# MONOPOLES AND THE LEBRUN-MASON TWISTOR CORRESPONDENCE

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## 1. Introduction

Twistor theory, originated by R. Penrose [16], has been studied for more than thirty years, and brought a large amount of harvests in many fields: differential geometry, complex geometry, mathematical physics and so on.

Penrose-type twistor theory study a moduli space $M$ of rational curves $C$ in a complex manifold $Z$, then $M$ often carries a natural structure which is the general solution of an interesting differential-geometric problem. Recently, in contrast, C. LeBrun and L. J. Mason investigated a new paradigm of twistor theory. In this new theory, we study a moduli space of *holomorphic curves-with-boundary* $C$ in $Z$ with boundaries lying on a totally real submanifold $P \subset Z$. This framework is well adapted to many problems in *global* differential geometry where the general solutions are of very *low regularity.*

At present, several cases of LeBrun-Mason twistor correspondences are known [8, 9, 10, 11, 14]. As a consequence of one of them, we find that there exist infinite-dimensionally many self-dual indefinite conformal structures on $S^2 \times S^2$ satisfying a certain strong global condition called *Zollfrei* condition. On the other hand, K. P. Tod [17] and H. Kamada [6] independently obtained infinitely many examples of self-dual indefinite metrics on $S^2 \times S^2$ via similar method as the Riemannian example obtained by LeBrun [7]. The natural question here is to determine whether the self-dual metrics obtained by Tod or Kamada are Zollfrei or not, and this turns out to be 'Yes' (Theorem 7.1).

In the investigation of the above problem, the author obtained, using the Radon transform, a general solution of the *wave equation* and the *monopole equation* over the de Sitter 3-space $S^3_1$ which is a Lorentzian manifold equipped with an Einstein metric. In future, as demonstrated in this article, LeBrun-Mason theory might take a good role for solving more complicated hyperbolic PDEs over pseudo-Riemannian manifolds, and for exploiting a new field of differential geometry.

## 2. LeBrun-Mason correspondence

The first result obtained by LeBrun and Mason is the following.

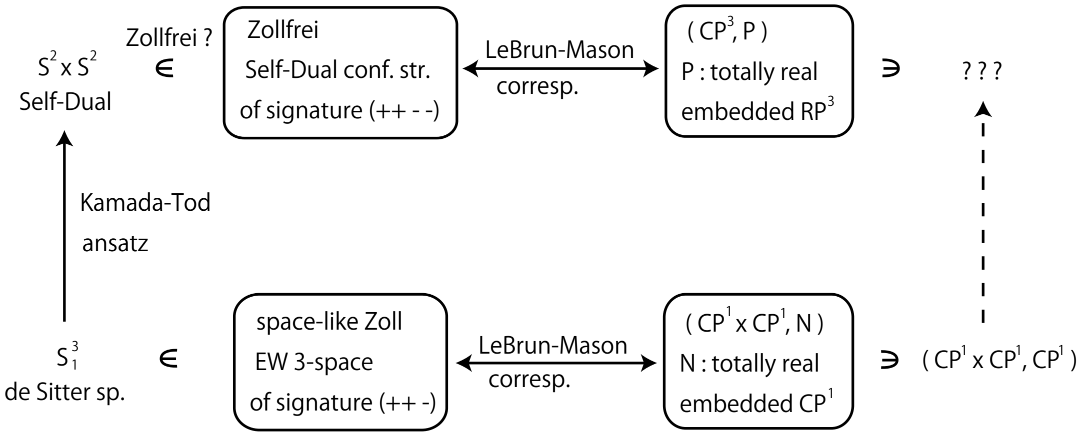
**Theorem 2.1** (LeBrun-Mason [9])**.** *There is a one-to-one correspondence between*

- *Zoll projective structures* $[\nabla]$ *on* $S^2$*, and*
- *totally real embeddings* $\iota : \mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$*,*

*in a neighborhood of the standard objects.*

A *projective structure* is an equivalence class of connections on the tangent bundle by which the notion of (un-parametrized) geodesic is well defined, and a projective structure is called **Zoll** iff all the maximal geodesics are closed. Notice that the Zoll condition is a *global* condition.

The correspondence in Theorem 2.1 is characterized in the following way. Suppose an embedding $\iota$ is given. Then we find a moduli space $M$ of **holomorphic disks** with boundaries lying on the totally real submanifold $\iota(\mathbb{RP}^2) \subset \mathbb{CP}^2$. If $\iota$ is sufficiently near to the standard embedding, $M$ turns out to be diffeomorphic to $S^2$. This moduli space $M$ is naturally equipped with a family of closed curves each of which corresponds to the set of disks with boundaries




passing through a fixed point $p \in \iota(\mathbb{RP}^2)$. Then the projective structure on $M$ is defined so that this family coincides to the family of geodesics.

The second result is as follows.

**Theorem 2.2** (LeBrun-Mason[10])**.** *There is a one-to-one correspondence between*

- *self-dual conformal structures* $[g]$ *of signature* $(++--)$ *on* $S^2 \times S^2$, *and*
- *totally real embeddings* $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}^3$,

*in a neighborhood of the standard objects.*

The correspondence is characterized similarly as the case of Theorem 2.1. LeBrun and Mason also proved the following theorem concerning to a *global* condition.

**Theorem 2.3** (LeBrun-Mason[10])**.** *Any self-dual metric $g$ of signature $(++--)$ on $S^2 \times S^2$ sufficiently near the standard metric is* **Zollfrei**, *i.e. all the maximal null geodesics of $g$ are closed.*

By Theorem 2.3, the self-dual metrics appearing in the statement of Theorem 2.2 turn out to be Zollfrei. This Zollfreiness is a key to prove the required correspondence.

Next results in the LeBrun-Mason theory are about the correspondence for Einstein-Weyl manifolds. An *Einstein-Weyl* manifold is a manifold equipped with a conformal structure $[g]$ and a compatible torsion-free connection $\nabla$ such that the trace-free symmetric part of the Ricci tensor of $\nabla$ vanish. It is known that the equations appearing in the 3-dimensional Einstein-Weyl condition is totally integrable (see [1, 4, 14]). LeBrun-Mason type result for the Einstein-Weyl manifolds is first obtained by the author as follows.

**Theorem 2.4** (N.[14])**.** *For any totally real embedding $\iota : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ sufficiently near* $\mathrm{id} \times \overline{\mathrm{id}}$*, we obtain an Einstein-Weyl manifold $(\mathbb{R} \times S^2, [g], \nabla)$ of signature $(-++)$ as a moduli space of holomorphic disks with boundaries on $\iota(\mathbb{CP}^1)$. Moreover $([g], \nabla)$ is* **space-like Zoll***.*

Here an Einstein-Weyl structure $([g], \nabla)$ is called *space-like Zoll* iff all the space-like (w.r.t. $[g]$) geodesics (w.r.t. $[\nabla]$) are closed. Soon after the above result, LeBrun and Mason established the following very strong theorem.

**Theorem 2.5** (LeBrun-Mason[11])**.** *There is a natural one-to-one correspondence between*

- *smooth, space-time-oriented, conformally compact, globally hyperbolic, Lorentzian Einstein-Weyl 3-manifolds $(M, [g], \nabla)$; and*
- *orientation-reversing diffeomorphisms $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$.*

The explanation for the conditions appearing in the above statement is omitted. We remark that the diffeomorphism $\psi$ defines a totally real embedding $\mathrm{id} \times \psi : \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$, and the correspondence is obtained by considering the family of holomorphic disks with boundaries on the image of this embedding. The most significant progress in Theorem 2.5 is that the complete one-to-one correspondence is established, in contrast in Theorem 2.1 or 2.2, the correspondence is established only for small neighborhoods of the standard objects. The conditions for Einstein-Weyl manifolds appearing in Theorem 2.5 are expected to be equivalent to the space-like Zoll condition, which is an interesting open problem.

## 3. Tod-Kamada ansatz

To construct self-dual metrics is an important problem in, for example, differential geometry, and many mathematicians are interested in this problem. One of the most striking results is obtained by C. LeBrun: he explicitly constructed circle-invariant positive-definite self-dual metrics on the connected sum of $\mathbb{CP}^1$([7]), which is called *LeBrun's hyperbolic ansatz*. In the indefinite case, K. P. Tod constructed infinitely many examples of self-dual metrics of signature $(++--)$ on $S^2\times S^2$ via an analogous technique with LeBrun ansatz ([17]).

On the other hand, H. Kamada investigated compact scalar-flat indefinite Kähler-surfaces with $S^1$-symmetry [6]. (We note that a Kähler-surface is self-dual iff scalar-flat.) Kamada proved that such structure is allowed only on $\mathbb{CP}^1\times\mathbb{CP}^1$, and introduced a method to construct such metrics. Kamada's method is essentially similar to Tod's construction but Kamada introduced a stronger result (e.g. Proposition 3.2). He also constructed many examples which contains all the examples given by Tod.

Now we recall Kamada's construction. Let $(S^3_1, g_{S^3_1})$ be the de Sitter 3-space defined by

$$\begin{aligned} S^3_1 &= \{(x_0,x_1,x_2,x_3)\in\mathbb{R}^4 \mid -x_0^2+x_1^2+x_2^2+x_3^2=1\},\\ g_{S^3_1} &= (-dx_0^2+dx_1^2+dx_2^2+dx_3^2)|_{S^3_1}. \end{aligned}$$

We remark that $g_{S^3_1}$ is Lorentzian, i.e. of signature $(-++)$, and is Einstein. We identify $S^3_1$ with $\mathbb{R}\times S^2$ via the bijection $\mathbb{R}\times S^2\to S^3_1$ given by

$$(t,v)\longmapsto (x_0,x_1,x_2,x_3)=(\sinh t,\cosh t\,(v_1,v_2,v_3)). \tag{3.1}$$

**Proposition 3.1** (Kamada[6]). *Let $V$ be a smooth positive function on $S^3_1$ such that $*dV/2\pi$ is a closed two-form on $S^3_1$ determining an integral class in $H^2(S^3_1;\mathbb{R})$. Let $\mathcal{M}\to S^3_1$ denote an $S^1$-bundle over $S^3_1$ with connection one-form $\Theta$ whose curvature is given by*

$$d\Theta = *dV. \tag{3.2}$$

*Then $g_{V,\Theta} := -V^{-1}\Theta\otimes\Theta + Vg_{S^3_1}$ is a self-dual indefinite metric on $\mathcal{M}$.*

**Proposition 3.2** (Kamada[6]). *Let $(V,\Theta)$ be a solution of (3.2) such that $V>0$ and $*dV$ is an exact two-form. Then the metric $\bar{g} := (\cosh t)^{-2}g_{V,\Theta}$ on $\mathcal{M}$ extends smoothly to the compactification $\overline{\mathcal{M}}\cong S^2\times S^2$ iff there exist smooth functions $F_\pm$ on $S^3_1\cong\mathbb{R}\times S^2$ such that*

$$V(t,v)=1+e^{2t}F_-(e^{2t},v) \quad \text{and} \quad V(t,v)=1+e^{-2t}F_+(e^{-2t},v), \tag{3.3}$$

*as $t\to-\infty$ and as $t\to+\infty$ respectively.*

If $*dV$ is exact, $\Theta$ is written as $\Theta = ds + A$ where $A$ is a one-form on $S^3_1$ and $s\in S^1$ is the fiber coordinate on $\mathcal{M}\to S^3_1$. Then the equation (3.2) is written as

$$dA = *dV \tag{3.4}$$

which we call the **monopole equation**. Kamada introduced infinitely many examples of self-dual indefinite metrics on $S^2\times S^2$ by constructing solutions $(V,A)$ of the monopole equation (3.4). Now the following natural question arises.

**Question 3.3.** *Are the metrics constructed above Zollfrei? If they are Zollfrei, can we establish the LeBrun-Mason twistor correspondence for them?*

The answer is 'Yes'. See Theorem 7.1.

## 4. Wave equation

The monopole equation (3.2) requires the following equation for $V$:

$$\Box V := *d*dV = 0, \tag{4.1}$$

which we call the **wave equation**. If we use the coordinate $(t, v)$ on $S^3_1$ as in (3.1), the wave equation (4.1) is written as

$$\left(-\frac{\partial^2}{\partial t^2} - 2\tanh t\,\frac{\partial}{\partial t} + (\cosh t)^{-2}\Delta_{S^2}\right)V = 0, \tag{4.2}$$

where $\Delta_{S^2}$ is the Laplace operator along the $S^2$-direction of $S^3_1 \simeq \mathbb{R}\times S^2$. Notice that the wave equation (4.2) is a *hyperbolic* partial differential equation.

The key for Question 3.3 is to solve the wave equation (4.1), and this is possible by using an 'extended Radon transform'. To introduce this transform, we define

$$\Omega_{(t,v)} := \{u \in S^2 \mid u\cdot v > \tanh t\}$$

for each $(t, v) \in S^3_1$. Notice that $\Omega_{(t,v)}$ is an open subset on $S^2$ bounded by a *small circle.* and that by the correspondence $(t, v) \leftrightarrow \partial\Omega_{(t,v)}$ the de Sitter space $S^3_1$ is identified with the set of oriented small circles on $S^2$. Now, for any smooth function $h$ on $S^2$, we define functions $Rh$ and $Qh$ on $S^3_1$ by

$$Rh(t,v) = \frac{1}{2\pi}\int_{\partial\Omega_{(t,v)}} h\,dS^1, \qquad Qh(t,v) = \frac{1}{2\pi}\int_{\Omega_{(t,v)}} h\,dS^2,$$

where $dS^1$ is a natural measure on the small circle $\partial\Omega_{(t,v)}$ of the total length $2\pi$, and $dS^2$ is the standard volume form on $S^2$. Notice that $R$ and $Q$ defines a linear transform $C^\infty(S^2)\to C^\infty(S^3_1)$. We remark that the transform $C^\infty(S^2)\to C^\infty(S^2)$ given by $h(v)\mapsto Rh(0,v)$ is called the (spherical) **Radon transform** or the **Funk transform**, and is studied deeply (see the textbook [3]).

Let us denote the antipodal map on $S^2$ by $A$. A function $h \in C^\infty(S^2)$ is called **even** [resp. **odd**] iff $h\circ A = h$ [resp. $h\circ A = -h$]. Similarly for the natural involution $\sigma : S^3_1 \to S^3_1 : (t,v)\mapsto(-t,-v)$, a function $f \in C^\infty(S^3_1)$ is called **even** [resp. **odd**] iff $f\circ\sigma = f$ [resp. $f\circ\sigma = -f$]. We also define

$$C^\infty_*(S^2) := \left\{h \in S^2 \;\middle|\; \int_{S^2} h dS^2 = 0\right\}, \qquad C^\infty_*(S^3_1) := \left\{f \in S^3_1 \;\middle|\; \int_{S^2} f(0,\cdot)dS^2 = 0\right\}.$$

The following Proposition is not so difficult to check.

**Proposition 4.1.** *For any function $h \in C^\infty_*(S^2)$,*

(1) *$f := Rh \in C^\infty(S^3_1)$ is even and satisfies the following equation:*

$$Lf := \left(-\frac{\partial^2}{\partial t^2} + (\cosh t)^{-2}\Delta_{S^2}\right)f = 0,$$

(2) *$V := Qh \in C^\infty(S^3_1)$ is odd and satisfies the wave equation $\Box V = 0$.*

What is more important is the converse. Making use of the uniqueness theorem for initial value problem of hyperbolic partial differential equations (see [2]) and so on, we can prove the following.

**Theorem 4.2.** (1) *Let $V \in C^\infty_*(S^3_1)$ be a solution of the wave equation $\Box V = 0$. Suppose $\lim_{t\to\pm\infty} V(t,v) = 0$ for every $v \in S^2$. Then $V$ is odd, and there exists a unique function $h \in C^\infty_*(S^2)$ satisfying $V = Qh$.*

(2) *Let $f \in C^\infty(S^3_1)$ be a solution of the equation $Lf = 0$. Suppose $h_\pm(v) = \lim_{t\to\pm\infty} f(t,v)$ exist and define smooth functions on $S^2$. Then $f$ is even and $f = Rh_+$ holds. Moreover, if $f \in C^\infty_*(S^3_1)$ then $h_+ \in C^\infty_*(S^2)$.*

## 5. Monopole equation

In this section, we study about the monopole equation $dA = *dV$. To adapt Proposition 3.2, we notice to the solutions $(V, A)$ such that $V = 1 + \tilde{V}$ and $\lim_{t\to\pm\infty} \tilde{V}(t, v) \equiv 0$. Since $\tilde{V}$ satisfies the wave equation $\Box\tilde{V} = 0$, $\tilde{V}$ is an odd function on $S^3_1$ by Theorem 4.2.

On the other hand, recall that $\Theta = ds + A$ is a connection 1-form on the $S^1$-bundle $\mathcal{M} \to S^3_1$. If we change the trivialization of $\mathcal{M}$ by multiplying $e^{i\phi}$ with a real function $\phi$, then $\Theta$ is changed into $\Theta + d\phi$. In this way, we obtain the transform $(V, A) \mapsto (V, A + d\phi)$ for the monopole solutions. We call this transform the *gauge transform*.

Let $\check{*}$ and $\check{d}$ be the fiberwise Hodge's operator and fiberwise exterior derivative on $S^2$-bundle $S^3_1 = \mathbb{R} \times S^2 \to \mathbb{R}$.

**Proposition 5.1.** *Let $(V, A)$ be a solution of the monopole equation $dA = *dV$, Suppose that $\lim_{t\to\pm\infty} V(t, v) = 1$. Then, by changing $(V, A)$ by a gauge transform, we can find a smooth function $f$ on $S^3_1$ satisfying*

(1) *$f$ solves the equation $Lf = 0$,*
(2) *$V = 1 + \partial_t f$, and $A = -\check{*}\,\check{d}\,f$.*

*Conversely, for any smooth function $f$ on $S^3_1$ satisfying $Lf = 0$ and $\lim_{t\to\pm\infty} f(t, v) = h_\pm(v) \in C^\infty(S^2)$, if we define $(V, A)$ as in (2) above, then $(V, A)$ solves the monopole equation.*

We call $f$ the **monopole potential**. Notice that we obtain $f = Rh_+$ by Theorem 4.2. Now we obtain the following.

**Theorem 5.2.** *There is a natural one-to-one correspondence between the following objects:*

1. **[generating function]**
   *smooth functions $h \in C^\infty_*(S^2)$,*
2. **[monopole potential]**
   *smooth functions $f \in C^\infty_*(S^3_1)$ satisfying $Lf = 0$ and $\lim\limits_{t\to\pm\infty} f(t, v) = h_\pm(v) \in C^\infty(S^2)$,*
3. **[equivalence class of monopoles]**
   *gauge equivalence classes of monopoles $[(V, A)]$ such that $\lim\limits_{t\to\pm\infty} V(t, v) = 1$.*

The correspondence is given by

$$f = Rh, \quad h = \lim_{t\to\infty} f, \quad \text{and} \quad (V, A) = \left(1 + \partial_t f, -\check{*}\,\check{d}\,f\right).$$

For the condition (3.3), the following hold.

**Proposition 5.3.** *Let $h \in C^\infty_*(S^2)$ be a generating function, and $(V, A) = (1 + \partial_t Rh, -\check{*}\,\check{d}\,Rh)$ be the induced monopole solution. Then the condition (3.3) in Proposition 3.2 is always satisfied. Thus any monopole $(V, A)$ induced from a generating function defines an self-dual metric on $S^2 \times S^2$.*

## 6. Standard model of the twistor correspondence

We now study the twistor space corresponding to the *standard indefinite metric* on $S^2 \times S^2$ with $S^1$-action. In fact, this situation is the one which is induced from the trivial monopole solution $(V, \Theta) = (1, 0)$.

Let $g_0$ be the product metric $g_0 = -p_1^* h + p_2^* h$ on $S^2 \times S^2$ where $p_i : S^2 \times S^2 \to S^2$ is the $i$-th projection and $h$ is the standard round metric on $S^2$. Let us define an $S^1$-action on $S^2 \times S^2$ by

$$\theta \cdot (v_1, v_2) = (\rho(\theta) v_1, v_2) \qquad \theta \in S^1$$

where $\rho(\theta) \in \mathrm{SO}(3)$ is a rotation of angle $\theta$ around a fixed axis. Let $p_\pm \in S^2$ be the 'north pole' and the 'south pole' of the rotation, then the fixed point set of this $S^1$-action is a disjoint union of the two spheres $S^2_\pm = \{p_\pm\} \times S^2$. Putting $\mathcal{M} := (S^2 \times S^2) \setminus (S^2_+ \sqcup S^2_-)$, we obtain the $S^1$-bundle $\mathcal{M} \to S^3_1$.

It is known that the twistor space corresponding to $(S^2 \times S^2, [g_0])$ in the sense of Theorem 2.2 is the pair $(\mathbb{CP}^3, \mathbb{RP}^3)$. Here $\mathbb{RP}^3$ is the standard real submanifold which can be written as

$$\mathbb{RP}^3 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0 = \bar{z}_3, z_1 = \bar{z}_2\}. \tag{6.1}$$

The $S^1$-action defined above induces a $\mathrm{U}(1)$-action on $\mathbb{RP}^3$ as

$$\mu \cdot [z_0 : z_1 : z_2 : z_3] = [\mu z_0 : \mu z_1 : z_2 : z_3] \qquad \mu \in \mathrm{U}(1). \tag{6.2}$$

This $\mathrm{U}(1)$-action extends to a holomorphic $\mathbb{C}^*$-action on $\mathbb{CP}^3$, and the fixed point set of the $\mathbb{C}^*$-action is the disjoint union of the two rational curves given by

$$\begin{aligned} L_+ &= \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0 = z_1 = 0\}, \\ L_- &= \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_2 = z_3 = 0\}. \end{aligned}$$

Let us put $Z := \mathbb{CP}^3 \setminus (L_+ \sqcup L_-)$. Then the $\mathbb{C}^*$-action is free on $Z$, and we obtain the quotient map

$$\pi : Z \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 : [z_0 : z_1 : z_2 : z_3] \longmapsto (\eta_1, \eta_2) = \left(\frac{z_1}{z_0}, \frac{z_3}{z_2}\right)$$

Notice that the real submanifold $\mathbb{RP}^3$ is mapped by $\pi$ to

$$\mathbb{CP}^1 = \{(\eta_1, \eta_2) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid \eta_1 = \bar{\eta}_2^{-1}\}. \tag{6.3}$$

In this way we obtain the pair $(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^1)$ which is in fact the twistor space in the sense of Theorem 2.4 or 2.5 corresponding to $S^3_1$ equipped with the natural Einstein-Weyl structure.

$$\begin{tikzcd} \mathcal{M} \arrow[d, "S^1"] & \subset & S^2 \times S^2 & & (Z, \mathbb{RP}^3) \arrow[d, "\pi"', "(\mathbb{C}^*, \mathrm{U}(1))"] & \subset & (\mathbb{CP}^3, \mathbb{RP}^3) \\ S^3_1 & & & & (\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^1) & & \end{tikzcd}$$

space-time twistor space

## 7. The twistor space

Here we explain the LeBrun-Mason twistor space for the self-dual metric on $S^2 \times S^2$ constructed in Proposition 5.3. Let $h$ be a real-valued smooth function on $S^2 \cong \mathbb{CP}^1$. Then we define

$$P_h := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0 = \bar{z}_3 e^{h(z_1/z_0)}, z_1 = \bar{z}_2 e^{h(z_1/z_0)}\}$$

which is a deformation of the standard real submanifold (6.1). Notice that the $\mathrm{U}(1)$-action defined in (6.2) preserves $P_h$, and the image $\pi(P_h) = \mathbb{CP}^1 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is not deformed from the standard case (6.3).

**Theorem 7.1.** *Let $h$ be a smooth function on $S^2$ and $(V, A)$ be the corresponding monopole solution. Suppose $V > 0$. Then the self-dual metric on $S^2 \times S^2$ induced by $(V, A)$ is Zollfrei, and its LeBrun-Mason twistor space is given by $(\mathbb{CP}^3, P_h)$.*

Exactly speaking, the correspondence is characterized in the following way. For given $h \in C^\infty_*(S^2)$, suppose $V > 0$ or equivalently $|\partial_t Rh(t, v)| < 1$. Then there exists a family of holomorphic disks $\mathcal{F} = \{D_x\}$ on $\mathbb{CP}^3$ with boundaries on $P_h$ such that

1. $\mathcal{F}$ is parameterized by $x \in S^2 \times S^2$;
2. each class $[D_x] \in H_2(\mathbb{CP}^3, P_h; \mathbb{Z}) \cong \mathbb{Z}$ gives a generator; and
3. $\mathcal{F}$ foliates $\mathbb{CP}^3 \setminus P_h$.

Moreover the following holds:

(i) $\mathcal{F}$ induces a self-dual Zollfrei conformal structure $[g]$ on $S^2 \times S^2$ in the sense of Theorem 2.2;
(ii) an $S^1$-action on $S^2 \times S^2$ is induced and is free on $\mathcal{M} := \{x \in S^2 \times S^2 \mid D_x \subset Z\}$;
(iii) the quotient $\mathcal{M}/S^1$ has a natural Einstein-Weyl structure which is equivalent to the standard $S^3_1$; and
(iv) the class $[g]$ contains the metric induced by the monopole solution $(V, A)$ corresponding to $h$.

Finally we notice to the case when the condition $|\partial_t Rh(t, v)| < 1$ does not satisfied. Even in this case, we can construct the family of holomorphic disks $\mathcal{F} = \{D_x\}$ on $\mathbb{CP}^3$ with boundaries on $P_h$ satisfying the above conditions (1) and (2). However, the condition (3) does not hold.

**Proposition 7.2.** *If $|\partial_t Rh(t, v)| > 1$ for some $(t, v) \in S^3_1$, then the family $\mathcal{F}$ does not give a foliation on $\mathbb{CP}^3 \backslash P_h$.*

This result insists that if the LeBrun-Mason twistor space $(\mathbb{CP}^3, P_h)$ is far from the standard one so that the critical condition $|\partial_t Rh(t, v)| < 1$ does not be satisfied, then the family of holomorphic disks 'degenerate', and the LeBrun-Mason correspondence does not work well at least in the sense so far.

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