Asymptotic questions on the Bergman kernels

By
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It is widely recognized that reproducing kernels often encode substantial information on geometric or physical objects. The Bergman kernel, which naturally arose in complex analysis and has been playing important roles in complex geometry and mathematical physics, is known as such an example.

Let us briefly review several open questions in the asymptotics of the Bergman kernels.

1. Continuity and asymptotic stability: Let M be a complex manifold, let M_k (k=1,2,...) be a sequence of complex manifolds converging to M in some nonnonsense way. Then an immediate question is whether or not the associated sequence of the Bergman kernels K_{M_k} of M_k converges to that of M in an appropriate sense.

Affirmative answers have been given in the following cases.

i) M is a bounded strongly pseudoconvex domain in a complex manifold X and, with respect to some differentiable family $\pi : \boldsymbol{M} \longrightarrow (-1,1)$ of smooth domains of X, $M = \pi^{-1}(0)$ and $M_k = \pi^{-1}(t_k)$ for some sequence t_k converging to 0 (Greene and Krantz [Gr-Kr]).

ii) M is a smooth weakly pseudoconvex Stein domain and M_k are the members of some special differentiable family $\pi : \boldsymbol{M} \longrightarrow (-1,1)$ with $M = \pi^{-1}(0)$ (Diederich and Ohsawa [Di-Oh-2,3]).

In virtue of Kohn's formula between the Bergman projection and the Neumann operator, a continuity theorem for the canonical solutions of the $\bar{\partial}$-equation in [Di-Oh-1] implies the continuity of the Bergman kernels as well.

On the other hand, in an interesting work of D.Kazhdan [Kd] on arithmetic varieties, a convergence theorem was suggested for the Bergman kernels on the towers of complex manifolds, from an observation on asymptotic stability for covering spaces (see also [M]). More precisely, assume that the group of biholomorphic automorphisms of M admits a discrete subgroup, say Γ, whose elements do not have fixed points in M, let $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots \supset \Gamma_k \supset \cdots$ be a decreasing sequence of subgroups of Γ satisfying $\Gamma_1 = \Gamma$ and $\bigcap_{k=1}^{\infty} \Gamma_k = \{id\}$, let $M_k = M/\Gamma_k$, and let $\pi_k : M \longrightarrow M_k$ be the projection. Then one asks whether or not the pullbacks of K_{M_k} by π_k converge to K_M. An immediate, trivial, but interesting affirmative example is given by $M = \mathbf{C}^n$ and $\Gamma_k = k(\mathbf{Z}^n + \sqrt{-1}\mathbf{Z}^n)$ (k=1,2,...). As for less trivial results and counterexamples, see [Oh].

2. The boundary asymptotics: Theories on the boundary behavior of the Bergman kernels are more or less classical and better understood than the above mentioned continuity properties. It is said that the earliest purpose of studying the boundary behavior of the Bergman kernel of strongly pseudoconvex domains in $\mathbf{C}^n$ was to solve the Levi problem (cf.[H-2]).

However, when the celebrated monograph of Bergman [B] was published, few things were known about the boundary asymptotics of the Bergman kernel, so that they were not at all available for solving the Levi problem. What was known at first was Oka's solution of the Levi problem without relying on the Bergman kernel (cf.[O]), and next its counterpart on compact manifolds by Kodaira with a reference to the Bergman kernel, but only in some application (cf.[K]). Therefore, the renowned works of Hörmander [H-1] and K.Diederich [Di], which explicitly describes the boundary asymptotics of the Bergman kernel in terms of the "Levi determinant", may well be said to sit in the picture drawn by Oka and Kodaira, especially from today's viewpoint.

Denken ist die Einschränkung auf einem Gedanken, der einst wie ein Stern am Himmmel der Welt stehen bleibt.

M.Heidegger

Nevertheless, if one regards the study of the boundary behavior of the Bergman kernel as a decoding process of getting information from the reproducing kernels, there should remain a lot to be found behind the scenery. Works of Fefferman [F-1,2] and K.Hirachi [Hi] may be counted as such instances. Geometric invariants in the asymptotics of the Bergman kernels are of interest because they are naturally attached to conformal maps in one variable. In such a spirit, it seems to make sense to investigate nonpositive terms in the asymptotic expansions of Fefferman and Hirachi in connection to a famous (or notorious) conjecture of N.Suita asserting that $\pi K_M(z,z) \geq c_\beta{}^2 |dz|^2$ holds if dim M = 1 and M admits the Green function. Here c_β denotes the logarithmic capacity. As for a recent progress on Suita's conjecture, see [Bł] or [Yd].

3. Asymptotics of the Bergman measures: Let M be a connected compact complex manifold of dimension n, and let $L \longrightarrow M$ be a holomorphic Hermitian line bundle. In this situation, one can explicitly state how an asymptotics of the (weighted) Bergman kernels decode the "Monge-Ampère type" geometry of L. Let us review such results following a recent work of R. Berman and S. Boucksom [Bm-Bs] (See also [Lb]).

In what follows, let us assume that the bundle L admits nonzero holomorphic sections over M, and take any fiber metric h of L. By $H^0(L)$, we shall denote the space of holomorphic sections of L over M. The length of $s \in H^0(L)$ will be denoted by $|s|_h$.

Given any compact subset K of M which is not locally pluripolar, and any probability measure say μ on K, define

$$\|s\|^2_{L^2(\mu,h)} = \int_M |s|_h^2 \, d\mu$$

and

$$\|s\|^2_{L^\infty(K,h)} = \sup_K |s|_h \, .$$

Clearly, $\|\cdot\|_{L^2} \leq \|\cdot\|_{L^\infty}$. Then we put

$$\varrho(\mu,h)(x) = \sup\{\, |s(x)|_h^2 / \|s\|^2_{L^2(\mu,h)};\ s \neq 0 \in H^0(L)\}.$$

Since $\dim H^0(L) = \int_M \varrho(\mu,h)d\mu$, one has a probability measure $\beta = \beta(\mu,h)$ defined by

$$\beta = \varrho\mu / \dim H^0(L).$$

We shall call β the Bergman measure for L with respect to (μ, h).

Under some regularity condition, one can see the following relationship between the Bergman measures and the Monge-Ampère measure.

Let Θ denote the curvature form of h.

Theorem 1. (Bouche-Catlin-Tian-Zelditch) Assume that μ and h as above are C^∞ and that $\mu > 0$ and $\sqrt{-1}\Theta > 0$ hold everywhere on M. Then, there exist C^∞ functions b_m on M such that one has the following asymptotic expansion for $\beta(\mu, h^k)$:

$$(\dagger) \qquad d\beta(\mu, h^k) = \sum_{m=0}^{\infty} b_m k^{-m}, \quad b_0 = MA(h)/V$$

where $MA(h) = (\sqrt{-1}\Theta)^n$ and $V = \int_M (\sqrt{-1}\Theta)^n$.

(For the references, see [Bch],[Ct],[Ti] and [Zd].)

We note that the initial term b_0 of the asymptotic expansion is the limit of the Bergman measures for L^k.

Recently, Theorem 1 was applied by B.Berndtsson [Bnt] to prove the uniform convergence of a sequence of iterated and scaled weighted Bergman kernels introduced by H.Tsuji [Tj] (Tsuji showed the L^1 convergence by a different method.)

J.-P. Demailly [Dm] introduced another scaling and proved a convergence theorem for the weighted Bergman kernels on pseudoconvex domains in $\mathbf{C}^n$.

Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let ϕ be a plurisubharmonic function on D. Let $z = (z_1, \dots, z_n)$ be the coordinate of $\mathbf{C}^n$.

We put for any $\varepsilon \geq 0$,

$$A^2_{\phi,\varepsilon}(D) = \{ f \mid f \text{ is holomorphic on D and } \int_D e^{-\phi}(1+|z_n|^2)^{-1-\varepsilon}|f|^2 d\lambda < \infty \}$$

and, by letting $D' = \{ z \in D \mid z_n = 0 \}$, put

$$A^2_\phi(D') = \{ f \mid f \text{ is holomorphic on } D' \text{ and } \int_{D'} e^{-\phi}|f|^2 \, d\lambda' < \infty \}.$$

Here λ and λ' denote respectively the Lebesgue measures on D and D'.

Theorem 2. ([O-T] (essentially)) For every $\varepsilon > 0$, there exist a constant C_ε and a bounded linear operator $I_\varepsilon : A^2_\phi(D') \longrightarrow A^2_{\phi,\varepsilon}(D)$ whose norm does not exceed C_ε, such that $I_\varepsilon(f)|D' = f$ holds for any $f \in A^2_\phi(D')$.

We note that Theorem 2 includes a parameter ε. The optimal number for C_ε is not known (cf. [Bł]). Obviously Theorem 2 is false for $\varepsilon = 0$. Theorem 2 was applied by J.-P.Demailly to approximate the function ϕ and its Lelong number in terms of the weighted Bergman kernels. Here the Lelong number $\nu(\phi,z_0)$ of ϕ at z_0 is defined by

$$\nu(\phi,z_0) := \liminf_{z \to z_0} \frac{\phi(z_0)}{\log|z-z_0|}$$

and the weighted Bergman kernel of D, denoted by K_ϕ for simplicity, is defined as the reproducing kernel of the space $A^2_\phi(D)$.

Theorem 3. ([Dm]) There exist constants C_1 and C_2 such that the following hold for any positive integer m.

(a) $\phi(z) - C_1/m \le (2m)^{-1}\log K_{2m\phi}(z,z) \le \sup_{|\zeta - z|<r} \phi(\zeta) + m^{-1}\log(C_2/r^n)$

if $z \in D$ and $r < \mathrm{dist}(z, \partial D)$ (=the euclidean distance from z to ∂D).

(b) $\nu(\phi,z_0) - n/m \le \nu((2m)^{-1}\log K_{2m\phi}(z,z), z_0) \le \nu(\phi,z_0), \quad z_0 \in D.$

In the situation of Theorem 1, it is readily seen from the asymptotic expansion (†) that the sequence of k-th roots of the Bergman measures for L^k, regarded as sections of $|\det T_M|^{-2/k}$, converges to 1. Here T_M denotes the tangent bundle of M. Therefore, the counterpart of Theorem 3 for compact manifolds is a rather trivial consequence of Theorem 1.

As for a generalization of Theorem 1, R.Berman obtained the following.

Theorem 4. (cf. [Bm]) Assume that μ and h are C^∞ and that $\mu > 0$ everywhere. Then, with respect to the fiber metric of L locally defined by $e^{-\phi_M}$ with

$$\phi_M = \sup\{\psi \,;\, \sqrt{-1}\partial\bar{\partial}\psi \geq 0 \text{ and } \psi \leq \varphi \text{ with } h = e^{-\varphi}\},$$

one has a weak convergence $\beta(\mu, e^{-k\varphi}) \longrightarrow MA(e^{-\phi_M})/V \quad (k \longrightarrow \infty)$.

Let us (locally) put

$$\phi_K = \sup\{\psi \,;\, \sqrt{-1}\partial\bar{\partial}\psi \geq 0 \text{ and } \psi|K \leq \varphi|K \text{ with } h = e^{-\varphi}\}$$

and

$$\mu_{eq}(K,h) = MA(e^{-\phi_K^*})/V,$$

where ϕ_K^* denotes the upper envelope of ϕ_K. $\mu_{eq}(K,h)$ is called the equilibrium measure for L with respect to (K,h).

Theorem 5. (cf. [Bm-Bs]) Let μ be the Bergman measure for L with respect to (K,h) (h is any fiber metric). Then

$$\beta(\mu, e^{-k\phi}) \longrightarrow \mu_{eq}(K,h) \quad \text{weakly.}$$

4. Scaling limits of Christoffel-Darboux kernels: The N-th Christoffel-Darboux kernel is, up to some weights, the reproducing kernel for the space of polynomials of degrees at most N. They satisfy asymptotic formulas which suggest those for the Bergman kernel. Let us review some of them.

Let $x = \mathrm{Id}_{\mathbb{R}}$,

$$H(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

$$\phi_k(x) = \frac{1}{(\sqrt{\pi}\, k!\, 2^k)^{1/2}} H_k(x) e^{-x^2/2}$$

and

$$K^{(N)}(x,y) = \sum_{k=0}^{N-1} \phi_k(x)\phi_k(y) \qquad (x, y \in \mathbb{R}).$$

The Christoffel-Darboux formula says

$$K^{(N)}(x,y) = \left(\frac{N}{2}\right)^{1/2} \frac{\phi_N(x)\,\phi_{N-1}(y) - \phi_{N-1}(x)\,\phi_N(y)}{x - y}.$$

By Adamov [Ad], it was shown that

$$H_k(x) \approx 2^{(k+1)/2} k^{k/2} e^{-k/2} e^{x^2/2} \cos(\sqrt{2k+1}\, x - \frac{k\pi}{2}), \quad k \to \infty.$$

From this, it is easy to see that

$$\lim_{N\to\infty} \frac{K^{(N)}(x,y)}{N} = 0.$$

Moreover, according to [Sh],

$$\lim_{N\to\infty} \frac{\pi}{\sqrt{2N}} K^{(N)}\left(\frac{\pi x}{\sqrt{2N}}, \frac{\pi y}{\sqrt{2N}}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)} \qquad \text{(scaling limit)}$$

and

$$\lim_{N\to\infty} \frac{1}{\sqrt{2}N^{1/6}} K^{(N)}\left(\sqrt{2N} + \frac{x}{\sqrt{2}N^{1/6}}, \sqrt{2N} + \frac{y}{\sqrt{2}N^{1/6}}\right) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y}$$

(double scaling limit) are known to hold, where

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos(xs + \frac{s^3}{3})ds.$$

(see also [Fo] and [T-W]).

Let $W(x)$ be a positive continuous function on the interval $[-1,1]$. Let $\{p_k\}_{k=1}^\infty$ be a sequence of orthonormal polynomials with respect to the measure $W(x)dx$, where $\deg p_k = k - 1$, let

$$K^N(z,w) = \sum_{k=0}^{N-1} p_k(z)\overline{p_k(w)}, \quad z, w \in \mathbf{C}$$

and let

$$\widetilde{K}^N(x,y) = W(x)^{1/2} W(y)^{1/2} K^N(x,y), \quad x, y \in [-1,1].$$

Then

$$\lim_{N\to\infty} \frac{K^N(x,x)}{N} = \frac{1}{\pi\sqrt{1-x^2}\,W(x)} \qquad \text{a.e. on } (-1,1)$$

(cf. [To]) and, for any $x \in (-1,1)$,

$$\lim_{N\to\infty} \frac{K^N\left(x+\frac{a}{\widetilde{K}^N(x,x)},\ x+\frac{b}{\widetilde{K}^N(x,x)}\right)}{K^N(x,x)} = \frac{\sin\pi(a-b)}{\pi(a-b)}$$

((bulk) universality limit) holds uniformly for a, b in compact subsets of **C** (cf. [S-1] and [L-1,2]).

The relationship of these formulas to random matrix theory and nuclear physics is described in [Sh].

As for the proof of the universality limit formula, according to [S-2] and [L-2], it is based on the theory of entire functions of exponential type, as in the case of the celebrated sampling theorem of Whittaker-Ogura-Kotel'nikov-Shannon-Someya. (See [B-F-H-S-S-S] for a historical aspect of the sampling theorem.) There is another approach using Riemann-Hilbert method (cf. [K-V]).

As these various scaling limit formulas suggest, one has formulas clarifying the relations between the asymptotics of the Bergman kernels and the energy minimizing sequences for the extremal metrics on complex manifolds. One instance is recently given by Berman and Boucksom. Their formula expresses the Mabuchi energy in terms of a scaling limit (see [Lb]).

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