A Unicity Theorem and Erdös' Problem for Polarized Semi-Abelian Varieties

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§1 Introduction

The following is a kind of the unicity problem in arithmetic recurrence:

Erdös' Problem (1988). *Let x, y be positive integers. Is it true that*

$$\{p; prime, p|(x^n - 1)\} = \{p; prime, p|(y^n - 1)\}, \forall n \in \mathbf{N}$$
$$\iff x = y \quad ?$$

The answer is Yes:

Theorem 1.1

(Corrales-Rodorigáñez and R. Schoof, JNT 1997)

1. $\{p; prime, p|(x^n - 1)\} \subset \{p; prime, p|(y^n - 1)\}, \forall n \in \mathbf{N}$

$$\Longrightarrow y = x^h \quad (\exists h \in \mathbf{N}).$$

2. *The elliptic version holds, too.*

In complex analysis, Yamanoi proved in Forum Math. 2004 the following striking unicity theorem:

Theorem 1.2

Let
$A_i, i = 1, 2$, *be abelian varieties;*
$D_i \subset A_i$ *be irreducible divisors such that*

$$\mathrm{St}(D_i) = \{a \in A_i; a + D_i = D_i\} = \{0\};$$

$f_i : \mathbf{C} \to A_i$ *be (algebraically) nondegenerate entire holomorphic curves.*
Assume that $f_1^{-1}D_1 = f_2^{-1}D_2$ *as sets.*
Then $\exists$ *isomorphism* $\phi : A_1 \to A_2$ *such that*

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^* D_2.$$

N.B.

1. The new point is that we can determine not only f, but the moduli point of a polarized abelian vareity (A, D) through the distribution of $f^{-1}D$ by a nondegenerate $f : \mathbf{C} \to A$.
2. The assumptions for D_i to be irreducible and the triviality of $\mathrm{St}(D_i)$ are not restrictive. There is a way of reduction.
3. For simplicity we assume them here.

§2 Main Results

We want to uniformize the results in the previous section.
Therefore we have to deal with semi-abelian varieties.
Let $A_i, i = 1, 2$ be semi-abelian varieties:

$$0 \to (\mathbf{C}^*)^{t_i} \to A_i \to A_{0i} \to 0.$$

Let D_i be an irreducible divisor on A_i such that

$$\mathrm{St}(D_i) = \{0\}$$

for simplicity.

Main Theorem

Main Theorem 2.1

Let $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ *be non-degenerate holomorphic curves. Assume that*

$$(2.2) \qquad \underline{\mathrm{Supp} f_1^* D_1}_{\infty} \subset \underline{\mathrm{Supp} f_2^* D_2}_{\infty} \text{ (germs at } \infty),$$

$$(2.3) \qquad N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2)||;$$

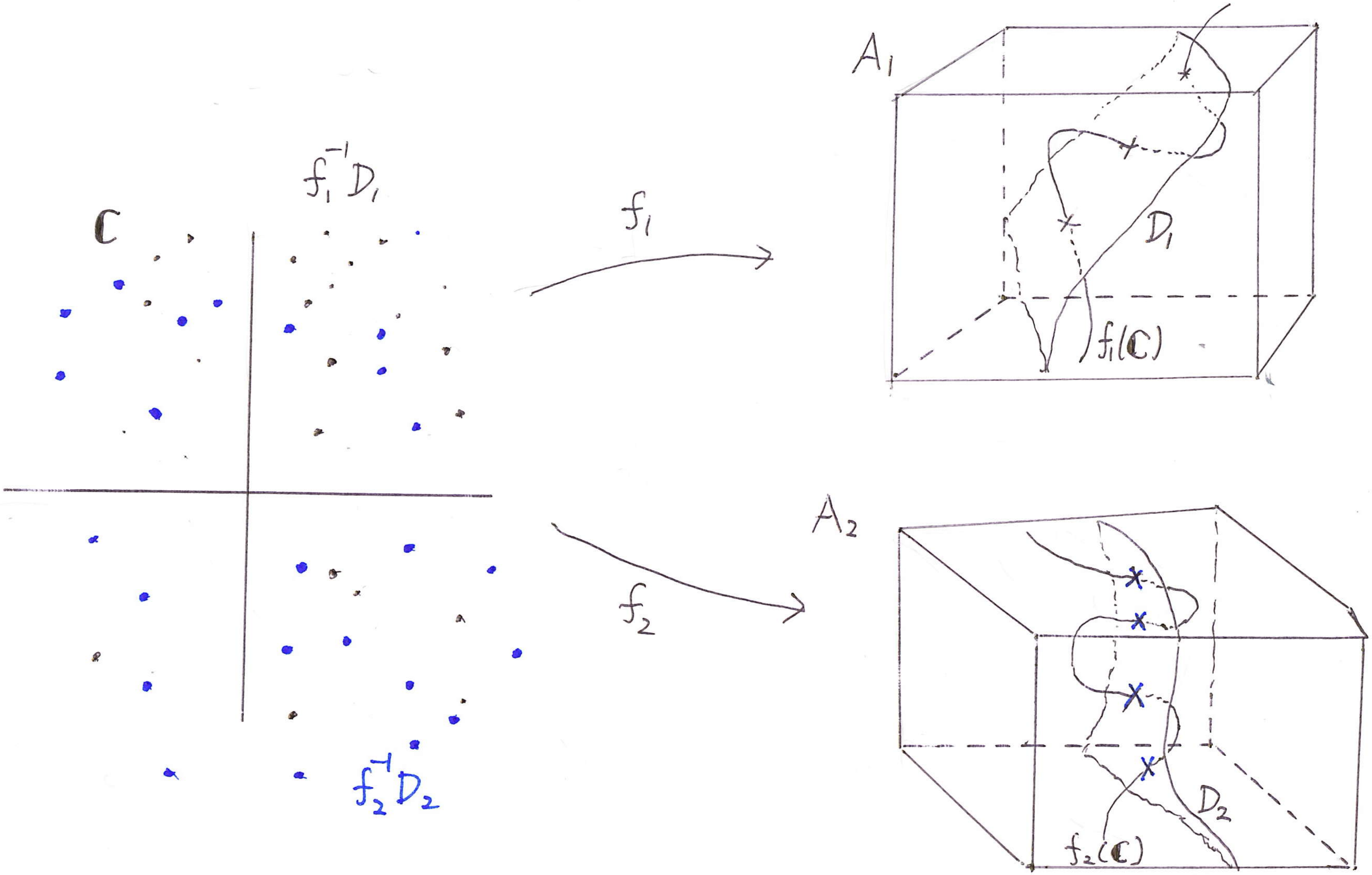
$$\text{i.e., } C^{-1} N_1(r, f_1^* D_1) \leq N_1(r, f_2^* D_2) \leq C N_1(r, f_1^* D_1)||, \ \exists C > 0.$$

Then there is a finite étale morphism $\phi : A_1 \to A_2$ *such that*

$$\phi \circ f_1 = f_2, \quad D_1 \subset \phi^* D_2.$$

If equality holds in (2.2), *then* ϕ *is an isomorphism and* $D_1 = \phi^* D_2$.

N.B. Assumption (2.3) is necessary by example.



$f_1^{-1}D_1$
$\mathbb{C}$
f_1
A_1
D_1
$f_1(\mathbb{C})$
A_2
f_2
$f_2^{-1}D_2$
D_2
$f_2(\mathbb{C})$

The following corollary follows immediately from the Main Theorem 2.1.

Corollary 2.4

1. *Let* $f : \mathbf{C} \to \mathbf{C}^*$ *and* $g : \mathbf{C} \to E$ *with an elliptic curve* E *be holomorphic and non-constant. Then*

$$\underline{f^{-1}\{1\}}_{\infty} \neq \underline{g^{-1}\{0\}}_{\infty}.$$

2. *If* $\dim A_1 \neq \dim A_2$ *in the Main Theorem 2.1, then*

$$\underline{f_1^{-1}D_1}_{\infty} \neq \underline{f_2^{-1}D_2}_{\infty}.$$

N.B.

1. The first statement means that the difference of the value distribution property caused by the quotient $\mathbf{C}^* \to \mathbf{C}^*/\langle\tau\rangle = E$ cannot be recovered by the choice of f and g, even though they are allowed to be *arbitrarily transcendental*.

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{f} & \mathbf{C}^* \\ & \searrow_{g} & \downarrow /\langle\tau\rangle \\ & & E \end{array}$$

2. The second statement implies that the distribution of $f_i^{-1}D_i$ about ∞ contains the topological informations of $\dim A_i$ and the compact or non-compactness of A_i;
   It is interesting to observe that this works even for one parameter subgroups with Zariski dense image.

Due to the well-known correspondence between Number Theory and Nevanlinna Theory, it is tempting to give a number-theoretic analogue of Theorem 2.1 as Pál Erdös Problem–Corrales-Rodorigáñez&Schoof Theorem.

A related problem asks to classify the cases where $x^n - 1$ divides $y^n - 1$ for infinitely many positive integers n.

We would like to deal with the case of a semi-abelian variety with a given divisor, i.e., a polarized semi-abelian variety.

We can prove an analogue of the Main Theorem 2.1 in the linear toric case, but not in the general case of semi-abelian varieties, that is left to be a *Conjecture*.

§3 Arithmetic Recurrence

Here is our result in the arithmetic case.

Theorem 3.1

Let
$\mathcal{O}_S$ *be a ring of* S*-integers in a number field* k*;*
$\mathbf{G}_1$, $\mathbf{G}_2$ *be linear tori;*
$g_i \in \mathbf{G}_i(\mathcal{O}_S)$ *be elements generating Zariski-dense subgroups.*
D_i *be reduced divisors defined over* k*, with defining ideals* $\mathcal{I}(D_i)$*, such that each irreducible component has a finite stabilizer and* $\mathrm{St}(D_2) = \{0\}$*.*
Suppose that for infinitely many $n \in \mathbf{N}$*,*

$$(g_1^n)^* \mathcal{I}(D_1) \supset (g_2^n)^* \mathcal{I}(D_2). \tag{3.2}$$

Then $\exists$ *étale morphism* $\phi : \mathbf{G}_1 \to \mathbf{G}_2$*, defined over* k*, and* $\exists h \in \mathbf{N}$ *such that* $\phi(g_1^h) = g_2^h$ *and* $D_1 \subset \phi^*(D_2)$*.*

N.B.

1. Theorem 3.1 is deduced from the main results of Corvaja-Zannier, Invent. Math. 2002.
2. By an example we cannot take $h = 1$ in general.
3. By an example, the condition on the stabilzers of D_1 and D_2 cannot be omitted.
4. Note that inequality (inclusion) (3.2) of ideals is assumed only for an infinite sequence of n, not necessarily for all large n. On the contrary, we need the inequality of ideals, not only of their *supports*, i.e. of the primes containing the corresponding ideals.
5. One might ask for a similar conclusion assuming only the inequality of supports. There is some answer for it, but it is of a weaker form.

Proof of the Main Theorem

The next two theorems are crucial in the proof.

Theorem 4.1

(**Log Bloch-Ochiai**, Nog. 1977 Hiroshima Math.J./81 Nagoya Math.J.)
Let $f : \mathbf{C} \to A$ be a holomoprhic curve into a semi-abelian variety A. Then $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a translate of a subgroup.

Theorem 4.2

(**NWY-II** (N.-Winkelmann-Yamanoi) 2008 Forum Math.)
Let $f : \mathbf{C} \to A$ be nondegenerate;
$Z \subset A$ be an algebraic cycle on A.

1. *If* $\mathrm{codim}\, Z = 1$, $\exists \bar{A} \supset \bar{Z}$ *s.t.* $T_f(r, L(\bar{Z})) = N_1(r, f^*Z) + o(T_f(r))||$.
2. *If* $\mathrm{codim}\, Z \geq 2$, $T_{f,\bar{Z}}(r) = N(r, f^*Z) + O(\log r T_f(r)) = o(T_f(r))||$.

Proof.

With the given f_i $(i = 1, 2)$ in the Main Theorem we set
$g = (f_1, f_2) : \mathbf{C} \to A_1 \times A_2$;
$A_0 = \overline{g(\mathbf{C})}^{\mathrm{Zar}}$ (semi-abelian variety by **Log Bloch-Ochiai**);
$p_i : A_0 \to A_i$ be the projections;
$E_i = p_i^* D_i$.

We apply **NWY-II**, Theorem 4.2 for g and E_i.

We deduce that

1. $E_1 \subset E_2$,
2. $\mathrm{St}(E_1) \subset \mathrm{St}(E_2)$, and are finite,
3. p_i are isogenies,
4. $A_1 \cong A_0/\mathrm{St}(E_1) \xrightarrow{\phi} A_0/\mathrm{St}(E_2) \cong A_2$.

Problem

Characterize a polarized algebraic variety (V, D)
in terms of $f^{-1}D$ or $f^{*}D$
by a nondegenerate holomorphic curve $f : \mathbf{C} \longrightarrow V$.

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