Asymptotic behaviors of metrics on the moduli space of curves

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$T_{g,n}$: the Teichmüller space of curves of genus g with n marked points $(2g-2+n>0)$

$C_{g,n}$: the Teichmüller curve over $T_{g,n}$ with projection $\pi : C_{g,n} \to T_{g,n}$ and n sections $\mathbf{P}_1, \ldots, \mathbf{P}_n$ corresponding to n marked points

$\Omega^1_{C_{g,n}}$(resp.$\Omega^1_{T_{g,n}}$) : the sheaf of holomorphic 1-forms on $C_{g,n}$ (resp. $T_{g,n}$)

$\omega_{C_{g,n}/T_{g,n}} := \Omega^1_{C_{g,n}}/\pi^*\Omega^1_{T_{g,n}}$

the sheaf of relative differential forms on $C_{g,n}$

For $l \in \mathbf{N}$,

$\lambda_l := \bigwedge^{\max} R^0\pi_*\omega^{\otimes l}_{C_{g,n}/T_{g,n}}((l-1)\mathbf{P}_1 + \cdots + (l-1)\mathbf{P}_n)$

the determinant line bundle on $T_{g,n}$

For a point $s \in T_{g,n}$,
$S := \pi^{-1}(s)$ a compact smooth curve
$S^0 := S - \{\mathbf{P}_1(s), \ldots, \mathbf{P}_n(s)\}$
$P_p := \mathbf{P}_p(s)$ $(p = 1, \ldots, n)$

$R^0\pi_*\omega^{\otimes l}_{C_{g,n}/T_{g,n}}((l-1)\mathbf{P}_1+\cdots+(l-1)\mathbf{P}_n)_s$
$=\Gamma(S, K_S^{\otimes l}\otimes\mathcal{O}_S(P_1+\cdots+P_n)^{\otimes(l-1)})$
$=\{$meromorphic l differentials on S with possibly poles of order at most $l-1$ only at the marked points$\}$

Pick a basis of local holomorphic sections $\phi_1,\ldots,\phi_{d(l)}$ for $R^0\pi_*\omega^{\otimes l}_{C_{g,n}/T_{g,n}}((l-1)\mathbf{P}_1+\cdots+(l-1)\mathbf{P}_n)$, where

$$d(l)=\begin{cases} g & (l=1)\\ (2l-1)(g-1)+(l-1)n & (l>1).\end{cases}$$

$$\langle\phi_i,\phi_j\rangle := \iint_{S^0}\phi_i\,\overline{\phi_j}\,\rho_{S^0}^{-(l-1)}\quad (i,j=1,\ldots d(l))$$

the Petersson product, where ρ_{S^0} is the hyperbolic area element on S^0.

$$Z_{S^0}(s):=\prod_{\{\gamma\}}\prod_{m=1}^{\infty}\left(1-e^{-(s+m)L(\gamma)}\right)$$

the Selberg Zeta function for S^0, Re $(s)>1$, where γ runs over all oriented primitive closed geodesics on S^0, and $L(\gamma)$ denotes the hyperbolic length of γ. It extends meromorphically to the whole plane in s.

$\|\phi_1\wedge\cdots\wedge\phi_{d(l)}\|_{L^2}:=|det(\langle\phi_i,\phi_j\rangle)|^{1/2}$

$\|\phi_1\wedge\cdots\wedge\phi_{d(l)}\|_Q:=\|\phi_1\wedge\cdots\wedge\phi_{d(l)}\|_{L^2}Z_{S^0}(l)^{-1/2}$

$(l\geq 2.$ For $l=1$, employ $Z'_{S^0}(1)$ in place of $Z_{S^0}(1)=0.)$

Here, $Z_{S^0}(l)$ denotes the special value of $Z_{S^0}(\cdot)$ on S^0 at l integer.

$\lambda_l \to T_{g,n}$ is a Hermitian holomorphic line bundle equipped with the Quillen metric $\|\cdot\|_Q$.

Theorem 1 (Belavin-Knizhnik (1986)+Wolpert(1986))**.**

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2}\omega_{WP} \quad (n = 0).$$

Theorem 2 (Takhtajan-Zograf (1988, 1991))**.**

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2}\omega_{WP} - \frac{1}{9}\omega_{TZ} \quad (n > 0).$$

Here, ω_{WP}, ω_{TZ} are the Kähler forms of the Weil-Petersson, the Takhtajan-Zograf metrics respectively.

$\mathrm{Mod}_{g,n}$: the mapping class group of curves of genus g with n marked points

$\mathcal{M}_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$: the moduli space of curves of genus g with n marked points

λ_l and all metrics we defined are compatible with the action of $\mathrm{Mod}_{g,n}$, then they all naturally descend to $\mathcal{M}_{g,n}$ as orbifold line sheaves and orbifold metrics respectively.

Theorem 3 (Weng (2001)**).**
We have an isometric decomposition of the determinant line bundle with appropriate hermitian metrics $(2g-2+n>0, n>0)$.

$$\lambda_l^{\otimes 12} \simeq \Delta_{WP}^{\otimes 6l^2-6l+1} \otimes \Delta_{TZ}^{-1},$$
$$c_1(\Delta_{WP}) = \frac{\omega_{WP}}{\pi^2}, \quad c_1(\Delta_{TZ}) = \frac{4}{3}\omega_{TZ}.$$

Theorem 4 (Wolpert (1986), Takhtajan-Zograf (1991)**).**
For $g>2$,

$$H^2(\mathcal{M}_g, \mathbf{Z}) \simeq \mathbf{Z} \simeq \left\langle \frac{[\omega_{WP}]}{\pi^2} \right\rangle,$$
$$H^2(\mathcal{M}_{g,1}, \mathbf{Z}) \simeq \mathbf{Z}^2 \simeq \left\langle \frac{[\omega_{WP}]}{\pi^2}, \frac{4}{3}[\omega_{TZ}] \right\rangle.$$

Here, $\mathcal{M}_g = \mathcal{M}_{g,0}$.

Theorem 5 (Weng (2001), Wolpert (2007)**).**
For $2g-2+n>0, n>0$,

$$c_1(\Delta_p) = \frac{4}{3}[\omega_p].$$

Here, Δ_p *denotes the line bundle associated with the* p*-th marked point over* $T_{g,n}$. ω_p *denotes the Kähler form of the Takhtajan-Zograf metric associated with the* p*-th marked point.*

Theorem 6 (Weng (2001), Wolpert (2007) + Harer)**.**
For $g > 2, n > 0$*,*

$$H^2(\mathcal{M}_{g,n}, \mathbf{Z}) \simeq \mathbf{Z}^{n+1} \simeq \left\langle \frac{[\omega_{WP}]}{\pi^2}, \frac{4}{3}[\omega_1], \frac{4}{3}[\omega_2] \ldots, \frac{4}{3}[\omega_n] \right\rangle.$$

$\overline{\mathcal{M}}_{g,n}$: the Deligne -Mumford compactification of $\mathcal{M}_{g,n}$

Theorem 7 (Saper (1993))**.**
For $g > 1, n = 0$*,*

$$H^*_{(2)}(\mathcal{M}_g, \omega_{WP}) \simeq H^*(\overline{\mathcal{M}}_g, \mathbf{R}).$$

Here, the left hand side is the L^2-*cohomology with respect to the W-P metric.*

Consider the asymptotic behavior of the W-P metric and the T-Z metric near the boundary of $\mathcal{M}_{g,n}$.

$D = \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$: the compactification divisor

$R_0 \in D$: a stable curve of genus g with n marked points and k nodes

(we regard the marked points as deleted from the surface.)

Each node q_i $(i = 1, 2, \ldots, k)$ has a neighborhood $N_i = \{(z_i, w_i) \in \mathbf{C}^2 \mid |z_i|, |w_i| < 1, z_i w_i = 0\}$.

W_p : a neighborhood of the marked point P_p $(p = 1. \ldots, n)$

τ_μ : a basis of Beltrami differentials with compact support in

$$R_0 / \bigcup_{i=1}^{k} N_i \bigcup \bigcup_{p=1}^{n} W_p \ (\mu = k+1, k+2, \ldots, 3g-3+n)$$

$f_s : R_0 \longrightarrow R_s$ is a homeomorphism with Beltrami differential $\tau(s) = \sum_{\mu=k+1}^{3g-3+n} s_\mu \tau_\mu$ $(s_\mu \in \mathbf{C})$, satisfying $\overline{\partial} f_s = \tau(s)\, \partial f_s$.

By a natural identification via f_s, z_i, w_i, N_i, W_p can be regarded as those for R_s.

$R_{t,s}$: the smooth surface gotten from R_s after cutting and pasting N_i to make $N_{i,t} = \{(z_i, w_i) | z_i w_i = t_i, |t_i| < 1\}$

Then, D is locally described as $\{t_1 = \cdots = t_k = 0\}$.

By the deformation theory of Kodaira-Spencer and the Hodge theory, for $[R_{t,s}] \in T_{g,n}$,

$$T_{[R_{t,s}]}T_{g,n} \simeq HB(R_{t,s}),$$

where $HB(R_{t,s})$ is the space of harmonic Beltrami differentials on $R_{t,s}$.

By the Serre duality,

$$T^*_{[R_{t,s}]}T_{g,n} \simeq Q(R_{t,s}),$$

where $Q(R_{t,s})$ is the space of holomorphic quadratic differentials on $R_{t,s}$ with finite the Petersson-norm, which is dual to $HB(R_{t,s})$.

The inner product of the Weil-Petersson metric at $T_{[R_{t,s}]}T_{g,n}$ is defined as ($\rho_{t,s}$ denotes the hyperbolic area element of $R_{t,s}$.)

$$\langle \alpha, \beta \rangle_{WP}(t,s) := \iint_{R_{t,s}} \alpha\overline{\beta}\ \rho_{t,s} \qquad (\alpha, \beta \in T_{[R_{t,s}]}T_{g,n}).$$

α_i (resp. β_μ): Beltrami differentials on $R_{t,s}$ representing $\partial/\partial t_i$ (resp. $\partial/\partial s_\mu$)

$$g_{i\bar{j}}(t,s) = \langle \alpha_i, \alpha_j \rangle_{WP}(t,s),$$
$$g_{i\overline{\mu}}(t,s) = \langle \alpha_i, \beta_\mu \rangle_{WP}(t,s),$$
$$g_{\mu\overline{\nu}}(t,s) = \langle \beta_\mu, \beta_\nu \rangle_{WP}(t,s),$$

$(i, j = 1, 2, \ldots, k,\ \mu, \nu = k+1, \ldots, 3g-3+n)$

the Riemannian tensors for the W-P metric.

The inner products of the Takhtajan-Zograf metrics are defined as

$$\langle \alpha, \beta \rangle_p(t,s) := \iint_{R_{t,s}} \alpha\overline{\beta} E_p(\cdot, 2)\ \rho_{t,s} \qquad (\alpha, \beta \in T_{[R_{t,s}]}T_{g,n})$$

$(p = 1, \dots, n)$. Here, $E_p(\cdot, 2)$ is the Eisenstein series associated with the p-th marked point with index 2.

$$\langle \alpha, \beta \rangle_{TZ}(t,s) := \sum_{p=1}^{n} \langle \alpha, \beta \rangle_p(t,s).$$

$$\begin{aligned} h_{i\bar{j}}(t,s) &= \langle \alpha_i, \alpha_j \rangle_{TZ}(t,s), \\ h_{i\bar{\mu}}(t,s) &= \langle \alpha_i, \beta_\mu \rangle_{TZ}(t,s), \\ h_{\mu\bar{\nu}}(t,s) &= \langle \beta_\mu, \beta_\nu \rangle_{TZ}(t,s), \end{aligned}$$

$(i, j = 1, 2, \dots, k, \quad \mu, \nu = k+1, \dots, 3g-3+n)$
the Riemannian tensors for the T-Z metric.

The Eisenstein series associated with the p-th marked point with index 2 is defined as

$$E_p(z, 2) := \sum_{A \in \Gamma_p \backslash \Gamma} \left\{ \mathrm{Im}(\sigma_p^{-1} A(z)) \right\}^2, \text{ for } z \in \mathbf{H},$$

where $\mathbf{H}$ is the upper-half plane, Γ is a uniformizing Fuchsian group and Γ_p is the parabolic subgroup associated with the p-th marked point, and $\sigma_p \in \mathrm{PSL}(2, \mathbf{R})$ is a normalizer.

$E_p(z, 2)$ is a positive subharmonic function on $R_{t,s}$, and assumes the infinity at the p-th marked point and vanishes at the other marked points.

The following theorem is a pioneering result for the asymptotic behavior of the W-P metric near the boundary of the moduli space.

Theorem 8 (Masur (1976))**.**
As $t_i, s_\mu \to 0$,

$$i)\quad g_{i\bar{i}}(t,s) \approx \frac{1}{|t_i|^2(-\log|t_i|)^3}$$
for $i \le k$,

$$ii)\quad g_{i\bar{j}}(t,s) = O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)$$
for $i, j \le k, i \ne j$,

$$iii)\quad g_{i\overline{\mu}}(t,s) = O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right)$$
for $i \le k, \mu \ge k+1$,

$$iv)\quad \lim_{(t,s)\to(0,0)} g_{\mu\overline{\nu}}(t,s) = g_{\mu\overline{\nu}}(0,0)$$
for $\mu, \nu \ge k+1$.

Recently, we updated Masur's result by improving Wolpert's formula for the asymptotic of the hyperbolic metric for degenerating Riemann surfaces.

Theorem 9 (O. and Wolpert (2008))**.**
As $t_i \to 0$,

$iv)'$

$$g_{\mu\overline{\nu}}(t,s) = g_{\mu\overline{\nu}}(0,s)$$
$$+\frac{4\pi^4}{3}\sum_{i=1}^{k}(\log|t_i|)^{-2}\Big\langle\beta_\mu,(E_{i,1}(\cdot,2)+E_{i,2}(\cdot,2))\beta_\nu\Big\rangle_{WP}(0,s)$$
$$+O\Big(\sum_{i=1}^{k}(\log|t_i|)^{-3}\Big) \qquad \textit{for } \mu,\nu \geq k+1.$$

Here, $E_{i,1}(\cdot,2), E_{i,2}(\cdot,2)$ *denote a pair of the Eisenstein series with index* 2*, associated with the* i*-th node of the limit surface.*

That is, the T-Z metrics have appeared from degeneration of the W-P metric!

Proof of Theorem 9

Step 1. Consider the pluming family of hyperbolic metrics

$$0<c<1, \quad \Delta=\{t\in \mathbf{C}||t|<c\}$$

$$\Pi := \{(z,w,t)\in \mathbf{C}^3 \mid zw=t, |z|,|w|,|t|<c\}$$

$$\Phi : \Pi \longrightarrow \Delta,\ \Phi(z,w,t)=t.$$

Define a metric ds_t on each fiber $\Phi^{-1}(t)$

$$ds_t^2=\begin{cases}\left(\dfrac{\pi}{\log|t|}\csc\left(\dfrac{\pi\log|z|}{\log|t|}\right)\left|\dfrac{dz}{z}\right|\right)^2, & t\neq 0\\ \left(\dfrac{|dz|}{|z|\log|z|}\right)^2, & t=0.\end{cases}$$

The annulus $\Phi^{-1}(t)$ embeds holomorphically into the neighborhood U of a pinching geodesic on the smooth surface R_t.

Step 2. The intrinsic hyperbolic metric ds^2_{hyp} on R_t and ds_t^2 are naturally interpolated around the neighborhood U to make the grafting metric ds^2_{graft} on R_t.

By analyzing the solution to Kazdan-Warner's curvature equation on S_t, we can show that as $t\longrightarrow 0$,

$$ds^2_{hyp}=ds^2_{graft}\left(1+\frac{4\pi^4}{3}(\log|t|)^{-2}(E^1+E^2)+O((\log|t|)^{-3})\right).$$

E^1, E^2 are the Eisenstein series the singular parts of which are cut-off. That is, a pinching geodesic yields a pair of Eisenstein series associated with a pair of cusps through degenaration. The above expansion formula gives the updated asymptotic formula of W-P metric through degenaration.

On the other hand, we have a result for asymptotics of the T-Z metric near the boundary of the moduli space.

Theorem 10 (O.-To-Weng (2008))**.**
As $t_i, s_\mu \to 0$, we have the following estimates.

i) For any $\varepsilon > 0$, there exists a constant $C_{1,\varepsilon}$ such that

$$h_{i\bar{i}}(t,s) \leq \frac{C_{1,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4-\varepsilon}} \quad \textit{for } i \leq k.$$

For any $\varepsilon > 0$, there exists a constant $C_{2,\varepsilon}$ such that

$$h_{i\bar{i}}(t,s) \geq \frac{C_{2,\epsilon}}{|t_i|^2(-\log|t_i|)^{4+\varepsilon}} \quad \textit{for } i \leq k \textit{ and the node } q_i$$

adjacent to punctures.

ii) $h_{i\bar{j}}(t,s) = O\left(\dfrac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)$
for $i, j \leq k, i \neq j$.

iii) $h_{i\overline{\mu}}(t,s) = O\left(\dfrac{1}{|t_i|(-\log|t_i|)^3}\right)$ *for $i \leq k, \mu \geq k+1$.*

iv) $\displaystyle\lim_{(t,s)\to(0,0)} h_{\mu\overline{\nu}}(t,s) = h_{\mu\overline{\nu}}(0,0) \qquad$ *for $\mu, \nu \geq k+1$.*

Here, a node q is called to be **adjacent to punctures** *if the component of $R_0 - \{$all nodes$\}$ containing q also contains at least one of the marked points.*

Proof of Theorem 10

Use the settings and the plumbing family in the proof of Thm 9.

We introduce the test function for Eisenstein series on each fiber $\Phi^{-1}(t)$,

$$E_t^*(z) = \begin{cases} -\dfrac{\pi}{\log|t|}\csc\Big(\dfrac{\pi\log|z|}{\log|t|}\Big), & t \neq 0 \\ -\dfrac{1}{\log|z|}, & t = 0. \end{cases}$$

The maximal principle for subharmonic function and etc. permits us to show that, around the neighborhood of the pinching geodesic, $E_t^*(z)$ approximates 'very nicely' the Eisenstein series $E_t(z)$ on the punctured smooth surface R_t.

We can use $E_t^*(z)$ on the the neighborhood of the pinching geodesic to evaluate the T-Z metric tensors $h_{ij}(t, s)$'s.

Open problems

1. Determine $H^*_{(2)}(\mathcal{M}_{g,n}, \omega_{TZ})$ for general (g, n). For that, we need more informations on precise asymptotics of degenerating Eisenstein series.
(cf. O., *RIMS Kôkyûroku Bessatsu* (2009), to appear)

2. Is the curvature of the Takhtajan-Zograf metric negative?

3. If the answer to the question 2. is YES, study $-\mathrm{Ric}\ \omega_{TZ}$.
– Recently, C.T. McMullen, and K. Liu, X. Sun & S.-T. Yau find good geometry of the moduli of curves using $-\mathrm{Ric}\ \omega_{WP}$.
(*Ann. of Math.* **151** (2000), *Publ. RIMS, Kyoto* **42** (2008))

4. Does the Takhtajan-Zograf Kähler form have a global representation formula?
– The Weil-Petersson Kähler form has a global representation formula in terms of the Fenchel-Nielsen coordinates.
(S.A. Wolpert, *Amer. J. Math* **107** (1985))

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