# LOGARITHMIC MODULI SPACES FOR SURFACES OF CLASS VII

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joint work with M. Toma, see arxiv and Math. Ann. 2008

# Introduction

$S$ compact complex surface, $b_1(S) = \dim_{\mathbb{R}} H^1(S, \mathbb{R}) = 1$ : class $\mathrm{VII}$

$S$ minimal : class $\mathrm{VII}_0$

$b_2(S) = \dim_{\mathbb{R}} H^2(S, \mathbb{R}) = 0$ and $S$ admits at least one compact curve : $S$ is a Hopf surface (Kodaira).

$b_2(S) = 0$ and $S$ admits no compact curve : $S \simeq (\mathbb{C} \times \mathbb{H})/\Gamma$ is an Inoue surface (Inoue, Bogomolov, Li-Yau-Zheng, Teleman).

$S$ minimal compact complex surface, $b_1(S) = 1$ and $b_2 > 0$ : class $\mathrm{VII}_0^+$. Then $a(S) = 0$, $\dim_{\mathbb{C}} H^1(S, \mathcal{O}) = 1$ and $\kappa(S) = -\infty$.

**A complete classification is not yet achieved for class $\mathrm{VII}_0^+$.**
Groundbreaking results were obtained in the 70's and 80's by Ma. Kato, Enoki, Nakamura and Dloussky.
All presently known surfaces of class $\mathrm{VII}_0^+$ contain *global spherical shells (GSS)* (Kato surfaces)

We give quickly the main idea of the construction.

**Construction of Hopf surfaces :**

Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^2$ and $f : \bar{\mathbb{B}} \to \mathbb{B}$ with $f(0) = 0$ and biholomorphic on its image.

Identifying the points on $S^3$ with the points on $f(S^3)$ by $f$, one obtains a compact complex surface, a (primary) Hopf surface.

The map $f$ can be biholomorphically conjugated to a map of the *normal form*

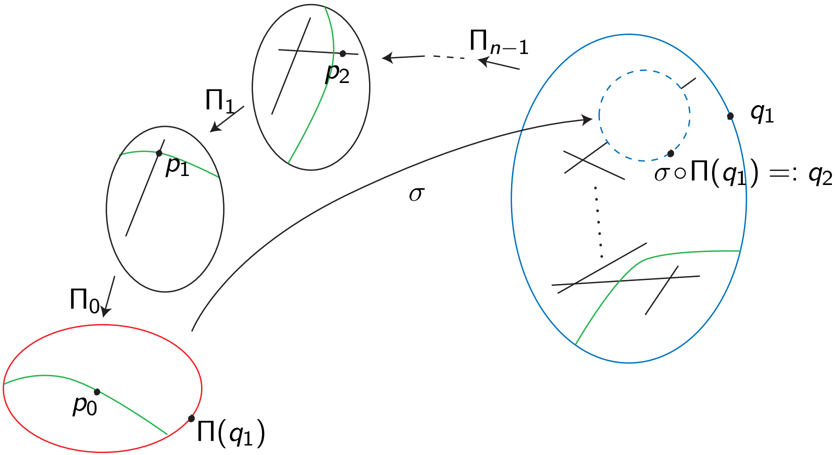
$$\varphi(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2),$$

with $m \in \mathbb{N}, \quad 0 < |\alpha_1| \leq |\alpha_2| < 1, \quad ({\alpha_2}^m - \alpha_1)\lambda = 0.$

Of course, these normal forms are essential in the study of Hopf surfaces : Deformations and moduli spaces these surfaces have been studied by Dabrowski and Wehler using them.

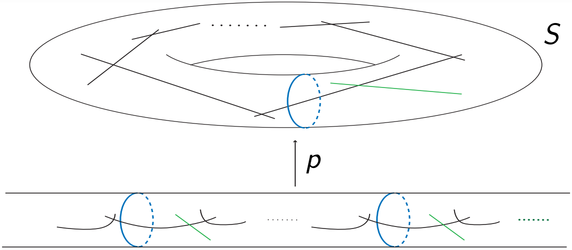
In 1977, Masahide Kato generalised this construction.

# Construction of Kato surfaces




$$\Pi := \Pi_0 \circ \cdots \circ \Pi_{n-1}, \text{ associated germ } \varphi : \begin{array}{ccc} (\mathbb{C}^2, 0) & \longrightarrow & (\mathbb{C}^2, 0) \\ z & \longmapsto & \Pi \circ \sigma(z) \end{array}$$

(Dloussky germ)

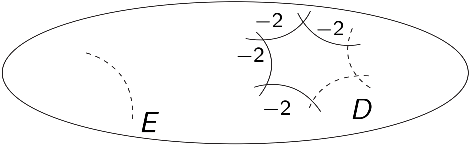



$\tilde{S}$ (universal covering of $S$)

We obtain the associated surface $S = S(\Pi, \sigma) = S(\varphi)$.
The fundamental group $\pi_1(S) \simeq \mathbb{Z}$.
All rational curves $C$ have $C^2 \leq -2$.
The second Betti number $b_2(S) = n$ where $n$ is the number of blowing ups performed in the above construction.
**Note : The germ $\varphi = \Pi \circ \sigma$ determines completely the surface. Therefore *normal* forms for it are very important.**

A finer subdivision Kato surfaces may be done by looking at the configuration (divisor) $D$ of rational curves on $S$.

1) Enoki surfaces including as a special case parabolic Inoue surfaces ($E$ is an elliptic curve (only in the Inoue case). $D$ is a cycle of rational curves with $D^2 = 0$). **All blow-ups are "generic".**

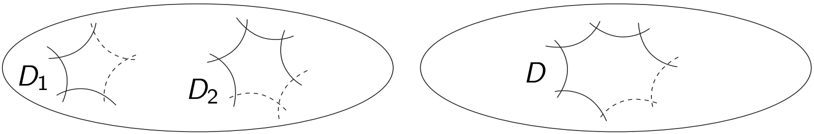



Normal form of the associated germ :

$$\varphi(z_1, z_2) = (\lambda^n z_1 z_2^n + \sum_{i=1}^{n} a_i \lambda^i z_2^i,\ \lambda z_2)$$

$0 < |\lambda| < 1,\ a_i \in \mathbb{C}$. (Inoue iff all $a_i = 0$.) Here $n = b_2(S)$.
$S \setminus D$ is an holomorphic $\mathbb{C}$-bundle (line bundle iff Inoue) over an elliptic curve $E$. The fundamental group $\pi_1(S \setminus D) \simeq \mathbb{Z}^2$ is abelian.
The universal covering $\widetilde{S \setminus D} \simeq \mathbb{C}^2$.

2) Inoue-Hirzebruch surfaces and half-Inoue surfaces ($D_1$, $D_2$ and $D$ are exceptional cycles of rational curves, in particular $D_1^2 < 0$, $D_2^2 < 0$ and $D^2 < 0$.) **All blowups are "non-generic".**

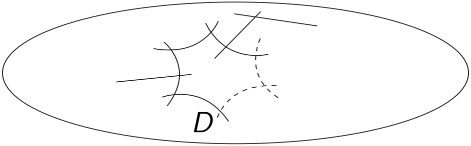



Normal form of the associated germ

$$\varphi(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d),$$

where $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ and $|\mathrm{tr}(A^2)| > 2$. The fundamental group $\pi_1(S \setminus (D_1 \cup D_2)) \simeq \mathbb{Z} \ltimes \mathbb{Z}^2$ is solvable. The universal covering $\widetilde{S \setminus (D_1 \cup D_2)}$ (or $\widetilde{S \setminus D}$) $\simeq \mathbb{H} \times \mathbb{C}$.

3) Intermediate surfaces ($D$ is an exceptional cycle of rational curves with at least one non-empty tree attached, in particular $D^2 < 0$.) **There are "generic" and "non-generic" blowups.**




The fundamental group $\pi_1(S \setminus D) \simeq \mathbb{Z} \ltimes \mathbb{Z}[\frac{1}{k}]$, with $k \geq 2$, is solvable. Here $k := 1 + \sqrt{|\det M(S)|} \in \mathbb{Z}, k \geq 2$, $M(S)$ being the intersection matrix of the $b_2(S)$ rational curves.

The universal covering $\widetilde{S \setminus D} \simeq \mathbb{H} \times \mathbb{C}$.

The normal form of the germ for intermediate surfaces will be discussed in detail in this talk.

Before doing so, we recall some recent results on class $\mathrm{VII}_0^+$ surfaces.

Every class $\mathrm{VII}_0^+$ surface $S$ admitting a non-trivial holomorphic vector field contains a GSS. (Dloussky-O-Toma)
In particular, this finishes the classification of compact complex surfaces with a strictly positive-dimensional automorphism group.

Every class $\mathrm{VII}_0^+$ surface $S$ admitting (at least) $b_2(S)$ rational curves contains a GSS, (Ma. Kato's conjecture), (Dloussky-O-Toma)

Every class $\mathrm{VII}_0^+$ surface $S$ with $b_2(S) = 1$ contains a GSS. (Teleman)

Every class $\mathrm{VII}_0^+$ surface $S$ with $b_2(S) = 2$ contains a cycle of rational curves. (Teleman)

We study *logarithmic moduli spaces*, i.e. parameter spaces of **intermediate surfaces** such that the maximal reduced divisor of rational curves $D$ is preserved. Logarithmic moduli spaces of Enoki surfaces were described by Dloussky-Kohler, whereas Inoue-Hirzebruch surfaces are logarithmically rigid (Nakamura).

**We shall examine the fundamental group of the complement of the maximal reduced divisor $D$, normal forms of the associated germs, the Dloussky sequence and the dual graph of the rational curves.**

**Finally results on moduli spaces will be presented.**

# INTERMEDIATE SURFACES : The index and the fundamental group of $S \setminus D$

Surface $S$ admitting a GSS.
One has

$$Pic^0(S) = (H^1(S, \mathcal{O}^*))^0 \simeq H^1(S, \mathbb{C}^*) \simeq \mathbb{C}^* \simeq Hom(\pi_1(S), \mathbb{C}^*).$$

For each $\lambda \in \mathbb{C}^*$ there is a unique associated flat line bundle denoted by $L_\lambda$.

For an intermediate surface $S$ there exists an integer $m \geq 1$, a flat line bundle $L$ and an effective divisor $D_m$ such that $(K_S \otimes L)^{\otimes m} = \mathcal{O}_S(-D_m)$. (Nakamura, Dloussky)

Smallest possible $m = m(S) = INDEX$ of the surface $S$

For each intermediate surface $S$ of index $m$ there is a unique intermediate surface $S'$ and a proper map $S' \to S$ which is generically finite of degree $m$ such that $S'$ is of index 1. Moreover $S' \setminus D' \to S \setminus D$ is a cyclic unramified covering of degree $m$.

The fundamental group of $S \setminus D$

**Surfaces of index** $1$

The universal cover $\widetilde{S \setminus D}$ of $S \setminus D$ is isomorphic to $\mathbb{C} \times \mathbb{H}_l$, where $\mathbb{H}_l := \{w \in \mathbb{C} \mid \Re e(w) < 0\}$ is the left half plane. Let $(z, w)$ be coordinates on $\mathbb{C} \times \mathbb{H}_l$. Then

$$\pi_1(S \setminus D) \simeq \mathbb{Z} \ltimes \mathbb{Z}[1/k].$$

Generated by :

$$\begin{cases} g_\gamma(z, w) & = & (z, w + 2\pi i) \\ g(z, w) & = & (\lambda z + a_0 + Q(e^{-w}), kw), \end{cases}$$

where $Q = Q(\zeta) := \sum_{m=l}^{\sigma} b_m \zeta^m$ is a polynomial such that $b_\sigma = 1$, $k \nmid \sigma$, $l := [\sigma/k] + 1$, $\gcd\{k, m \mid b_m \neq 0\} = 1$ and $(\lambda - 1)a_0 = 0$.

One has $g \circ g_\gamma \circ g^{-1} = g_\gamma^k$.

The fundamental group of $S \setminus D$

**Surfaces of higher index**

All surfaces of higher index are holomorphic quotients of surfaces of index 1 by finite cyclic groups induced by maps of the form

$$(z, w) \mapsto \left( e^{\frac{\sigma 2\pi i}{q}} z, w - \frac{2\pi i}{q} \right).$$

The universal cover $\widetilde{S \setminus D}$ of $S \setminus D$ is again isomorphic to $\mathbb{C} \times \mathbb{H}_l$.
The fundamental group of $S \setminus D$ is still isomorphic to $\mathbb{Z} \ltimes \mathbb{Z}[1/k]$.

# INTERMEDIATE SURFACES : Contracting germs of holomorphic mappings

In his thesis in 2000, Favre constructed normal forms for contracting germs of holomorphic mappings $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, in particular for those germs which give rise to surfaces with global spherical shells.

His result which follows is essential for our considerations.

Remark that in 1994, Hubbard and Oberste-Vorth, in their study of Hénon automorphisms of $\mathbb{C}^2$, already encountered certain of these normal forms.

## Theorem

*To every intermediate surface is associated a polynomial germ in the following normal form :*

$$(CG) \qquad \varphi(z,\zeta) := (\lambda\zeta^s z + P(\zeta) + c\zeta^{\frac{sk}{k-1}}, \zeta^k),$$

*where* $k, s \in \mathbb{Z}$, $k > 1$ *as before,* $s > 0$, $\lambda \in \mathbb{C}^*$,
$P(\zeta) := c_j\zeta^j + c_{j+1}\zeta^{j+1} \ldots + c_s\zeta^s$ *is a complex polynomial,*
$0 < j < k$, $j \leq s$, $c_j = 1$, $c_{\frac{sk}{k-1}} := c \in \mathbb{C}$ *with* $c = 0$ *whenever* $\frac{sk}{k-1} \notin \mathbb{Z}$ *or* $\lambda \neq 1$ *and* $\gcd\{k, m \mid c_m \neq 0\} = 1$.
*Moreover, two polynomial germs in normal form (CG)* $\varphi$ *and*

$$\tilde{\varphi} := (\tilde{\lambda}\zeta^{\tilde{s}} z + \tilde{P}(\zeta) + \tilde{c}\zeta^{\frac{\tilde{s}\tilde{k}}{\tilde{k}-1}}, \zeta^{\tilde{k}})$$

*are conjugated if and only if there exists* $\varepsilon \in \mathbb{C}$ *with* $\varepsilon^{k-1} = 1$ *and* $\tilde{k} = k$, $\tilde{s} = s$, $\tilde{\lambda} = \varepsilon^s\lambda$, $\tilde{P}(\zeta) = \varepsilon^{-j}P(\varepsilon\zeta)$, $\tilde{c} = \varepsilon^{\frac{sk}{k-1}-j}c$, *with conjugating map* $(z,\zeta) \mapsto (\varepsilon^j z, \varepsilon\zeta))$.

## Remark

*Intermediate surfaces of index one correspond precisely to germs $\varphi$ in normal form (CG) such that $(k-1)|s$. Under this necessary condition there is a holomorphic vector field ($= \zeta^{\frac{s}{k-1}}\frac{\partial}{\partial z}$) on the surface iff $\lambda = 1$.*

## Remark

*The index $m(S)$ of an intermediate surface associated to a polynomial germ in normal form (CG) is*

$$m(S) = \frac{k-1}{\gcd(k-1, s)}.$$

**Motivating questions :**

1) Determine the parameters in the normal form which are essential for the configuration of the rational curves on $S$.

2) Given an intermediate surface $S$ by a polynomial germ in normal form :

$$(CG) \qquad \varphi(z,\zeta) := (\lambda\zeta^s z + P(\zeta) + c\zeta^{\frac{sk}{k-1}}, \zeta^k),$$

produce an algorithm to calculate $b_2(S)$.

3) Construct moduli spaces of intermediate surfaces with a fixed dual graph of the rational curves.

**Type** of a polynomial germ in normal form (CG).

## Definition

*For fixed* $k$ *and* $s$ *and for a polynomial germ* $\varphi(z,\zeta) := (\lambda\zeta^s z + P(\zeta) + c\zeta^{\frac{sk}{k-1}}, \zeta^k)$ *in normal form (CG) with* $P(\zeta) := \zeta^j + c_{j+1}\zeta^{j+1} + \dots + c_s\zeta^s$ *we define inductively the following finite sequences of integers* $j =: m_1 < \dots . < m_t \leq s$ *and* $k > i_1 > j_2 \dots > i_t = 1$ *by :*

i) $m_1 := j$, $i_1 := \gcd(k, m_1)$ ;

ii) $m_\alpha := \min\{m > m_{\alpha-1} \mid c_m \neq 0, \gcd(i_{\alpha-1}, m) < i_{\alpha-1}\}, i_\alpha := \gcd(k, m_1, \dots, m_\alpha) = \gcd(i_{\alpha-1}, m_\alpha)$ ;

iii) $1 = i_t := \gcd(k, m_1, \dots, m_{t-1}, m_t) < \gcd(k, m_1, \dots, m_{t-1})$.

*We call* $(m_1, \dots, m_t)$ *the* type *of* $\varphi$ *and* $t$ *the* length of the type. *If* $t = 1$ *we say that* $\varphi$ *is of* simple type. *This is the case, if and only if* $k$ *and* $j$ *are relatively prime,* $\gcd(k, j) = 1$.

One also sets

$$\varepsilon(k, m_1, ..., m_t, s) := \left\lfloor \frac{m_2 - m_1}{i_1} \right\rfloor + ... + \left\lfloor \frac{m_t - m_{t-1}}{i_{t-1}} \right\rfloor + s - m_t.$$

This is the number of coefficients of $P$ whose vanishing or non-vanishing does not affect the type. All other coefficients $< m_t$ necessarilly vanish except $m_1, ..., m_t$ which are non-zero.

The type is preserved by the conjugations appearing in the previous theorem.

For fixed $k$, $s$ and fixed type $(m_1, \ldots, m_t)$ we consider the following parameter spaces for the coefficients $(\lambda, c_{j+1}, \ldots, c_s, c)$ appearing in (CG) :

- when $(k-1)$ does not divide $s$

$$U_{k,s,m_1,\ldots,m_t} = \mathbb{C}^* \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)},$$

- in case $(k-1) \mid s$

$$U^{\lambda \neq 1, c=0}_{k,s,m_1,\ldots,m_t} = \mathbb{C} \setminus \{0,1\} \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)},$$

$$U^{\lambda=1}_{k,s,m_1,\ldots,m_t} = (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)} \times \mathbb{C}.$$

In the second case we have considered separately the parameter spaces for surfaces without and with holomorphic vector fields.

# INTERMEDIATE SURFACES : Decomposition of germs

## Proposition (simplified version)

*Let* $\varphi(z,\zeta) := (\lambda\zeta^s z + P(\zeta) + c\zeta^{\frac{sk}{k-1}}, \zeta^k)$ *be a germ in normal form and whose type has length* $t$. *Then* $\varphi$ *admits a canonical decomposition* $\varphi = \varphi_1 \circ \ldots \circ \varphi_t$ *into* $t$ *polynomial germs in normal form and of simple type.*

## Remark

*If the germ is of simple type, then there is exactly one tree in the configuration of rational curves of the associated intermediate surface.*

*In general, the number of trees in this configuration is given by the length* $t$ *of the type of the germ.*

# INTERMEDIATE SURFACES : The blow-up process and the Dloussky sequence

A *simple* Dloussky sequence is associated to an intermed. surface with exactely one tree in the rational curve configuration and is of the form

$$[\mathrm{DlS}] \;=\; [\; \overbrace{a_1+2, \underbrace{2,..,2}_{a_1-1}}^{singular\ subsequence}\;, ..., \overbrace{a_{q+1}+2, \underbrace{2,..,2}_{a_{q+1}-1}}^{singular\ subsequence},\;\; \overbrace{\underbrace{2,..,2}_{m}}^{regular\ subs.}\;],$$

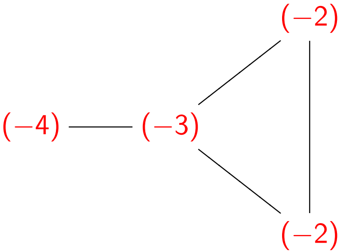
with $q \geq 0$, $a_i \geq 1$ for $1 \leq i \leq q+1$ and $m \geq 1$. The entries of the sequence represent the rational curves in the order of creation by the blow-up process, the integer giving the negative of the self-intersection number.

A general Dloussky sequence is of the form $[\mathrm{DlS}_1, ...., \mathrm{DlS}_t]$, where $[\mathrm{DlS}_j], j = 1, ..., t$ are simple Dloussky sequences. One has $t$ trees.

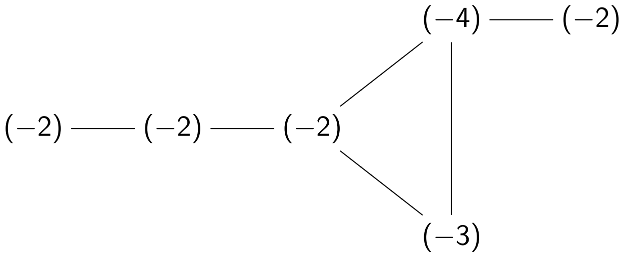
**Construction of the dual graph :** The entries of the sequence represent the knots of the graph. An entry with value $\alpha + 2$ is connected with the entry following $\alpha + 1$ places after it at the right hand (with the entries in cyclic order !).

Conversely, the dual graph of the rational curve configuration of a surface of intermediate type determines a sequence which is unique up to a cyclic permutation of its simple subsequences.

**Example 1 :** The simple sequence $[3, 4, 2, 2]$ produces the graph



In the same way the non-simple sequence
$[\mathrm{DlS}_1, \mathrm{DlS}_2] = [\underbrace{3, 2}_{\mathrm{DlS}_1}, \underbrace{4, 2, 2, 2}_{\mathrm{DlS}_2}]$ produces the graph



Let now $\varphi(z,\zeta) := (\lambda\zeta^s z + P(\zeta) + c\zeta^{\frac{sk}{k-1}}, \zeta^k)$ where $P(\zeta) := c_j\zeta^j + c_{j+1}\zeta^{j+1} ... + c_s\zeta^s$ be a germ in normal form and suppose that $\gcd(j,k) = 1$, that is, $\varphi$ is of simple type.
We use the two standard blow up coordinate charts $\eta : (u,v) \to (uv, v)$ and $\eta' : (u',v') \to (v', u'v')$ to "resolve" the germ in the form $\Pi \circ \sigma$. (The new curve is given always by $v = 0$ respectively by $v' = 0$.)

**Description of the algorithm**

A singular subsequence of length $a$ starts with $\eta'$, continues with $(a-1)$ times $\eta$ and is stopped by either $\eta'$ which starts a new singular subsequence or by a map of the form $(u,v) \to ((u+x)v, v), x \neq 0$ which starts a regular subsequence in the end of a simple sequence.

## Simple example

$$\sigma(z,\zeta) = \left(\frac{-2z-\zeta z^2}{(1+\zeta z)^3}, \zeta(1+\zeta z)\right) \stackrel{\eta}{\mapsto} \left(\frac{-2z\zeta-\zeta^2 z^2}{(1+\zeta z)^2}, \zeta(1+\zeta z)\right) =$$

$$= \left(\frac{1}{(1+\zeta z)^2} - 1, \zeta(1+\zeta z)\right) \stackrel{(u,v)\mapsto((u+1)v,v)}{\longmapsto} \left(\frac{\zeta}{(1+\zeta z)}, \zeta(1+\zeta z)\right)$$

$$\stackrel{\eta'}{\mapsto} (\zeta^2 z + \zeta, \zeta^2) = \varphi(z,\zeta). \text{ Dloussky sequence } [3,2,2].$$

In fact, this germ $\varphi$ is the normal form of the germ given by the birational Henon morphism

$$[x:y:t] \mapsto [x^2 - 2yt : xt : t^2]$$

at the superattractive fix point $[1:0:0] \in \mathbb{P}_2(\mathbb{C})$.

In general, this "resolution" induces the division algorithm for $k$ and $j$ :

$$\begin{aligned} k &= \alpha_1 j + \beta_1 \\ j &= \alpha_2 \beta_1 + \beta_2 \\ \beta_1 &= \alpha_3 \beta_2 + \beta_3 \\ &\vdots \\ \beta_{q-2} &= \alpha_q \beta_{q-1} + 1 \\ \beta_{q-1} &= \alpha_{q+1} \cdot 1 + 0 \end{aligned}$$

with the convention $k = \beta_{-1}, j = \beta_0$.

This algorithm shows that the sequence of the associated surface is simple and given by

$$[\mathrm{DlS}] = [\alpha_1+2, \underbrace{2,..,2}_{\alpha_1-1}, ..., \alpha_q+2, \underbrace{2,..,2}_{\alpha_q-1}, \alpha_{q+1}+1, \underbrace{2,..,2}_{\alpha_{q+1}-2}, \underbrace{2,..,2}_{s-j+1}] =$$

$$= [s_{\alpha_1}, ..., s_{\alpha_q}, s_{\alpha_{q+1}-1}, r_{s-j+1}].$$

Furthermore this algorithm gives the second Betti number of the surface as

$$b_2 = (\sum_{i=1}^{q+1} \alpha_i - 1) + (s-j+1) = (\sum_{i=1}^{q+1} \alpha_i) + (s-j)$$

which is the *length* of $[\mathrm{DlS}]$.

For the general case let $\varphi = \varphi_1 \circ \varphi_2 \circ ... \circ \varphi_t$ be the decomposition of a germ $\varphi$ into germs of simple type and $\mathrm{DlS}$ resp. $\mathrm{DlS}_i$ the associated Dloussky sequences of $\varphi$ resp. $\varphi_i, i = 1, ..., t$. A calculation shows that

$$[\mathrm{DlS}] = [\mathrm{DlS}_1, ...., \mathrm{DlS}_t],$$

i.e. the operations of composition of germs and concatenations of Dloussky sequences are compatible and that there is an algorithm to calculate the second Betti number for a given germ $\varphi$.
**The following three objects associated to an intermediate surface are algorithmically computatble from each other one : the dual graph of the rational curves, the Dloussky sequence, the type of the contracting germ in normal form.**

# INTERMEDIATE SURFACES : Moduli Spaces

We have seen before that for fixed $k$, $s$ and fixed type $(m_1, \ldots, m_t)$ we get parameter spaces for germs in normal form (CG)

$$U_{k,s,m_1,\ldots,m_t} = \mathbb{C}^* \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)},$$

when $(k-1) \nmid s$ and

$$U^{\lambda\neq 1, c=0}_{k,s,m_1,\ldots,m_t} = \mathbb{C} \setminus \{0,1\} \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)},$$

$$U^{\lambda=1}_{k,s,m_1,\ldots,m_t} = (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)} \times \mathbb{C},$$

when $(k-1) \mid s$. In the case $(k-1) \mid s$ the spaces $U^{\lambda\neq 1, c=0}_{k,s,m_1,\ldots,m_t}$ and $U^{\lambda=1}_{k,s,m_1,\ldots,m_t}$ appear as subspaces of

$$U_{k,s,m_1,\ldots,m_t} = \mathbb{C}^* \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)} \times \mathbb{C},$$

which parameterizes germs $(z,\zeta) \mapsto (\lambda\zeta^s z + P(\zeta) + c\zeta^{\frac{sk}{k-1}}, \zeta^k)$ in normal form.

By the preceding section all these germs have the same *logarithmic type*, i.e. the same dual graph of rational curves.
On the other hand, every intermediate surface with such a configuration of curves corresponds to a germ of this type.

Now, one may perform the blow-ups and the glueing simultaneously over the parameter space $U_{k,s,m_1,\dots,m_t}$ thus obtaining a holomorphic family $\mathcal{S}_{k,s,m_1,\dots,m_t} \to U_{k,s,m_1,\dots,m_t}$ of intermediate surfaces over $U_{k,s,m_1,\dots,m_t}$.
Then $\mathcal{S}_{k,s,m_1,\dots,m_t}$ is a complex manifold of dimension $\dim U_{k,s,m_1,\dots,m_t} + 2 = t + \varepsilon(k, m_1, \dots, m_t, s) + \delta + 2$, where $\delta = 1$ if $(k-1) \mid s$ and $\delta = 0$ otherwise.

Consider now an intermediate surface $S$ with maximal effective reduced divisor $D$. We would like to consider families of logarithmic deformations (Kawamata, 1978).
A *family of logarithmic deformations* for the pair $(S, D)$ is a 6-tuple $(\mathcal{S}, \mathcal{D}, \pi, V, v, \psi)$, where $\mathcal{D}$ is a divisor on $\mathcal{S}$, $\pi : \mathcal{S} \to V$ is a proper smooth morphism of complex spaces, which is locally a projection as well as its restriction to $\mathcal{D}$, $v \in V$ and $\psi : S \to \pi^{-1}(v)$ is an isomorphism restricting to an isomorphism $S \setminus D \to \pi^{-1}(v) \setminus \mathcal{D}$.

Let $T_S(-\log D)$ be the *logarithmic tangent sheaf* of $(S, D)$ (sheaf of vector fields tangent to $D$). Versal logarithmic deformations of intermediate surfaces $(S, D)$ exist, since $D$ has only simple normal crossings and their tangent space is $H^1(S, T_S(-\log D))$. (Kawamata) The space $H^2(S, T_S(-\log D)) = 0$ (Nakamura). Therefore the basis of the versal logarithmic deformation of a pair $(S, D)$ is smooth. On the other side $H^0(S, T_S(-\log D)) = H^0(S, T_S)$ and this space is at most one dimensional.

## Theorem

*With the above notations we have :*

- *If* $(k-1)$ *does not divide* $s$ *the family*

$$\mathcal{S}_{k,s,m_1,\dots,m_t} \to U_{k,s,m_1,\dots,m_t}$$

  *is logarithmically versal around every point of* $U_{k,s,m_1,\dots,m_t}$.
- *If* $(k-1) \mid s$, *the restriction of the family*

$$\mathcal{S}_{k,s,m_1,\dots,m_t} \to U_{k,s,m_1,\dots,m_t}$$

  *to* $U^{\lambda\neq 1, c=0}_{k,s,m_1,\dots,m_t}$ *is logarithmically versal around every point of* $U^{\lambda\neq 1, c=0}_{k,s,m_1,\dots,m_t}$.
- *If* $(k-1) \mid s$, *the family* $\mathcal{S}_{k,s,m_1,\dots,m_t} \to U_{k,s,m_1,\dots,m_t}$ *is logarithmically versal around every point of* $U^{\lambda=1}_{k,s,m_1,\dots,m_t}$.

Proof :

Take $(S, D)$ a pair as above and $\varphi(z,\zeta) := (\lambda\zeta^s z + P(\zeta), \zeta^k)$ an associated germ in normal form (CG). We see $\varphi$ as a point in $U_{k,s,m_1,\dots,m_t}$. Let $(\mathcal{S}', \mathcal{D}', \pi, V, v, \psi)$ be the logarithmically versal deformation of the pair $(S, D)$. Since there are no holomorphic vector fields, this deformation is modular around each point, $(*)$.

Then the family $\mathcal{S}_{k,s,m_1,\dots,m_t} \to U_{k,s,m_1,\dots,m_t}$ is obtained from $\mathcal{S}' \to V$ by base change by means of a map $F : (U_{k,s,m_1,\dots,m_t}, \varphi) \to (V, v)$. Now $F$ has finite fibres. Thus $\dim U_{k,s,m_1,\dots,m_t} \leq \dim V$. Because of $(*)$, one gets $\dim U_{k,s,m_1,\dots,m_t} = \dim V$.

Next we show that $F$ is injective near $\varphi$ hence locally biholomorphic. Assume the contrary.

Then there exists a root of unity $\varepsilon$ of order $q$ with $1 \neq q \mid (k-1)$ and a sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n(z,\zeta) := (\lambda_n \zeta^s z + P_n(\zeta), \zeta^k)$, converging to $\varphi$ in $U_{k,s,m_1,\dots,m_t}$ such that for $\tilde{\varphi}_n(z,\zeta) := (\varepsilon^s \lambda_n \zeta^s z + \varepsilon^{-m_1} P_n(\varepsilon\zeta), \zeta^k)$ one has $\tilde{\varphi}_n \to \varphi$ and $F(\tilde{\varphi}_n) = F(\varphi_n)$ for all $n \in \mathbb{N}$. Let $r = -m_1 - \lceil \frac{m_1}{q} \rceil q$ and $\chi(z,\zeta) = (\varepsilon^{-r} z, \varepsilon\zeta)$. Then $\tilde{\varphi}_n = \chi^{-1} \circ \varphi_n \circ \chi$ for all $n \in \mathbb{N}$. Thus $\chi$ induces an automorphism a non-trivial automorphism of the surface $S = S_\varphi$. On the other side $\chi$ induces for each $n \in \mathbb{N}$ the isomorphisms $S_{\varphi_n} \cong S'_{F(\varphi_n)} \cong S_{\tilde{\varphi}_n}$. This gives a contradiction. The second assertion of the theorem has a completely analogous proof.

We recall now that there is a natural action of $\mathbb{Z}/(k-1)\mathbb{Z}$ on $U_{k,s,j=m_1,\dots,m_t}$ given by the conjugation with the map $(z,\zeta)\mapsto(\varepsilon^j z,\varepsilon\zeta)$, which permutes conjugated germs : through a generator of $\mathbb{Z}/(k-1)\mathbb{Z}$ a germ

$$\varphi(z,\zeta) := (\lambda\zeta^s z + P(\zeta) + c\zeta^{\frac{sk}{k-1}}, \zeta^k)$$

is mapped to

$$\varphi(z,\zeta) := (\varepsilon^s\lambda\zeta^s z + \varepsilon^{-m_1}P(\varepsilon\zeta) + \varepsilon^{\frac{sk}{k-1}-m_1}c\zeta^{\frac{sk}{k-1}}, \zeta^k),$$

where $\varepsilon$ is a primitive $(k-1)$-th root of the unity.

## Theorem

*Fix $k$, $s$ and a type $(m_1, \ldots, m_t)$ for polynomial germs in normal form (CG). Set $j = m_1$ as before.*

- *When $j < \max(s, k-1)$ the natural action of $\mathbb{Z}/(k-1)$ on*

$$U_{k,s,m_1,\ldots,m_t}, U^{\lambda\neq 1, c=0}_{k,s,m_1,\ldots,m_t} \text{ and } U^{\lambda=1}_{k,s,m_1,\ldots,m_t}$$

  *is effective.*
- *In the remaining case, i.e. when $j = k-1 = s$, the natural action of $\mathbb{Z}/(k-1)$ is effective on $U^{\lambda=1}_{k,s,m_1,\ldots,m_t} = \mathbb{C}$ and trivial on $U^{\lambda\neq 1, c=0}_{k,s,m_1,\ldots,m_t} = \mathbb{C} \setminus \{0, 1\}$.*

## Theorem

*Fix $k$, $s$ and a type $(m_1, ..., m_t)$ for polynomial germs in normal form (CG). Set $j = m_1$ as before.*

*The quotient spaces $U_{k,s,m_1,...,m_t}/(\mathbb{Z}/(k-1))$ when $(k-1) \nmid s$, and $U^{\lambda\neq 1, c=0}_{k,s,m_1,...,m_t}/(\mathbb{Z}/(k-1))$, $U^{\lambda=1}_{k,s,m_1,...,m_t}/(\mathbb{Z}/(k-1))$, when $(k-1) \mid s$, are coarse logarithmic moduli spaces for intermediate surfaces of the given logarithmic type without, respectively with, non-trivial holomorphic vector fields.*

*These spaces are fine moduli spaces if and only if either the corresponding action of $\mathbb{Z}/(k-1)$ is trivial or this action is free. The natural action of $\mathbb{Z}/(k-1)$ on either of the spaces*

$$U_{k,s,m_1,...,m_t}, U^{\lambda\neq 1, c=0}_{k,s,m_1,...,m_t} \quad \text{and} \quad U^{\lambda=1}_{k,s,m_1,...,m_t}$$

*is free if and only if $\gcd(k-1, s, m_2 - j, ..., m_t - j) = 1$.*