On a curvature property of effective divisors and its application to sheaf cohomology

(based on a paper to appear in Publ. RIMS)

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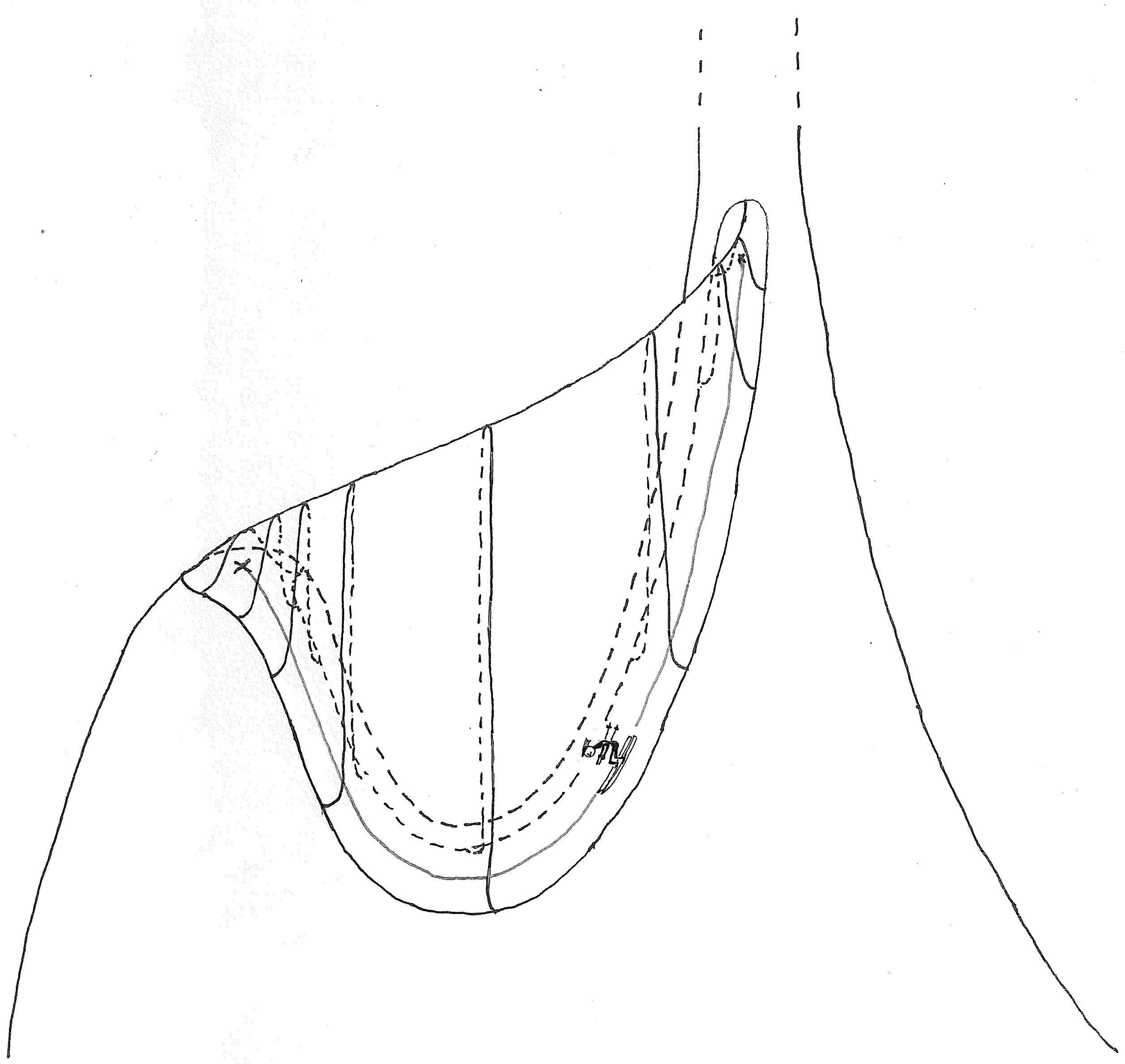
Let X be a complex analytic space of dimension n which is countable at infinity. It is known that X is n-complete in the sense of Andreotti-Grauert [A-G] if every irreducible component of X is noncompact (cf. [O] and [Dm]). This shows that the vanishing theorem for the cohomology groups of top degrees, due to Y.-T. Siu [S], is essentially contained in [A-G].

In [F], based on the 1-completeness of noncompact Riemann surfaces, an elementary proof was given to a basic fact that, for any Riemann surface R and for any point $p \in R$, the line bundle [p] associated to the divisor p, is positive.

The purpose of the present note is to extend the paper [F] to establish the following.

Theorem 1. Let X be a compact complex analytic space of dimension n and let D be an effective Cartier divisor of X such that $|D|$, the support of D, intersects every n-dimensional irreducible component of X. Then the line bundle $[D]$ admits a fiber metric whose curvature form has everywhere at most $n-1$ nonpositive eigenvalues on the regular points of X.

Theorem 1 supplements [O] and [Dm]. Let us recall that the main point in the proof of the n-completeness theorem in [O] is that, given any smooth exhaustion function φ on M whose critical points are nondegenerate, and given any point p at which φ takes its local maximum, one can modify φ on a neighborhood U of a curve connecting p with a noncritical point q of φ satisfying φ(q) > φ(p), in such a way that the modified function does not have any local maximum on the closure of U. This procedure is quite elementary as the following figure shows.



By applying Theorem 1, we obtain the following Serre type vanishing theorem.

Theorem 2. Let M be a complex manifold, let Z be a complex analytic space, let $f : M \longrightarrow Z$ be a proper holomorphic map, let D be an effective divisor of M, let $z \in f(|D|)$, and let n be any positive integer exceeding the dimension of any compact irreducible component of $(f^{-1}(z) \setminus |D|) \cup (f^{-1}(z) \cap |D|)$. Then, for any holomorphic vector bundle $E \longrightarrow M$, there exists a positive number m_0 such that

$$(R^n f_* \mathcal{O}(E \otimes [D]^m))_z = 0$$

holds if $m \geq m_0$. Here $\mathcal{O}(E \otimes [D]^m)$ denotes the sheaf of the germs of holomorphic sections of $E \otimes [D]^m$, and $R^n f_* \mathcal{O}(E \otimes [D]^m)$ the n-th direct image of $\mathcal{O}(E \otimes [D]^m)$ by f.

For the proof of Theorem 2, we need results from [B] and [O].

Further we have a refined version of Theorem 2 when E is the canonical bundle K_M of M.

Theorem 3. In the above situation, suppose moreover that M admits a Kähler metric. Then

$$(R^n f_* \mathcal{O}(K_M \otimes [D]))_z = 0$$

holds.

Theorem 3 may well be regarded as a supplement to the vanishing theorem of Grauert-Riemenschneider [G-R].

In view of Theorem 2, the following conjecture seems to make sense.

Conjecture : Let M be a projective algebraic variety of dimension n and let L be a holomorphic line bundle over M such that $L \cdot C > 0$

holds for almost all curves C in M. Then, for any coherent analytic sheaf $\mathcal{F}$ over M, there exists a positive integer m_0 such that

$$H^n(M, \mathcal{F} \otimes L^m) = 0$$

holds for all $m \geq m_0$.

Further, recalling Grauert's conjecture on the cohomological characterisation of q-complete spaces, the converse of the above conjecture seems plausible, too.

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