A successive iteration method for the Cauchy-Riemann equation and its applications

By

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1. The principle: By solving the inhomogeneous Cauchy-Riemann equations (= the $\bar{\partial}$-equations) one can produce holomorphic objects in various situations and deduce definite answers to important existence questions in several complex variables, with estimates in many cases.

Several methods are known to solve the $\bar{\partial}$-equations on complex manifolds. Among such, L^2 estimates of Hörmander (cf. [H-1,2]) is most powerful and flexible. It is based on a general principle that given a closed operator T from a Hilbert space H_1 to another Hilbert space H_2, an element $v \in H_2$ belongs to the image of T if and only if

$$(\dagger) \quad \|(v,w)_{H_2}\| \leq C \| T^*w \|_{H_1}$$

holds for any $w \in \mathrm{Dom}\, T^*$, where $(\ , \ )_{H_2}$ denotes the inner product on H_2, $\| \ \|_{H_1}$ the norm on H_1, T^* the adjoint of T, and C is a constant which is independent of w. In applying this, it is important that whenever v satisfies this criterion, the unique element $u \in \mathrm{Dom}\, T$ with $Tu = v$ and $u \perp \mathrm{Ker}\, T$, which is called the canonical solution, satisfies $\| u \|_{H_1} \leq C$.

The purpose of the present note is to emit a remark that one can combine this principle with a successive iteration method to construct a global solution for the $\bar{\partial}$-equation from the canonically well-estimated local solutions.

There are applications in two directions. One is to a stability question for the Bergman kernels suggested by Kazhdan [Kj]. A partial affirmative result was given in [O-4], which amounts to an iteration in a large scale. However, instead of presenting this more or less abstract result, we shall announce a counterexample recently found in [O-6]. The other is a generalization of Kodaira's embedding theorem to holomorphically foliated manifolds. A result obtained in [O-5] will be presented without proof. Concerning this question, there is an advantage of our method of iteration (this time in a small scale), because it can be applied to laminations. In fact, by applying our method, it is easy to show that one can remove the homological restriction from an embedding theorem stated by Gromov [Gr]. Another application in this direction is given in [O-T] to embed "holomorphically woven" manifolds.

It may be worthwhile to note that our strategy of constructing an approximate global solution by patching the local data has already been suggested in Serre's work [S-2] which established the categorical equivalence between coherent analytic sheaves and coherent algebraic sheaves on projective algebraic varieties based on the "local" results on affine varieties in [S-1].

It is well understood that the à priori estimate (†) for $T = \bar{\partial}$ is a consequence of the existence of certain potential functions and that the link is realized in the presence of a complete Kähler metric on the manifold (cf. [Dm]. See also [O-2]). Let us briefly recall it.

2. Nakano's identity and L² canonical solutions: Let M be a complex manifold, let $B \longrightarrow M$ be a holomorphic line bundle, let $C_0^{p,q}(B)$ be the space of B-valued C^∞ (p,q)-forms with compact support, and let

$$\bar{\partial} : C_0^{p,q}(B) \longrightarrow C_0^{p,q+1}(B)$$

be the complex exterior derivative of type (0,1). Assume that M admits a Kähler metric, say g, and let h be a C^∞ fiber metric of B.

Then the complex Laplacian $\square$ for $\bar{\partial}$ with respect to g and h satisfies Nakano's identity

$$\square - \overline{\square} = \sqrt{-1}[e(\Theta_h),\Lambda],$$

where $\overline{\square}$ denotes the "properly defined" complex conjugate of $\square$, $e(\cdot)$ the exterior multiplication by $\cdot$, Θ_h the curvature form of h, Λ the adjoint of $e(\omega)$ for the fundamental form ω of g, and $[\ ,\ ]$ the commutator (cf. [Dm] or [O-2]).

Since $\Theta_{mh} = m\Theta_h$ holds for any integer m, one has by integration by parts, the inequalities

$$(\#) \qquad \| \bar\partial u \|^2 + \| \bar\partial^* u \|^2 \geq m c_0 \| u \|^2$$

for any $u \in C_0^{p,q}(B^m)$, where the constant c_0 depends only on (M,g) and (B,h), and $c_0 > 0$ if $p = n$, $q \geq 1$ and $\sqrt{-1}\Theta_h > c\omega$ for some positive constant c. Actually one may take c as c_0 in these cases.

From the inequality (#) one can deduce, via à priori estimates of type (†), a basic existence theorem for the $\bar\partial$-equation. In this argument, a limit process comes into play as follows.

In what follows, let (M,g) and (B,h) be as above with $\sqrt{-1}\,\Theta_h \geq c\omega$ for some positive constant c. Let $L^{p,q}_{(2)}(B)_{g,h}$ denote the space of B-valued square integrable (p,q)-forms on M, with the inner product $(\ ,\ )_{g,h}$ and the norm $\|\ \|_{g,h}$, with respect to g and h which are not referred to if there is no fear of confusion.

If there exists a complete Kähler metric say g_0 on M, the $\bar\partial$-equations $\bar\partial u = v$ for $v \in \mathrm{Ker}\,\bar\partial \cap L^{n,q}_{(2)}(B)_{g,h}$ $(q \geq 1)$ are solvable in $L^{p,q-1}_{(2)}(B)$ for any $\varepsilon > 0$.

In fact, in virtue of (#) it is easy to verify that one has for each ε an à priori estimate

$$c\,|(v,w)_{g+\varepsilon g_0,h}|^2 \leq \| v \|^2_{g,h} \, \| \bar\partial^* w \|^2_{g+\varepsilon g_0,h}$$

holding for any $w \in \mathrm{Dom}\,\bar\partial \cap L^{n,q}_{(2)}(B)_{g+\varepsilon g_0,h}$. Hence there exists $u_\varepsilon \in L^{n,q-1}_{(2)}(B)_{g+\varepsilon g_0,h}$ for each ε such that $\bar\partial u\ \ = v$ holds in the sense of distribution.

By letting $\varepsilon \longrightarrow 0$, taking account of that

$$c_0 \| u_\varepsilon \|^2_{g+\varepsilon g_0,h} \leq \| v \|^2_{g,h}$$

holds if $u_\varepsilon \perp \mathrm{Ker}\,\bar\partial$, one may take a weakly convergent sequence u_{ε_k} (k=1, 2, . . .) with $\varepsilon_k \longrightarrow 0$.

Consequently one has

Theorem 1. (cf. [Dm], [O-2]) For any holomorphic Hermitian line bundle (B,h) over M such that $\sqrt{-1}\,\Theta_h > 0$, $\mathrm{Ker}\,\bar\partial \cap L^{n,q}_{(2)}(B)_{g,h} = \bar\partial(\mathrm{Dom}\,\bar\partial \cap L^{n,q-1}_{(2)}(B)_{g,h})$ holds if $q \geq 1$ and $\omega = \sqrt{-1}\,\Theta_h$, provided that M admits a complete Kähler metric. Moreover, in this situation, the solution to $\bar\partial u = v$ with $u \perp \mathrm{Ker}\,\bar\partial$ satisfies $\| u \| \leq \| v \|$.

Complete Kähler metrics always exist on Stein manifold. Conversely, applying the solvability of the $\bar\partial$-equation with L^2 estimates as above in a specific situation, it was proved in [O-1] that C^1-smooth domains with complete Kähler metrics are pseudoconvex.

3. The iteration: From now on, we assume that $\sqrt{-1}\,\Theta_h = \omega$. Let $\{U_i\}$ be a locally finite open covering of M and let $\{\rho_i\}$ be a system of nonnegative C^∞ functions on M such that $\mathrm{supp}\,\rho_i \subset U_i$, $\Sigma\,\rho_i = 1$ and $K := \sup_i \sup_{U_i} |\bar\partial\rho_i|^2 < \infty$.

Given any $v \in \mathrm{Ker}\,\bar\partial \cap L^{n,q}_{(2)}(B^m)$ $(q \geq 1)$, one has, by Theorem 1, solutions to $\bar\partial u_i = v \mid U_i$ satisfying $mc_0\| u_i \|^2 \leq \| v \|^2$.

We put $u^{(1)} = \Sigma\,\rho_i u_i$. Then

$$\bar\partial u^{(1)} = \Sigma\,\bar\partial\rho_i u_i + \Sigma\,\rho_i\bar\partial u_i = \Sigma\,\bar\partial\rho_i u_i + v,$$

so that $\| v - \bar\partial u^{(1)} \|^2 = \| \Sigma\,\bar\partial\rho_i u_i \|^2 \leq K \| v \|^2/(mc_0)$.

Hence, for sufficiently large m, one can solve the equations

$$\bar\partial u_{i,k} = (v - \bar\partial u^{(k-1)}) \mid U_i \quad (u_{i,1} = u_i,\ u^{(0)} = 0)$$

inductively on k, with estimates

$$\| u_{i,k} \| + \| v - \bar\partial u^{(k)} \| \leq 2^{-k} \| v \|,$$

where $u^{(k)} = u^{(k-1)} + \sum_i \rho_i u_{i,k}$.

Summing up, we obtain the following.

Theorem 2. In the above situation, $u^{(k)}$ converges in $L^{n,q}_{(2)}(B^m)$ to a solution of $\bar\partial u = v$ if m is sufficiently large.

We note that Theorem 2 tells us something new even in local analytic geometry. For instance, for any regularly embedded p-dimensional complex manifold N of a bounded domain in $\mathbf{C}^n$, it follows immediately from Theorem 2 that its L^2 $\bar{\partial}$-cohomology groups of type (p,q) ($q \geq 1$) with respect to the induced metric vanish if N is locally Stein, i.e. if every point $x \in \overline{N} \setminus N$ admits a Stein neighborhood in $\mathbf{C}^n$ whose intersection with N is Stein. Since it is an open question whether or not such a manifold N is Stein, vanishing of the L^2 cohomology groups of N is not a straightforward consequence of Theorem 1.

The L^2 cohomology of locally closed complex submanifolds of $\mathbf{C}^n$ is of interest also in connection to the intersection cohomology (cf. [O-3]).

4. Asymptotics of the Bergman kernels on towers: Let G be a subgroup of AutM acting freely and properly discontinuously on M, and let $G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_k \supset \cdots$ be a decreasing sequence of subgroups of G satisfying $G_1 = G$ and $\bigcap_k G_k = \{id\}$, let $M_k = M/G_k$, and let $\pi_k : M \longrightarrow M_k$ be the projection. Then one asks whether or not the pullbacks of the Bergman kernels of M_k by π_k converge to that of M.

It is written in [Y] that Kazhdan proved the following:

> *If $[G_k , G_{k+1}] < \infty$ and M admits the Bergman metric ds^2_M, then the pullbacks of the Bergman metrics $ds^2_{M_k}$ of M_k converge on M to ds^2_k.*

Note that it is implicitly assumed that M_k carries the Bergman metric, since otherwise there is a counterexample with $M = \{z \in \mathbf{C};\ \mathrm{Im}\, z > 0\}$ and G an elliptic modular group. However the situation becomes subtler if the question is stated in the following way.

Let $H = \{z \in \mathbf{C};\ \mathrm{Im}\, z > 0\}$ and let Γ be a Fuchsian group acting on H. In [R], J. A. Rhodes studied the asymptotic behavior of the Bergman kernels associated to the towers of compact Riemann surfaces $\{S_k\}$ given by $S_k = H/\Gamma_k$, with $\Gamma_1 = \Gamma \supset \Gamma_2 \supset \cdots$, and $\bigcap_k \Gamma_k = \{ id \}$. He showed that, letting $\pi_k : H \longrightarrow S_k$ be the projections, the Bergman kernels K_{S_k} of S_k restricted to the diagonal satisfy

(1) $$\lim_{k\to\infty} \pi_k^* K_{S_k} = \frac{|dz|^2}{4\pi (\mathrm{Im}\, z)^2}$$

and

(2) $$\lim_{k\to\infty} \partial\bar{\partial} \log \pi_k^* K_{S_k} = \frac{dz \wedge d\bar{z}}{2(\mathrm{Im}\, z)^2}$$

if either the following (a) or (b) holds.

(a) Γ_k is normal in Γ for all k.

(b) The smallest nonzero eigenvalue of the Laplace-Beltrami operator associated to the Poincaré metric on S is bounded away from zero as k varies.

This fact supports the validity of the following, which was first stated by D. Mumford [Mf] attributing to D. Kazhdan's work [Kj] :

> *Suppose* $\{S_k\}$ *is a sequence of compact Riemann surfaces,* $S_k = H/\Gamma_k$, $\Gamma_1 \supset \Gamma_2 \supset \cdots$, *and* $\bigcap_k \Gamma_k = \{ id \}$. *Then, letting* ds_k^2 *denote the pullback of the Bergman metric of* S_k *to H, with suitably chosen scalars* λ_k,
>
> $$\lim_{k\to\infty} \lambda_k\, ds_k^2 = \frac{|dz|^2}{(\mathrm{Im}\, z)^2}.$$

Rhodes made the following remark in [R].

> *There is no reason to think that the statement would be false if neither conditions (a) nor (b) held ; they are needed to overcome technical problems.*

Unfortunately there exists a counterexample. In fact, it is easy to construct $\{S_k\}$ such that neither (1), (2), (a) nor (b) hold, by taking a sequence of covering spaces of hyperelliptic Riemann surfaces whose members are all hyperelliptic (cf.[O-6]).

5. Embedding partially holomorphic structures: The celebrated embedding theorem of Kodaira [K] says that every positive line bundle over a compact complex manifold is ample. In view of Matsushima's (last) work [M] on the embedding of certain nonalgebraic tori by partially holomorphic maps, it looks worthwhile to extend Kodaira's theorem to the manifolds endowed with partially complex structures. For that, we shall focus our attention at first to an abstract question of embedding foliated manifolds with complex leaves.

Let X be a C^∞ manifold countable at infinity. A foliation on X is by definition a submanifold $\mathfrak{F}$ of X such that the inclusion $\iota : \mathfrak{F} \longrightarrow X$ induces a bijection between $\iota_* T\mathfrak{F}$ and a C^∞ subbundle of TX. If $\mathfrak{F}$ is a complex manifold, we say that $\mathfrak{F}$ is a complex foliation on X. X will be called holomorphically foliated if it is equipped with a complex foliation.

Let $(X, \mathfrak{F}, \iota)$ be a holomorphically foliated manifold and let (L,h) be a C^∞ complex Hermitian line bundle over X. We say that L is tangentially holomorphic if $\iota^* L$ is holomorphic, and positive if $\iota^* h$ has everywhere positive curvature. A section s of L over X is said to be tangentially holomorphic if $s \circ \iota$ is holomorphic.

In [O-5] the following is proved by solving the $\iota^* L$-valued $\bar\partial$-equations with regularity conditions on X. The method is by a successive iteration explained as above.

Theorem 3. Let $(X, \mathfrak{F}, \iota)$ be a compact holomorphically foliated manifold equipped with a positive tangentially holomorphic Hermitian line bundle (L,h). Then, for any nonnegative integer k, there exists a positive integer m_0 such that, one can find, for any integer $m \geq m_0$, tangentially holomorphic sections $s_0, s_1, \ldots, s_N$ of L such that the ratio $(s_0 : s_1 : \cdots : s_N)$ is a C^k embedding of X into the complex projective space.

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