphic/rational dynamical systems.

## 1. The Invariants

In this lecture, we will be interested in the following four invariants.

1.1. **Topological Entropy.** Let $f : X \to X$ be a continuous map of a compact topological space $X$. For an open covering $\mathcal{U}$ of $X$, we denote by $\mathrm{card}^*(\mathcal{U})$ the minimum number of elements in $\mathcal{U}$ to cover $X$. Given two coverings $\mathcal{U}$ and $\mathcal{V}$, let us write $\mathcal{U} \vee \mathcal{V} = \{U \cap V\}_{U \in \mathcal{U}, V \in \mathcal{V}}$ and $\mathcal{U}_n = \mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-(n-1)}(\mathcal{U})$, where $f^{-1}(\mathcal{U}) = \{f^{-1}(U)\}_{U \in \mathcal{U}}$. Then,

$$h_{\mathrm{top}}(f) \equiv \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{1}{n} \log \mathrm{card}^*(\mathcal{U}_n)$$

exists and is called the *topological entropy* of $f$. Topological entropy is shown to be invariant under topological conjugacy.

Intuitively, the topological entropy represents the growth of the number of orbits generated by $f$. To see this, we give an alternative definition of the topological entropy here. Let $(X, d)$ be a compact metric space, and let $f : X \to X$ be a continuous map. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we say that two distinct points $x \neq y \in X$ are $(n, \varepsilon)$-*separated* if there exists $0 \leq k \leq n-1$ so that $d(f^k(x), f^k(y)) \geq \varepsilon$. This means that the two strings $\{f^i(x)\}_{i=1}^{n-1}$ and $\{f^i(y)\}_{i=0}^{n-1}$ of length $n$ are "distinguishable" with the resolution of $\varepsilon > 0$. Let $N(n, \varepsilon)$ be the maximum number of mutually $(n, \varepsilon)$-separated points in $X$. Then, an alternative definition of the topological entropy is given by

$$h_{\mathrm{top}}(f) \equiv \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon).$$

Thus, the topological entropy can be seen as the growth rate of the number of $n$-strings up to the resolution of $\varepsilon \to 0$.

When $X$ is a complex projective manifold and $f : X \dashrightarrow X$ is a rational map given by its graph $\Gamma_f \subset X \times X$, we define $\Gamma_f^\infty \equiv \{x = (x_0, x_1, \cdots) \in X^{\mathbb{N}} : (x_i, x_{i+1}) \in \Gamma_f\}$ with the topology induced from the product topology in $X^{\mathbb{N}}$. Then, the shift map $\sigma_f : \Gamma_f^\infty \to \Gamma_f^\infty$, $\sigma(x_0, x_1, \cdots) \equiv (x_1, x_2, \cdots)$ becomes continuous on the compact space $\Gamma_f^\infty$, thus we can define the topological entropy of a rational map $f$ by $h_{\mathrm{top}}(f) \equiv h_{\mathrm{top}}(\sigma_f)$.

1.2. **Metric Entropy.** Let $(X, \mathcal{B}, \nu)$ be a probability space, $f : X \to X$ be a measurable map and $\nu$ be a $f$-invariant probability measure, i.e. $\nu(X) = 1$ and $f_*\nu = \nu$. For a finite partition $\mathcal{U} = \{U_1, \cdots, U_N\}$ of $X$ (this means that $U_i$'s are mutually disjoint measurable sets and their union becomes $X$), we put

$$H_\nu(\mathcal{U}) \equiv \sum_{i=1}^{N} -\nu(U_i) \log \nu(U_i)$$

with the convention $0 \log 0 = 0$. Keeping the previous definition of $\mathcal{U}_n$,

$$h_\nu(f) \equiv \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} H_\nu(\mathcal{U}_n),$$

where the supremum is taken over all finite partitions of $X$, exists and is called the *metric (measure theoretic) entropy* of $f$ with respect to $\nu$.

1.3. **Volume Growth Rate.** Let $X$ be a compact Riemannian manifold of dimension $k$, $\Gamma \subset X \times X$ be an $m$–dimensional submanifold of $X \times X$. Write $\Gamma^n \equiv \{x = (x_0, x_1, \cdots, x_{n-1}) \in X^n : (x_i, x_{i+1}) \in \Gamma\}$. Then, we put

$$\operatorname{lov}(\Gamma) \equiv \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Vol}_m(\Gamma^n),$$

where $\operatorname{Vol}_m$ is the $m$–dimensional Hausdorff measure in $X^n$ induced from the Riemannian metric in $X$. When $f$ is a rational endomorphism of a projective manifold $X$ given by its graph $\Gamma_f \subset X \times X$, we define $\operatorname{lov}(f) \equiv \operatorname{lov}(\Gamma_f)$ and call it the *volume growth rate* of $f$.

1.4. **Algebraic Entropy.** Let $X$ be a complex projective manifold of dimension $k$ with the standard Kähler form $\omega$ (normalized as $\int_X \omega^k = 1$) and let $f : X \dashrightarrow X$ be a dominating rational map. For each $1 \leq l \leq k$, the *dynamical degree of order* $l$ is given by

$$\lambda_l(f) \equiv \lim_{n\to\infty} \left( \int_X (f^n)^*(\omega^l) \wedge \omega^{k-l} \right)^{1/n}.$$

The dynamical degrees are shown to be invariant under birational conjugacy.

Now, let $f : \mathbb{CP}^k \to \mathbb{CP}^k$ be a rational map of a complex projective space. We let $\deg(f)$ be the maximum degree of the polynomials which express $f$ in the homogeneous coordinates. The *algebraic entropy* of $f$ is then defined as

$$h_{\mathrm{alg}}(f) \equiv \lim_{n\to\infty} \frac{1}{n} \log \deg(f^n).$$

One can show that $h_{\mathrm{alg}}(f) = \log \lambda_1(f)$.

1.5. **Main Question.** So far we have defined four kinds of invariants: topological entropy $h_{\mathrm{top}}(f)$ (a topological invariant), metric entropy $f_\nu(f)$ (a measure theoretic invariant), volume growth rate $\operatorname{lov}(f)$ (a geometric invariant) and algebraic entropy $h_{\mathrm{alg}}(f)$ (an algebraic invariant). Once some of these four invariants are defined for certain class of dynamical systems, it is natural to ask how they are related. In the next section, we discuss the relationship in three classes of holomorphic/rational dynamical systems.

## 2. Their Relations

2.1. **Topological versus Metric Entropies.** We start with a general fact, which is a main background of the results presented in this section. Consider a dynamical system $f : X \to X$ under the topological and the measure theoretic settings. Let $M(f)$ be the set of all $f$-invariant Borel probability measures. Then, a well–known classical fact is the so–called

**Variational Principle:**

$$h_{\mathrm{top}}(f) = \sup_{\nu \in M(f)} h_\nu(f).$$

This fact then suggests the following two questions.

*Question 1.* Is the "sup" above attained by some measure? (If so, such a measure is called a *maximal entropy measure.*)

*Question 2.* If such a measure exists, is it unique? (If so, such a measure is called the *unique maximal entropy measure.*)

2.2. **Polynomial Diffeomorphisms of $\mathbb{C}^2$.** One of the first recent results concerning these questions in higher dimensional complex dynamical systems is

**Theorem (Bedford–Smillie, Bedford–Lyubich–Smillie).** *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial diffeomorphism of $\mathbb{C}^2$ with algebraic degree $d > 1$.*

(i) *There exists a unique maximal entropy measure $\mu$ for $f$, i.e. $\mu$ is the unique $f$-invariant probability measure with $h_\mu(f) = h_{\mathrm{top}}(f)$* [BS3, BLS1].

(ii) *$\mu$ describes the limit distribution of saddle periodic points of $f$, i.e.*

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{z \in \mathrm{SP}_n(f)} \delta_z = \mu$$

*in the weak topology, where $\mathrm{SP}_n(f)$ denotes the set of saddle periodic points of period $n$ for $f$* [BLS2].

(iii) *There exists a positive closed $(1,1)$-current $\mu^+$ which describes the limit distribution of preimages of a generic complex one–dimensional disk $M$ in $\mathbb{C}^2$, i.e. there is a constant $c = c_M > 0$ so that*

$$\lim_{n\to\infty} \frac{1}{d^n} f^{*n}[M] = c\mu^+$$

*in the weak topology for a "reasonably chosen" $M$, and $[M]$ denotes its current of integration. A similar result holds for $\mu^-$* [BS1, BS2].

In fact, we define $\mu \equiv \mu^+ \wedge \mu^-$ and the $(1,1)$-currents $\mu^\pm$ have been explicitly constructed from the Green functions:

$$G^\pm(x,y) \equiv \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^{\pm n}(x,y)\|$$

as $\mu^\pm \equiv \frac{1}{2\pi} dd^c G^\pm$ (Bedford and Sibony; see Section 1 of [BS1]). Key ingredients of the proof are pluripotential theory and Pesin theory for non–uniformly hyperbolic dynamical systems. Similar results for holomorphic automorphisms of compact complex surfaces have been recently obtained by Cantat [C].

2.3. **Holomorphic Endomorphisms of $\mathbb{CP}^k$.** For holomorphic endomorphisms of $\mathbb{CP}^k$, the following result has been established.

**Theorem (Briend–Duval).** *Let $f : \mathbb{CP}^k \to \mathbb{CP}^k$ be a holomorphic endomorphism of $\mathbb{CP}^k$ with algebraic degree $d > 1$.*

(i) *There exists a unique maximal entropy measure $\mu$ for $f$, i.e. $\mu$ is the unique $f$-invariant probability measure with $h_\mu(f) = h_{\mathrm{top}}(f)$* [BrDu2].

(ii) *$\mu$ describes the limit distribution of repelling periodic points of $f$, i.e.*

$$\lim_{n\to\infty} \frac{1}{(d^k)^n} \sum_{z\in \mathrm{RP}_n(f)} \delta_z = \mu$$

*in the weak topology, where $\mathrm{RP}_n(f)$ is the set of repelling periodic points of period $n$ for $f$* [BrDu1].

(iii) *$\mu$ describes the limit distribution of preimages of a generic point $w$ in $\mathbb{CP}^k$, i.e. there exists a proper algebraic subset $E \subset \mathbb{CP}^k$ so that*

$$\lim_{n\to\infty} \frac{1}{(d^k)^n} \sum_{z\in f^{-n}(w)} \delta_z = \mu$$

*in the weak topology for all $w \in \mathbb{CP}^k \setminus E$* [BrDu2].

The measure $\mu$ in the setting above has been again explicitly constructed from the Green function:

$$G(z) \equiv \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(z)\|$$

defined through the lift $F : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{C}^{k+1} \setminus \{0\}$ of $f$ by the canonical projection $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{CP}^k$ as $\mu \equiv dd^c(G\circ s) \wedge dd^c(G\circ s) \wedge \cdots \wedge dd^c(G\circ s)$, where the wedge products are taken $k$ times and $s : \mathbb{CP}^k \supset U \to \mathbb{C}^{k+1} \setminus \{0\}$ is a local holomorphic section (Hubbard–Papadopol [HP], Fornæss–Sibony [FS1, FS2]). Key ingredients of the proof are Bezout theorem and an argument *à la* Lyubich [L] with area-diameter inequality.

2.4. **Rational Endomorphisms of Projective Manifolds.** Assume that $X$ is a complex projective manifold and consider a dominating rational map $f : X \dashrightarrow X$. In this setting, all of the invariants introduced in the previous section except for the metric entropy can be considered.

**Theorem (Gromov, Dinh–Sibony).** *Let $X$ be a complex projective manifold of dimension $k$. If $f : X \dashrightarrow X$ is a dominating rational map, then*

(i) $h_{\mathrm{top}}(f) \leq \mathrm{lov}(f)$ [G],

(ii) $\mathrm{lov}(f) = \max_{1\leq l\leq k} \log \lambda_l(f)$ [DS].

In fact, Gromov [G] has proved $h_{\mathrm{top}}(f) \leq \mathrm{lov}(f)$ for holomorphic endomorphisms of $\mathbb{CP}^k$, but his proof applies to the setting of the theorem above as well without modification. Key ingredients of the proofs of the theorem above are an inequality of Lelong (for Gromov part) and careful analysis of positive closed currents (for Dinh–Sibony part).

As an immediate consequence of this theorem, we have

**Corollary.** *For a birational map $f$ of $\mathbb{CP}^2$, we have $h_{\mathrm{top}}(f) \leq h_{\mathrm{alg}}(f)$.*

Suggested by this result, we propose the

**Conjecture.** *For a birational map $f$ of $\mathbb{CP}^2$, we have $h_{\rm top}(f) = h_{\rm alg}(f)$.*

A possible approach to this conjecture might be to employ the missing cast "metric entropy" in this setting, that is, to construct a reasonable measure like $\mu$ in the previous theorems and show that $h_\mu(f) = \log \lambda_1(f)$. This would then imply $h_{\rm alg}(f) \geq h_{\rm top}(f) \geq h_\mu(f) \geq h_{\rm alg}(f)$. Also consult recent partial results towards this direction by Bedford–Diller [BeDi], Dujardin [D], etc for some birational maps of surfaces.

2.5. **Non–Integrability of Discrete Systems.** Several discrete systems such as discrete Painlevé equations can be regarded as a non–autonomous iterations of rational maps of $\mathbb{CP}^2$. Unlike the Liouville–Arnold formulation for continuous systems, the concept of "integrability" is not yet well–established for such discrete systems. There is a criterion for integrability of discrete systems called the *singularity confinement test.* However, (i) there exists a system which passes this test but which presents a chaotic behavior (Hietarinta–Viallet [HV]), and (ii) there exists a system which is solvable by elementary functions but it does not pass the test (Nakamura [N]). Note that the notion of algebraic entropy has been introduced in this context and it is claimed that the positivity of algebraic entropy should be related to non–integrability of a discrete system [BV].

Here, we have the following observation based on the conjecture in the previous subsection. First recall that for a smooth dynamical system in dimension two, we have

**Theorem (Katok** [K]**).** *Let $f : X \to X$ be a $C^{1+\alpha}$–diffeomorphism of a two–dimensional compact Riemannian manifold $X$ for some $\alpha > 0$. Then, $h_{\rm top}(f) > 0$ if and only if $f^N$ has a horseshoe for some $N > 0$ (thus, the dynamics of $f$ is "chaotic").*

We can not immediately apply this theorem to rational dynamics under consideration, since the smoothness of $f$ is essential in the proof. However, if the conjecture in the previous subsection holds, then the above theorem suggests that the positivity of the algebraic entropy is equivalent to the existence of chaos for rational dynamical systems in $\mathbb{CP}^2$.

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## ANALOGUES OF THE HOLOMORPHIC MORSE INEQUALITIES IN CR GEOMETRY

RAPHAËL PONGE

This talk is a preliminary report about a joint project with George Marinescu on extending to the CR setting Demailly's holomorphic Morse inequalities together with some applications to complex geometry, including a generalization of the Grauert-Riemenschneider criterion to the noncompact setting.

The talk is divided into 3 sections. In Section 1 we briefly review the holomorphic Morse inequalities. In Section 2 we recall the main definitions and properties concerning CR manifolds, CR vector bundles, CR connections and the $\overline{\partial}_b$-complex. In Section 3 we present our main results.

### 1. Holomorphic Morse Inequalities

By Kodaira's embedding theorem a compact complex manifold is projective algebraic iff it carries a positive holomorphic line bundle. The Grauert-Riemenschneider conjecture was an attempt to generalize Kodaira's embedding theorem to compact Moishezon manifolds. Recall that the latter are compact complex manifolds which are projective algebraic up to a proper modification or, equivalently, have maximal Kodaira dimension.

**Conjecture** (Grauert-Riemenschneider). *A compact complex manifold is Moishezon if it carries a holomorphic line bundle which is positive on a dense open set.*

This was conjecture was first proved by Siu ([Si1], [Si2]) using elliptic estimates together with the Hirzbruch-Riemann-Roch formula. Subsequently, Demailly [De] gave an alternative proof based on a holomorphic version of the classical Morse inequalities as follows.

Let $M^n$ be a complex manifold and let $L$ be a Hermitian holomorphic line bundle over $M$ with curvature $F^L$. It is convenient to identify $F^L$ with the section of $\operatorname{End} T_{0,1}$ such that $\frac{\partial}{\partial \bar{z}^j} \rightarrow F(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k})\frac{\partial}{\partial \bar{z}^k}$.

For $q = 0, \ldots, n$ we let $O_q$ denote the open set consisting of points $x \in M$ such that $F^L(x)$ has $q$ negative eigenvalues and $n-q$ positive eigenvalues and we set $O_{\leq q} = O_0 \cup \ldots \cup O_q$.

**Theorem 1.1** (Demailly). *As $k \rightarrow \infty$ the following asymptotics hold.*

*(i) Weak Holomorphic Morse Inequalities:*

$$\dim H^{0,q}(M, L^k) \leq (-1)^q (\frac{k}{2\pi})^n \int_{O_q} \det F^L + o(k^n). \tag{1.1}$$

*(ii) Strong Holomorphic Morse Inequalities:*

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^{0,j}(M, L^k) \leq (-1)^q (\frac{k}{2\pi})^n \int_{O_{\leq q}} \det F^L + o(k^n). \tag{1.2}$$

*(iii) Asymptotic Hirzbruch-Riemann-Roch formula:*

$$(1.3)\qquad \chi(M,L^k)=\sum_{j=0}^{n}(-1)^j\dim H^{0,j}(M,L^k)=(\frac{k}{2\pi})^n\int_M \det F^L+o(k^n),$$

*where* $\chi(M,L^k)$ *is the holomorphic Euler characteristic with coefficients in* $L^k$.

In particular, for $q=1$ we get

$$(1.4)\qquad -\dim H^{0,0}(M,L^k)+H^{0,1}(M,L^k)\le\frac{-1}{n!}(\frac{k}{2\pi})^n\int_{O_{\le 1}}\det F^L+o(k^n).$$

Thus,

$$(1.5)\qquad \dim H^{0,0}(M,L^k)\ge\frac{1}{n!}(\frac{k}{2\pi})^n\int_{O_{\le 1}}\det F^L+o(k^n).$$

If $\int_{O_{\le 1}}\det F^L>0$ (e.g. if $L$ is semi-positive and is $>0$ at a point) then we get:

$$(1.6)\qquad \dim H^{0,0}(M,L^k)\gtrsim k^n,$$

which implies that $M$ has maximal Kodaira dimension, i.e., $M$ is Moishezon.

In [Bi] Bismut gave a heat kernel proof of Demailly's inequalities. Bismut's approach can be divided into 2 main steps.

*Step 1:* For $q=0,\ldots,n$ let $\Delta^{0,q}_{L^k}$ denote the Dolbeault Laplacian acting on sections of $\Lambda^{0,q}T^*M\otimes L^k$. We let $\mathcal{F}^L$ be the Clifford lift of $F^L$, i.e, the section of $\mathrm{End}(\Lambda^{0,*}T^*M)$ so that locally we have $\mathcal{F}^L=F(\frac{\partial}{\partial z^j},\frac{\partial}{\partial \bar{z}^k})\varepsilon(d\bar{z}^j)\iota(dz^k)$. Then Bismut proved:

**Theorem 1.2** (Bismut). *For any* $t>0$ *we have*

$$(1.7)\qquad \operatorname{Tr} e^{-\frac{t}{k}\Delta^{0,q}_{L^k}}=(\frac{k}{2\pi})^n\int_M\det[\frac{F^L}{1-e^{-tF^L}}]\operatorname{Tr}_{|\Lambda^{0,q}}e^{-t\mathcal{F}^L}+o(k^n).$$

*Step 2:* By taking the limit as $t\to\infty$ in the integral in (1.7) Bismut recovered the inequalities (1.1)–(1.3), via linear-algebraic arguments similar to that of his earlier proof of the Morse inequalities.

## 2. CR MANIFOLDS AND THE $\overline{\partial}_b$-COMPLEX

2.1. **CR Manifolds.** A CR structure on an orientable manifold $M^{2n+1}$ is given by a rank $n$ vector bundle $T_{1,0}\subset T_{\mathbb{C}}M$ such that:

(i) $T_{1,0}$ is integrable in Froebenius' sense;

(ii) $T_{1,0}\cap T_{0,1}=\{0\}$, where $T_{0,1}=\overline{T_{1,0}}$.

The main examples of CR manifolds include:

- Boundaries of complex domains;
- Circle bundles over complex manifolds;
- Boundaries of complex hyperbolic spaces.

Given be a global non-vanishing real 1-form $\theta$ annihilating $T_{1,0}\oplus T_{0,1}$ the associated Levi form is given by

$$(2.1)\qquad L_\theta(Z,W)=-id\theta(Z,\overline{W}),\quad Z,W\in C^\infty(M,T_{1,0}).$$

We say that $M$ is *strictly pseudoconvex* when we can choose $\theta$ so that at every point $L_\theta$ is positive definite. Similarly, we say $M$ is $\kappa$*-strictly pseudoconvex* when we can choose $\theta$ so that at every point $L_\theta$ has exactly $\kappa$-negative eigenvalues and $n-\kappa$ positive eigenvalues.

2.2. **The $\overline{\partial}_b$-complex.** Let $\mathcal{N}$ be a supplement of $T_{1,0}\oplus T_{0,1}$ in $T_{\mathbb{C}}M$ and define:

$$\Lambda^{1,0} = \text{annihilator in } T^*_{\mathbb{C}}M \text{ of } T_{0,1}\oplus\mathcal{N},$$
$$\Lambda^{0,1} = \text{annihilator in } T^*_{\mathbb{C}}M \text{ of } T_{1,0}\oplus\mathcal{N},$$
$$\Lambda^{p,q} = (\Lambda^{1,0})^p\wedge(\Lambda^{0,1})^q, \quad p,q=0,\ldots,n.$$

This gives rise to the splitting,

$$\Lambda^*T^*_{\mathbb{C}}M = (\bigoplus_{p,q=0}^{n}\Lambda^{p,q})\oplus(\theta\wedge\Lambda^*T^*_{\mathbb{C}}M). \tag{2.2}$$

If $\alpha\in C^\infty(M,\Lambda^{0,q})$, then we can write

$$d\alpha = \partial_b\alpha+\overline{\partial}_b\alpha+\theta\wedge\beta, \tag{2.3}$$

with $\partial_b\alpha\in C^\infty(M,\Lambda^{1,q})$ and $\overline{\partial}_b\alpha\in C^\infty(M,\Lambda^{0,q+1})$.

We have $\overline{\partial}_b^2=0$, so $\overline{\partial}_b : C^\infty(M,\Lambda^{0,*})\rightarrow C^\infty(M,\Lambda^{0,*+1})$ is a chain complex whose cohomology groups are denoted $H_b^{0,q}(M)$, $q=0,\ldots,n$.

Endowing $T_{\mathbb{C}}M$ with a Hermitian metric, the Kohn Laplacian is

$$\Box_b = \overline{\partial}_b^*\overline{\partial}_b+\overline{\partial}_b\overline{\partial}_b^*. \tag{2.4}$$

**Proposition 2.1.** *We have $H_b^{0,q}(M)\simeq\ker\Box_{b,q}$.*

For $x\in M$ let $\kappa_+(x)$and $\kappa_-(x)$ be the number of positive and negative eigenvalues of the Levi form $L_\theta$ at $x$.

**Definition 2.2** (Condition $Y(q)$). *The condition $Y(q)$ is satisfied when for all $x\in M$ we have:*

$$q\notin\{\kappa_-(x),\ldots,n-\kappa_+(x)\}\cup\{\kappa_+(x),\ldots,n-\kappa_-(x)\}. \tag{2.5}$$

*Examples.* 1) If $M$ is strictly pseudoconvex then the condition $Y(q)$ means $q\neq 0,n$.

2) If $M$ is $\kappa$-strictly pseudoconvex then the condition $Y(q)$ means $q\neq\kappa,n-\kappa$.

3) The condition $Y(0)$ means that $L_\theta$ has at least one positive and one negative eigenvalue.

**Proposition 2.3** (Kohn). *Under condition $Y(q)$ the operator $\Box_{b,q}$ is hypoelliptic with gain of 1 derivative, i.e., for any compact $K\subset M$ we have estimates,*

$$\|u\|_{s+1}\leq C_{Ks}\|\Box_{b,q}u\|_s \quad \forall u\in C_K^\infty(M,\Lambda^{0,q}). \tag{2.6}$$

**Corollary 2.4.** *If the condition $Y(q)$ holds then $\dim H_b^{0,q}(M)<\infty$.*

2.3. **CR vector bundles and CR connections.** In the sequel we say that a map $\phi=(\phi_{kl}) : M\rightarrow M_p(\mathbb{C})$ is CR when $\overline{\partial}_b\phi_{kl}=0$.

**Definition 2.5.** *A CR vector bundle $\mathcal{E}$ over $M$ is a vector bundle given by a covering of $M$ by trivializations $\tau_i : \mathcal{E}_{|U_j}\rightarrow U_j\times\mathbb{C}^p$ whose transition maps $\tau_{ij}=\tau_i\circ\tau_j^{-1} : U_i\cap U_j\rightarrow \mathrm{GL}_p(\mathbb{C})$ are CR maps.*

Given a vector bundle $\mathcal{E}$ over $M$ for $p,q=0,\ldots,n$ we let $\Lambda^{p,q}(\mathcal{E})=\Lambda^{p,q}\otimes\mathcal{E}$.

**Proposition 2.6.** *If $\mathcal{E}$ is a CR vector bundle then there exists a unique operator,*

$$\overline{\partial}_{b,\mathcal{E}} : C^\infty(M, \Lambda^{0,*}(\mathcal{E})) \to C^\infty(M, \Lambda^{0,*+1}(\mathcal{E})), \tag{2.7}$$

*such that $\overline{\partial}^2_{b,\mathcal{E}} = 0$ and for any local CR frame $e_1, \ldots, e_p$ of $\mathcal{E}$ and any section $s = \sum s_i e_i$ we have*

$$\overline{\partial}_{b,\mathcal{E}} s = \sum (\overline{\partial}_b s_i) \otimes e_i. \tag{2.8}$$

The cohomology groups of the complex $\overline{\partial}_{b,\mathcal{E}} : C^\infty(M, \Lambda^{0,*}(\mathcal{E})) \to C^\infty(M, \Lambda^{0,*+1}(\mathcal{E}))$ are denoted $H^{0,q}_b(M,\mathcal{E})$, $q = 0, \ldots, n$. As before if the condition $Y(q)$ holds then $\dim H^{0,q}_b(M,\mathcal{E}) < \infty$.

Next, let $\mathcal{E}$ be a CR vector bundle endowed with a Hermitian metric and let $\nabla : C^\infty(M,\mathcal{E}) \to C^\infty(M, T^*M \otimes \mathcal{E})$ be a connection. Recall that $\nabla$ is said to be unitary when we have

$$d\langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle \tag{2.9}$$

for sections $\xi$ and $\eta$ of $\mathcal{E}$.

On the other hand, thanks to the splitting we can write:

$$\nabla = \nabla^{1,0} + \nabla^{0,1} + \theta \wedge D, \tag{2.10}$$

where $= \nabla^{1,0}$ and $\nabla^{0,1}$ map to sections of $\Lambda^{1,0}(\mathcal{E})$ and $\Lambda^{0,1}(\mathcal{E})$ respectively.

**Definition 2.7.** *$\nabla$ is a CR connection when $\nabla^{0,1} = \overline{\partial}_{b,\mathcal{E}}$.*

Now, let $\operatorname{End}_{\mathrm{sa}} \mathcal{E}$ the bundle of selfadjoint endomorphisms of $\mathcal{E}$. Then we have:

**Proposition 2.8.** *The space of unitary CR connections is a non-empty affine space modelled on $i\theta \otimes C^\infty(M, \operatorname{End}_{sa} \mathcal{E})$.*

## 3. CR Morse Inequalities

Let $M^{2n+1}$ be a compact CR manifold together with a Hermitian metric on $T_{\mathbb{C}}M$ (not necessarily a Levi metric) and with a global real non-vanishing 1-form $\theta$ annihilating $T_{1,0} \oplus T_{0,1}$ and let $L$ is a Hermitian CR line bundle over $M$ with unitary CR connection of curvature $F^L$.

Our goal is to obtain analogues of the asymptotics (1.1)–(1.7) in this setting. There are several earlier related results in this direction.

First, in [Ge] Getzler proved an analogue of heat kernel asymptotics (1.7) for strictly pseudoconvex CR manifolds with Levi metric and conjectured that such an asymptotics should hold for more general CR manifolds. Nevertheless, he didn't derive asymptotic inequalities for $\dim H^{0,q}_b(M, L^k)$. There seems to be a mistake in Getzler's final formula (compare Theorem 3.1 below).

Later on, as a consequence of his version of the holomorphic Morse inequalities for pseudoconcave complex manifolds, Marinescu [Ma] obtained a lower bound for $H^{0,0}_b(M, L^{\otimes^k})$ when $M$ is the boundary of a strictly $q$-concave domain on a $q$-concave complex manifold $X^{2n}$ with $n \geq 3$ and $q \leq n-2$.

In addition, Berman [Be] proved a version of Demailly's inequalities for complex manifold with nondegenerate boundary and Fu has announced during his talk at the symposium analogues of the weak holomorphic Morse inequalities (1.1) on bounded finite type pseudoconvex domains in $\mathbb{C}^2$.

3.1. **Heat kernel version.** Let $\square^{0,q}_{b,L^k}$ be the Kohn Laplacian acting on sections of $\Lambda^{0,q}(L^k)$. As before it will be convenient to identify $F^L$ and $L_\theta$ with the sections of $\operatorname{End}_{\mathbb{C}} T_{1,0}$ such that, for any orthonormal frame $Z_1, \ldots, Z_n$ of $T_{1,0}$, we have:

$$F^L Z_j = F^L(Z_j, Z_{\bar{k}})Z_k, \quad L_\theta Z_j = L_\theta(Z_j, Z_k)Z_k. \tag{3.1}$$

Furthermore, for $\mu \in \mathbb{R}$ we set

$$F^L_\theta(\mu) = F^L - \mu L_\theta, \tag{3.2}$$

and we let $\mathcal{F}^L_\theta(\mu)$ denote the Clifford lift of $F^L_\theta(\mu)$ to $\Lambda^{0,*}$, i.e., the section of $\operatorname{End}_{\mathbb{C}} \Lambda^{0,*}$ such that, for any orthonormal frame $Z_1, \ldots, Z_n$ of $T_{1,0}$ with dual coframe $\theta^1, \ldots, \theta^n$, we have

$$\mathcal{F}^L_\theta(\mu) = [F^L(Z_j, \overline{Z_k}) - \mu L_\theta(Z_j, Z_k)]\varepsilon(\theta^{\bar{j}})\iota(\theta^k). \tag{3.3}$$

**Theorem 3.1** (GM+RP). *Assume that the condition $Y(q)$ holds. Then for any $t > 0$ we have*

$$\operatorname{Tr} e^{-t\square^{0,q}_{b,L^k}} = (\frac{k}{4\pi})^{n+1} \int_M G^{0,q}(x,t)d\nu(x) + O(k^n), \tag{3.4}$$

$$G^{0,q}(x,t) = \int_{-\infty}^{\infty} \det[\frac{F^L_\theta(\mu)}{1 - e^{-tF^L_\theta(\mu)}}] \operatorname{Tr} e^{-t\mathcal{F}^L_\theta(\mu)} d\mu, \tag{3.5}$$

*where $d\nu(x)$ denotes the volume form of $M$.*

*Remark* 3.2. We actually have a complete and local asymptotics in $k$, so this might yield a CR analogue of the Tian-Yau-Zelditch-Catlin asymptotics on $(0,q)$-forms.

3.2. **Cohomological version (in progress).** We make the following extra assumptions:
- $M$ is $\kappa$-strictly pseudoconvex;
- We can choose $F^L$ and $d\theta$ and the Hermitian metric of $T_{\mathbb{C}}M$ so that we have

$$[F^L, L_\theta] = 0. \tag{3.6}$$

This condition is automatically satisfied when $M$ is strictly pseudoconvex by taking the metric to be the Levi metric.

**Proposition 3.3** (GM+RP). *Under the above assumptions for $q \neq \kappa, n-\kappa$ we have:*

$$\lim_{t\to\infty} G^{0,q}(x,t) = (-1)^q \int_{\lambda_q(x)}^{\lambda_{q+1}(x)} \det(L_\theta^{-1}F^L(x) - \mu)d\mu, \tag{3.7}$$

*where $\lambda_j(x)$ denotes the $j$'th eigenvalue of $L_\theta^{-1}F^L(x)$ counted with multiplicity.*

Thanks to this result we may argue as in [Bi] to get:

**Proposition 3.4** (GM+RP). *Under the same assumptions for $q \neq \kappa, n-\kappa$ we have:*

*1) If $\lambda_{q+1}(x_0) > \lambda_q(x_0)$ for some $x_0 \in M$, then we have:*

$$\dim H^{0,q}(M, L^k) \gtrsim k^{n+1}. \tag{3.8}$$

*2) If $\lambda_{q+1}(x) = \lambda_q(x)$ at every point, then we have:*

$$\dim H^{0,q}(M, L^k) = 0(k^n). \tag{3.9}$$

### 3.3. **Application to complex geometry (in progress).** As an application to the previous results we obtain:

**Theorem 3.5.** *Let $M$ be a complex manifold (not necessarily compact) together with a Hermitian holomorphic line bundle $L$ such that:*

*(i) $L$ is positive outside a Stein domain;*

*(ii) $F^L$ degenerates with multiplicity at least 2 on $\partial D$.*

*Then $M$ is Moishezon.*

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# SPECTRAL ANALYSIS ON COMPLEX HYPERBOLIC SPACES

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## 1. Introduction

Our main goal is to develop a spectral and scattering analysis of the automorphic Laplace-Beltrami operator on discrete quotients of the complex hyperbolic space $\mathbf{CH}^n$. The complex hyperbolic space $\mathbf{CH}^n$ is the rank one Hermitian symmetric space of noncompact type, $\mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$. A standard model of the complex hyperbolic space is the complex unit ball $B^n = \{z \in \mathbf{C}^n;\ |z| < 1\}$ with the Bergman metric $g = \sum_{j,k=1}^n g_{j,k}(z)dz_j \otimes d\bar{z}_k$, where $g_{j,k} = \mathrm{const}\cdot \partial_j\overline{\partial}_k \log(1-|z|^2)$. This model is the bounded realization of the Hermitian symmetric space $\mathbf{CH}^n$. We shall use mainly the unbounded hyperquadric model of the complex hyperbolic space, that is $D^n = \{z \in \mathbf{C}^n;\ \Im m z_n > \frac{1}{2}\sum_{j=1}^{n-1}|z_j|^2\}$. The complex hyperbolic Laplace-Beltrami operator on the unit ball is given by

$$\Delta_{\mathbf{CH}^n} = \frac{4}{n+1}(1-|z|^2)\sum_{j,k=1}^{n}(\delta_{jk} - z_j\bar{z}_k)\frac{\partial^2}{\partial z_j \partial \bar{z}_k}.$$

The quotient, $\mathbf{CH}^n/\Gamma$, is formed by a discrete subgroup $\Gamma$ of the holomorphic automorphism group of the complex hyperbolic space. We mainly interested in subgroups with a noncompact fundamental domain of finite invariant volume or geometrically finite subgroups with infinite volume. Our starting point is the simplest, but realistic example for a automorphism subgroup with a noncompact fundamental domain of finite invariant volume: the Picard modular groups.

The Picard modular groups are

$$SU(n,1;\mathcal{O}_d),$$

where $\mathcal{O}_d$ is the ring of algebraic integers of the imaginary quadratic extension $\mathbf{Q}(i\sqrt{d})$ for any positive square-free integer $d$ (see [H1]). We are interested in the simplest case perhaps: $d = 1$, that is, $\mathcal{O}_d = \mathbf{Z}[i]$, the Picard modular group with Gaussian

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integer entries. The Picard modular group $SU(n,1;\mathbf{Z}[i])$ is a discontinuous holomorphic automorphism subgroup of $\mathbf{CH}^n$ with Gaussian integer entries. It is a higher dimensional analogue of the modular group, $PSL(2,\mathbf{Z})$, in $\mathbf{C}^n$. Our first goal in this direction is to analyze the geometric and spectral properties of the Picard modular group $\Gamma \equiv SU(2,1;\mathbf{Z}[i])$ acting on the complex hyperbolic space $\mathbf{CH}^2$.

Geometric and spectral properties of lattices in symmetric spaces attracted much attention during the last decades. Although remarkable progress has been achieved, several important problems related to arithmeticity, existence of embedded eigenvalues in the continuous spectrum etc., are still open. The general structure of a fundamental domain for lattices is well known since the work of Garland-Raghunathan [GR], for example. However there are very few fundamental domains known completely explicitly. This is especially true for complex hyperbolic spaces. The case of complex hyperbolic spaces is a particularly difficult case. This phenomenon is well known since the work of Mostow [M]. Recently very strong progress has been made in constructing explicit fundamental domains for discrete subgroups of complex hyperbolic spaces; see for example, the work of Cohn [C], Holzapfel [H1], [H2], Goldman [G], Goldman-Parker [GP], Falbel-Parker [FP1], [FP2], Schwartz [Sch], Francsics-Lax [FL1], [FL2]. However, explicit fundamental domains do not seem to be known in the literature for the Picard modular groups, except in the case $d = 3$ (Falbel-Parker [FP2]), see the comment in [FP2], on page 2. Moreover, very little is known about the spectral properties of the automorphic complex hyperbolic Laplace-Beltrami operator, see the work of [EMM], [R], [LV].

The holomorphic automorphism group of $\mathbf{CH}^n$, $\mathbf{Aut}(\mathbf{CH}^n)$, consists of rational functions $g = (g_1, \dots, g_n) : D^n \mapsto D^n$,

$$g_j(z) = \frac{a_{j+1,1} + \sum_{k=2}^{n+1} a_{j+1,k} z_{k-1}}{a_{1,1} + \sum_{k=2}^{n+1} a_{1,k} z_{k-1}},$$

$j = 1, \dots, n$. These automorphisms act linearly in homogeneous coordinates $\zeta_0, \dots, \zeta_n$, $z_j = \frac{\zeta_j}{\zeta_0}$, $j = 1, \dots, n$. The corresponding matrix $A = [a_{jk}]_{j,k=1}^{n+1}$ satisfies the condition

$$A^* C A = C, \tag{1}$$

where

$$C \equiv \begin{pmatrix} 0 & 0 & i \\ 0 & I_{n-1} & 0 \\ -i & 0 & 0 \end{pmatrix}$$

and $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix. The determinant of the matrix $A$ is normalized to be equal to 1. The matrix $C$ is the matrix of the quadratic form of the defining function of $D^n$ written in homogeneous coordinates. Three important classes of holomorphic automorphisms are Heisenberg translations, dilations, and rotations. The Heisenberg translation by $a \in \partial D^2$, $N_a \in \mathbf{Aut}(\mathbf{CH}^2)$ is defined as

$N_a(z_1, z_2) = (z_1 + a_1, z_2 + a_2 + iz_1\bar{a}_1)$. The holomorphic automorphism of $D^2$, $A_\delta(z) = (\delta z_1, \delta^2 z_2)$ is called dilation with parameter $\delta > 0$. Rotation in the first variable by $e^{i\varphi}$, $M_{e^{i\varphi}}(z_1, z_2) = (e^{i\varphi}z_1, z_2)$ is a holomorphic automorphism of $D^2$ with $\varphi \in \mathbf{R}$. The holomorphic involution $J(z_1, z_2) = (iz_1/z_2, -1/z_2)$ will also play significant role.

In a metric space $(X, d)$ the subset $S \subset X$ is a Siegel set for an isometry group $G$ of $X$ if (i) for all $x \in X$ there is $g \in G$ such that $gx \in S$, (ii) the set $\{g \in G;\ g(S) \cap S \neq \emptyset\}$ is finite. Let $S_{1/4}$ be the set

$$\begin{aligned} S_{1/4} &= \{z \in D^2; 0 \leq \Re z_1,\ 0 \leq \Im z_1,\ \Re z_1 + \Im z_1 \leq 1,\ |\Re z_2| \leq \frac{1}{2} \\ &\quad \Im m z_2 - \frac{1}{2}|z_1|^2 \geq \frac{1}{4}\}. \end{aligned} \tag{2}$$

We introduce horoshperical coordinates $(x_1, x_2, x_3, y) \in \mathbf{R}^3 \times \mathbf{R}_+$ as $z_1 = x_1 + ix_2$, $z_2 = x_3 + i(y + (x_1^2 + x_2^2)/2)$ on $D^2$. Then the complex hyperbolic Laplace-Beltrami operator in horospherical coordinates is given by

$$\begin{aligned} \Delta_{\mathbf{CH}^2} &= \frac{y}{2}(\partial_{x_1}^2 + \partial_{x_2}^2) + \frac{y}{2}(2y + x_1^2 + x_2^2)\partial_{x_3}^2 + \\ &\quad y^2\partial_y^2 + yx_2\partial_{x_1}\partial_{x_3} - yx_1\partial_{x_2}\partial_{x_3} - y\partial_y. \end{aligned}$$

We will use the notation

$$H_{\mathcal{F}}(f) = 2\int_{\mathcal{F}} |f|^2 \frac{1}{y^3} dx dy$$

for the invariant $L_2$-integral, and $D_{\mathcal{F}}$ for the corresponding invariant Dirichlet integral, that is

$$\begin{aligned} D_{\mathcal{F}}(f) &= \int_{\mathcal{F}} \Big[ y|f_{x_1}|^2 + y|f_{x_2}|^2 + y(2y + x_1^2 + x_2^2)|f_{x_3}|^2 + 2y^2|f_y|^2 + \\ &\quad x_2 y(f_{x_1}\overline{f_{x_3}} + f_{x_3}\overline{f_{x_1}}) - x_1 y(f_{x_2}\overline{f_{x_3}} + f_{x_3}\overline{f_{x_2}}) \Big] \frac{1}{y^3} dx dy. \end{aligned}$$

## 2. Statement of the results

Our first result is a semi-explicit fundamental domain for the Picard modular group with Gaussian integers.

**Theorem 1.** *The set $S_{1/4}$ is a Siegel set for the Picard modular group $\Gamma$ in $\mathbf{C}^2$. Let $H_1 \equiv J$, $H_2$, ..., $H_N$ be all the holomorphic automorphisms $H \in \Gamma \setminus \Gamma_\infty$ such that $S_{1/4} \cap H(S_{1/4}) \neq \emptyset$. Then the set*

$$\mathcal{F} \equiv \{z \in S_{1/4};\ |\det H_j'(z)|^2 \leq 1,\ 1 \leq j \leq N\} \tag{3}$$

*is a fundamental domain of the Picard modular group with Gaussian integers.*

At this point the transformations $H_2$, ..., $H_N$ are not known explicitly, moreover not all of them are necessary for defining the fundamental domain $\mathcal{F}$. However, this form of the fundamental domain is already useful for obtaining important geometric and spectral properties.

Three important geometric properties of the fundamental domain $\mathcal{F}$ are stated in the following theorem.

**Theorem 2.** *(i) The Siegel set $S_{1/4}$ is invariant under the involutive transformation $S(z_1, z_2) = (i\bar{z}_1, -\bar{z}_2)$. Moreover, the set of transformations $H_1, \ldots, H_N$ is invariant under the conjugation*

$$H \mapsto SHS.$$

*Therefore the fundamental domain $\mathcal{F}$ is invariant under the involutive transformation $S(z_1, z_2) = (i\bar{z}_1, -\bar{z}_2)$.*

*(ii) The two dimensional edge of the fundamental domain $\mathcal{F}$ at $z_1 = 0$ is identical to the standard fundamental domain for the modular group.*

*(iii) The fundamental domain $\mathcal{F}$ has a product structure near infinity, that is,*

$$\mathcal{F} \cap \{z \in \mathbf{C}^2;\ \Im m z_2 \geq a\} = S_{1/4} \cap \{z \in \mathbf{C}^2;\ \Im m z_2 \geq a\}$$

*for large $a > 0$.*

We exploit the involutive transformation $S$ to obtain spectral information on the Laplace operator of the Picard modular group $\Gamma$ in the next theorem. We recall that the continuous spectrum of the Laplace-Beltrami operator $\Delta_\Gamma$ is $(\frac{n^2}{4}, \infty)$ and an eigenvalue in the continuous spectrum is called embedded eigenvalue or Maass cusp form.

**Theorem 3.** *The invariance of the fundamental domain $\mathcal{F}$ under the transformation $S$ in Theorem 2 implies the existence of infinitely many embedded eigenvalues in the continuous spectrum of the associated automorphic complex hyperbolic Laplace-Beltrami operator.*

Our next goal is to determine a completely explicit fundamental domain for the Picard modular group $\Gamma$ based on Theorem 1. In the next theorem we obtain a surprisingly simple description of $\mathcal{F}$ in terms of boundary defining functions.

**Theorem 4.** *A fundamental domain for the Picard modular group is*

$$\begin{aligned}\mathcal{F} \equiv\ & \{z \in S_{1/4};\ |z_2|^2 \geq 1,\\ & |r + i - (1+i)z_1 + z_2|^2 \geq 1,\ r = -1, 0, 1,\\ & |r + i - 2iz_1 + 2z_2|^2 \geq 2,\ r = -1, 1,\\ & |r + i - 2z_1 + 2z_2|^2 \geq 2,\ r = -1, 1\}.\end{aligned}$$

Theorem 5 contains more precise description of the structure of the fundamental domain $\mathcal{F}$.

**Theorem 5.** *There are eight holomorphic automorphims $G_1 = J, G_2, \ldots, G_8$ in the Picard modular group $\Gamma$, described below in equations (4), (5), (6), (7), such that the* *set*

$$\begin{aligned}\mathcal{F} &\equiv \{z \in S_{1/4};\ \ |z_2|^2 \geq 1, \\ &\qquad |\det G_j'(z)|^2 \leq 1,\ j = 2, \ldots, 8\}\end{aligned}$$

*is a fundamental domain of the Picard modular group acting on the complex hyperbolic space* $\mathbf{CH}^2$. *All eight transformations are needed. The holomorphic transformations* $G_1, \ldots, G_8$ *can be described as follows:*

*There are four transformations with dilation parameter* 1*:*

$$\begin{aligned}G_1(z_1, z_2) &\equiv J(z_1, z_2) \\ &\equiv \left(\frac{iz_1}{z_2}, -\frac{1}{z_2}\right),\end{aligned} \tag{4}$$

$$\begin{aligned}G_{r+3} &= J \circ P_{r+3} \\ &= J \circ N_{(1+i, r+i)} \circ M_{-1},\end{aligned} \tag{5}$$

*with* $r = -1, 0, 1$.

*There are four transformations with dilation parameter* $\sqrt{2}$*:*

$$\begin{aligned}G_{5+\frac{1+r}{2}} &\equiv N_{(1, \frac{r+i}{2})} \circ J \circ P_{5+\frac{1+r}{2}} \\ &= N_{(1, \frac{r+i}{2})} \circ J \circ N_{(-1+ri, r+i)} \circ \\ &\quad A_{\sqrt{2}} \circ M_{\frac{-1+ri}{\sqrt{2}}},\end{aligned} \tag{6}$$

*and*

$$\begin{aligned}G_{7+\frac{1+r}{2}} &\equiv N_{(i, \frac{r+i}{2})} \circ J \circ P_{7+\frac{1+r}{2}} \\ &= N_{(i, \frac{r+i}{2})} \circ J \circ N_{(-r-i, r+i)} \circ \\ &\quad A_{\sqrt{2}} \circ M_{\frac{-1+ri}{\sqrt{2}}},\end{aligned} \tag{7}$$

*where* $r = -1, 1$.

The precise definition of the holomorphic automorphisms $P$, $J$, $N$, $A$, and $M$ is described in the introduction. We mention that the inequalities in the description of $\mathcal{F}$ in Theorem 4 are simplified explicit versions of the inequalities

$$|\det G_j'(z)|^2 \leq 1$$

of Theorem 5.

We mention that the method used in Theorem 3 for analyzing the discrete spectrum can be extended for a class of automorphism groups containing the Picard modular group with Gaussian integer entries.

**Definition 1.** *The class* $\mathcal{G}$ *of automorphism groups consists of holomorphic automorphism groups* $\Gamma \subset \mathbf{Aut}(\mathbf{CH}^2)$ *with the following properties:*

*(1) $\Gamma$ has a fundamental domain $F$ with finite invariant volume and with one cusp at infinity.*

*(2) There is a nonholomorphic isometry $S: D^2 \to D^2$ such that $S^2 = I$, $S$ preserves the height function $h(z) = \Im z_2 - \frac{1}{2}|z_1|^2$, that is $h(S(z)) = h(z)$, $S(F) \subset F$, and $S\Gamma S \subset \Gamma$.*

**Theorem 6.** *Let $\Gamma$ be a holomorphic automorphism group in the class $\mathcal{G}$. The complex hyperbolic Laplace-Beltrami operator $\Delta_{\mathbf{CH}^2}$ acting on $\Gamma$-automorphic functions has infinitely many embedded eigenvalues in the continuous spectrum.*

The restriction that $\Gamma$ has only one cusp is not important. We expect that the class of automorphism groups, $\mathcal{G}$, contains more groups than the Picard modulr group with Gaussian integers. It is likely that the conditions defining $\mathcal{G}$ can be weakened, it may be enough to assume that $\Gamma$ has finite invariant volume and there is a nonholomorphic isometry of $\mathbf{CH^2}$, $S$ such that $S^2 = I$, and $S\Gamma S \subset \Gamma$.

## 3. Outline of the method

The main building block in our fundamental domain construction is the Siegel $S_{1/4}$. The triangular shape of $S_{1/4}$ in the $z_1$ variable is the consequence of the fact that a Heisenberg translation $N_a$ is in $\Gamma$ if and only if $\Re a_1, \Im a_1, \Re a_2 \in \mathbf{Z}$ and $|a_1|^2$ is even. The finitness property of $S_{1/4}$ is obtained by using the transformation formula of the Bergman kernel function and the the involution $J$. We build a semi-explicit fundamental domain $\mathcal{F}$ from the Siegel set $S_{1/4}$ in the following way. Let $H_1 \equiv J$, $H_2$, $\dots$, $H_N$ be all the holomorphic automorphisms $H \in \Gamma \backslash \Gamma_\infty$ such that $S_{1/4} \cap H(S_{1/4}) \neq \emptyset$. Then we prove that the set

$$\{z \in S_{1/4};\ |\det H_j'(z)|^2 \leq 1,\ 1 \leq j \leq N\}.$$

is a fundamental domain for the Picard modular group acting on the complex hyperbolic space $\mathbf{CH^2}$. At this point the transformations $H_j$, $j = 2, \dots, N$ are not known explicitly, moreover not all of them contribute to the fundamental domain $\mathcal{F}$.

The key observation in obtaining the spectral properties of $\Delta_\Gamma$ is that the transformation $S$ splits the space of $L_2$ automorphic functions into even and odd automorphic functions with respect to the transformation $S$. One can prove that the resolvent of $\Delta_\Gamma$ is compact on the space of odd automorphic functions. This step uses a Poincaré inequality in the $x$-variables. Near infinity the fundamental domain has a compact cross section, that is, the cross section written in horospherical coordinates, $K_a = \mathcal{F} \cap \{y = a\}$, is compact for large $a > 0$. The Poincaré inequality is applied on the cross section $K_a$.

The basic idea of the explicit construction in Theorems 4 and 5 can be described easily.

Let $\mathcal{F}_1 \equiv S(L) \cap \{z \in \mathbf{C}^2; |z_2| \geq 1\} \equiv S(L) \cap \{z \in \mathbf{C}^2; |\det H_1'(z)|^2 \leq 1\}$. Clearly $\mathcal{F} \subset \mathcal{F}_1$. We will prove that if $H$ is one of the transformations $H_2, \dots, H_N$ in the description of $\mathcal{F}$ in (1) then either

(1) $|\det H'(z)| \leq 1$ for all $z \in \mathcal{F}_1$;
or
(2) there is a transformation $G_j$, $j = 2, \dots, 8$ appearing in (5), (6) and (7) such that $|\det H'(z)| \leq |\det G'_j(z)|$ for all $z \in \mathcal{F}_1$. In either case, the transformation $H$ does not contribute to the fundamental domain $\mathcal{F}$.

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