## Recent Results in Pluripotential Theory

### 1. (Unweighted) Pluripotential Theory.

Let $E \subset \mathbf{C}^d$ be a bounded Borel set. The global extremal function of $E$ is given by

$$V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta) \text{ where}$$

$$V_E(z) := \sup\{u(z) : u \in L(\mathbf{C}^d),\ u \leq 0 \text{ on } E\}.$$

Here, $u \in L(\mathbf{C}^d)$ if $u \in PSH(\mathbf{C}^d)$ and

$$u(z) - \log|z| = 0(1),\ |z| \to \infty.$$

If $E$ is compact,

$$V_E(z) = \sup\{\frac{1}{deg(p)} \log|p(z)| : ||p||_E \leq 1\}.$$

We call the Monge-Ampere measure of $V_E^*$,

$$\mu_E := \frac{1}{(2\pi)^d}(dd^c V_E^*)^d,$$

the *extremal measure* for $E$ if $E$ is not pluripolar. In this case, $V_E^* \in L^+(\mathbf{C}^d)$ where

$$L^+(\mathbf{C}^d) = \{u \in L(\mathbf{C}^d) : u(z) \geq \log^+|z| + C_u\}.$$

## 2. Weighted Pluripotential Theory.

Let $K \subset \mathbf{C}^d$ be closed and let $w$ be an admissible weight function on $K$: $w$ is a nonnegative, usc function with $\{z \in K : w(z) > 0\}$ nonpluripolar; if $K$ is unbounded, we assume

$$|z|w(z) \to 0 \text{ as } |z| \to \infty,\ z \in K.$$

Let $Q := -\log w$ and define the weighted extremal function

$$V_{K,Q}^*(z) := \limsup_{\zeta \to z} V_{K,Q}(\zeta) \text{ where}$$

$$V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbf{C}^d),\ u \leq Q \text{ on } K\}.$$

Then $S_w := \mathrm{supp}(\mu_{K,Q})$ is compact where

$$\mu_{K,Q} := \frac{1}{(2\pi)^d}(dd^c V_{K,Q}^*)^d;$$

$$V_{K,Q}(z) = \sup\{\frac{1}{deg(p)} \log |p(z)| : ||w^{deg(p)}p||_{S_w} \leq 1\};$$

and

$$||w^{deg(p)}p||_{S_w} = ||w^{deg(p)}p||_K.$$

We call $w^{deg(p)}p = e^{-deg(p)Q}p$ a *weighted polynomial.*

## 3. Example.

(1) Let $K = \{z : |z| \leq 1\}$ and $Q(z) = |z|^2$. Then

$$V_{K,Q} = Q \text{ on the ball } \{z : |z| \leq 1/\sqrt{2}\}$$

and

$$V_{K,Q}(z) = \log|z| + 1/2 - \log(1/\sqrt{2})$$

outside this ball.

(2) Let $K = \mathbf{C}^d$ and the same weight function $Q(z) = |z|^2$. One obtains the **same** weighted extremal function $V_{\mathbf{C}^d,Q}$. In particular, $S_w$ is compact.

In the examples, $S_w = \{V_{K,Q} = Q\}$. In general,

$$S_w \subset S_w^* := \{z \in K : V_{K,Q}^*(z) \geq Q(z)\}.$$

Notation: We write $\mathcal{P}_n$ for the polynomials of degree at most $n$ in $\mathbf{C}^d$ and

$$N = dim\mathcal{P}_n = \binom{n+d}{d}.$$

**4. Bernstein-Markov Measures.**

Given a compact set $K \subset \mathbf{C}^d$ and a measure $\nu$ on $K$, we say that $(K, \nu)$ satisfies a *Bernstein-Markov property* if for all $p_n \in \mathcal{P}_n$,

$$||p_n||_K \leq M_n ||p_n||_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1.$$

If $(K, \nu)$ satisfies the Bernstein-Markov inequality then

$$\lim_{n\to\infty} \frac{1}{2n} \log K_n^\nu(z,z) = V_K(z) \ (\textit{weak asymptotics})$$

locally uniformly on $\mathbf{C}^d$ where

$$B_n^\nu(z) := K_n^\nu(z,z) = \sum_{j=1}^{N} |q_j^{(n)}(z)|^2$$

is the $n-th$ *Bergman function* of $K, \nu$ and

$$K_n^\nu(z,\zeta) := \sum_{j=1}^{N} q_j^{(n)}(z)\overline{q_j^{(n)}(\zeta)}$$

where $\{q_j^{(n)}\}_{j=1,\ldots,N}$ is an orthonormal basis for $\mathcal{P}_n$ with respect to $L^2(\nu)$.

**5. Weighted Bernstein-Markov Measures.**

For $K \subset \mathbf{C}^d$ compact, $w = e^{-Q}$ an admissible weight function on $K$, and $\nu$ a measure on $K$, we say that the triple $(K, \nu, Q)$ satisfies a *weighted Bernstein-Markov property* if for all $p_n \in \mathcal{P}_n$,

$$||w^n p_n||_K \leq M_n ||w^n p_n||_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1.$$

For such $(K, \nu, Q)$ we have that

$$\lim_{n\to\infty} \frac{1}{2n} \log K_n^{\nu,w}(z,z) = V_{K,Q}(z) \ (\textit{weak asymptotics})$$

locally uniformly on $\mathbf{C}^d$ where

$$B_n^{\nu,w}(z) := K_n^{\nu,w}(z,z) w(z)^{2n} = \sum_{j=1}^{N} |q_j^{(n)}(z)|^2 w(z)^{2n}$$

is the $n-th$ *Bergman function* of $K, w, \nu$ and

$$K_n^{\nu,w}(z,\zeta) := \sum_{j=1}^{N} q_j^{(n)}(z) \overline{q_j^{(n)}(\zeta)}.$$

Here, $\{q_j^{(n)}\}_{j=1,\ldots,N}$ is an orthonormal basis for $\mathcal{P}_n$ with respect to the weighted $L^2-$norm $||w^n p_n||_{L^2(\nu)}$.

## 6. Strong Bergman Asymptotics.

Strong Bergman asymptotics says that if $(K,\mu,w)$ satisfies a weighted Bernstein-Markov inequality, then

$$\frac{1}{N}B_n^{\mu,w}d\mu \to \mu_{K,Q} \text{ weak-} * . \qquad (SBA)$$

(SBA) follows from the main result of Berman and Boucksom (Theorem 2). A key ingredient in the proof of Theorem 2 is a special case of (SBA):

**Theorem 1 (Berman, IUMJ).** *Strong Bergman asymptotics holds if* $K = \mathbf{C}^d$, $Q(z) \geq (1+\epsilon)\log|z|$, $Q \in C^{1,1}(\mathbf{C}^d)$. *That is,*

$$\frac{1}{N}B_n^{\omega_d,w}\omega_d \to \frac{1}{(2\pi)^d}(dd^cV_{\mathbf{C}^d,Q})^d \text{ weak} - *; \text{ indeed,}$$

$$\frac{1}{N}B_n^{\omega_d,w}\cdot \chi_D\omega_d \to \frac{1}{(2\pi)^d}(dd^cV_{\mathbf{C}^d,Q})^d \text{ weak} - *$$

*where* $D = \{V_{\mathbf{C}^d,Q} = Q\}$ *and* $\omega_d$ *is Lebesgue measure on* $\mathbf{C}^d$.

Moreover, $V_{\mathbf{C}^d,Q} \in C^{1,1}(\mathbf{C}^d)$ and $(dd^cV_{\mathbf{C}^d,Q})^d$ is absolutely continuous.

What is Theorem 2? Sorry – first we need some more definitions!

## 7. Ball Volume Ratios and Gram Determinants.

Given an $N-$dimensional vector space $V$ (e.g., $V = \mathcal{P}_n$), and two subsets $A, B$ in $V$, we write

$$[A : B] := \log \frac{vol(A)}{vol(B)}.$$

Here, "vol" denotes any (Haar) measure on $V$. If $V$ is equipped with two Hermitian inner products $h, h'$, and $B, B'$ are the corresponding unit balls, then

$$[B : B'] = \log \det[h'(e_i, e_j)]_{i,j=1,\ldots,N}$$

where $e_1, ..., e_N$ is an $h-$orthonormal basis for $V$. Thus $[B : B']$ is a *Gram determinant* with respect to the $h'$ inner product relative to the $h-$orthonormal basis.

For $E \subset \mathbf{C}^d$ closed and $w = e^{-Q}$ admissible weight on $E$, let

$$\mathcal{B}^{\infty}(E, nQ) := \{p_n \in \mathcal{P}_n : |p_n(z)e^{-nQ(z)}| \leq 1 \text{ on } E\}$$

be a weighted $L^{\infty}-$ball in $\mathcal{P}_n$. If $\mu$ is a measure on $E$,

$$\mathcal{B}^2(E, \mu, nQ) := \{p_n \in \mathcal{P}_n : \int_E |p_n|^2 e^{-2nQ} d\mu \leq 1\}$$

is a weighted $L^2-$ball in $\mathcal{P}_n$.

**8. Asymptotics of Ball Volume Ratios: Theorem 2.**
(Remark: Ball volume ratios trivially satisfy a cocycle condition: $[A : B] + [B : C] + [C : A] = 0$.)

Given $Q, Q'$ admissible weights on $E, E'$, consider the sequence

$$\{\frac{1}{2nN}[\mathcal{B}^\infty(E, nQ) : \mathcal{B}^\infty(E', nQ')]\}_{n=1,...}$$

and if $\mu$ and $\mu'$ are measures on $E$ and $E'$ with both $(E, \mu, Q)$ and $(E', \mu', Q')$ satisfying a weighted BM property, consider the sequence

$$\{\frac{1}{2nN}[\mathcal{B}^2(E, \mu, nQ) : \mathcal{B}^2(E', \mu', nQ')]\}_{n=1,...}.$$

**Theorem 2.** *Both sequences converge to*

$$\frac{1}{(d+1)(2\pi)^d}\mathcal{E}(V^*_{E,Q}, V^*_{E',Q'}).$$

**Question: What is $\mathcal{E}$?**

**9. A Mabuchi-Aubin-type Energy: $\mathcal{E}$.**

For $u, v \in L^+(\mathbf{C}^d)$,

$$\mathcal{E}(u,v) := \int_{\mathbf{C}^d} (u-v) \sum_{j=0}^{d} (dd^c u)^j \wedge (dd^c v)^{d-j}.$$

Fixing $v$, $u \to \mathcal{E}(u,v)$ is a primitive for the complex Monge-Ampere operator $(dd^c u)^d$: if $u' \in L^+(\mathbf{C}^d)$ and for $0 \leq t \leq 1$ we define

$$f(t) := \mathcal{E}(u + t(u'-u), v),$$

then $f'(t)$ exists for $0 \leq t \leq 1$ and

$$f'(t) = (d+1) \int_{\mathbf{C}^d} (u'-u) \big(dd^c[u + t(u'-u)]\big)^d.$$

In particular,

$$f'(0) = (d+1) \int_{\mathbf{C}^d} (u'-u)(dd^c u)^d.$$

Note that $f'(t)$ is independent of $v$. As a corollary, we obtain the cocycle property: Let $u, v, w \in L^+(\mathbf{C}^d)$. Then

$$\mathcal{E}(u,v) + \mathcal{E}(v,w) + \mathcal{E}(w,u) = 0.$$

**10. Differentiability of $\mathcal{E} \circ P$.**

New notation: given an admissible weight $w = e^{-Q}$ on $K$, write

$$P(Q) = P_K(Q) := V^*_{K,Q}.$$

**Theorem 3.** *Let $v \in L^+(\mathbf{C}^d)$. For $w = e^{-Q}$ an admissible weight on $K$ and $u \in C(K)$, let*

$$F(t) := \mathcal{E}(P(Q + tu), v)$$

*for $t \in \mathbf{R}$. Then*

$$F'(t) = (d+1) \int_{\mathbf{C}^d} u(dd^c P(Q + tu))^d.$$

*In particular,*

$$F'(0) = (d+1) \int_{\mathbf{C}^d} u(dd^c P(Q))^d.$$

Again, $F'(t)$ is independent of $v$. This is the key ingredient in a variational approach to complex Monge-Ampere (Berman, Boucksom, Guedj, Zeriahi).

## 11. Proof of Differentiability of $\mathcal{E} \circ P$.

Take $F(t) := \mathcal{E}(P(Q+tu), P(Q))$. Then $F(0) = 0$ and we want to show

$$F'(0) = (d+1) \int_{\mathbf{C}^d} u(dd^c P(Q))^d.$$

It suffices to show

$$\mathcal{E}(P(Q+tu), P(Q)) = (d+1)t \int_{\mathbf{C}^d} u(dd^c P(Q))^d + o(t).$$

This follows from two ingredients:

$$\mathcal{E}(P(Q+tu), P(Q)) = \qquad (A)$$

$$(d+1) \int_{\mathbf{C}^d} [P(Q+tu) - P(Q)](dd^c P(Q))^d + o(t)$$

and

$$\lim_{t \to 0} \int_{D(0)-D(t)} (dd^c P(Q))^d = 0 \qquad (B)$$

where

$$D(t) := \{z \in \mathbf{C}^d : \; P(Q+tu)(z) = (Q+tu)(z)\}.$$

**12. Proof of Differentiability of $\mathcal{E} \circ P$ (cont'd).**
We have $\mathrm{supp}(dd^c P(Q))^d \subset D(0)$, hence:

$$\mathcal{E}(P(Q+tu), P(Q)) = \text{ (from (A))}$$

$$(d+1)\int_{\mathbf{C}^d}[P(Q+tu)-P(Q)](dd^cP(Q))^d + o(t)$$

$$= (d+1)\int_{D(0)-D(t)}[P(Q+tu)-P(Q)](dd^cP(Q))^d$$

$$+(d+1)\int_{D(0)\cap D(t)}[P(Q+tu)-P(Q)](dd^cP(Q))^d + o(t)$$

$$= (d+1)\int_{D(0)-D(t)}[P(Q+tu)-P(Q)](dd^cP(Q))^d$$

$$+(d+1)t\int_{D(0)\cap D(t)} u(dd^cP(Q))^d + o(t)$$

since $P(Q+tu)-P(Q) = tu$ on $D(0)\cap D(t)$. Now use (B) together with the observation that

$$|P(Q+tu)-P(Q)| = 0(t)$$

on the *bounded* set $D(0)-D(t)$.

## 13. Proof of Theorem 2.

We will use an integrated version of the previous result:

$$\mathcal{E}(P(Q+u),P(Q)) = (d+1)\int_{t=0}^{1} dt \int_{\mathbf{C}^d} u(dd^c P(Q+tu))^d. \qquad (I)$$

Given a closed set $K$, an admissible weight $w = e^{-Q}$ on $K$, and a function $u \in C(K)$, we consider the perturbed weight $w_t := w\exp(-tu)$. Equivalently, $Q_t := Q + tu$. Let $\{\mu_n\}$ be a sequence of measures on $K$. We set

$$f_n(t) := -\frac{(d+1)}{2dnN} \log \det(G_n^{\mu_n,w_t})$$

where

$$G_n^{\mu_n,w_t} = G_n^{\mu_n,w_t}(\beta_n) := [\langle p_i, p_j\rangle_{\mu_n,w_t}] \in \mathbf{C}^{N\times N}$$

is the *Gram matrix* with respect to the weighted inner product

$$\langle f, g\rangle_{\mu_n,w_t} := \int_K f(z)\overline{g(z)}w_t(z)^{2n} d\mu_n$$

and a fixed basis $\beta_n := \{p_1, p_2, \cdots, p_N\}$ of $\mathcal{P}_n$. Then

$$f_n'(t) = \frac{d+1}{dN}\int_K u(z) B_n^{\mu_n,w_t}(z) d\mu_n. \qquad (II)$$

**14. Sketch of Proof of (II).**

Fixing a basis $\beta_n := \{p_1, ..., p_N\}$ of $\mathcal{P}_n$, we set

$$f_n(t) := -\frac{(d+1)}{2dnN} \log \det(G_n^{\mu_n, w_t})$$

where $G_n^{\mu_n, w_t} = G_n^{\mu_n, w_t}(\beta_n)$ and we want to show

$$f_n'(t) = \frac{d+1}{dN} \int_K u(z) B_n^{\mu_n, w_t}(z) d\mu_n. \qquad (II)$$

Using $\log \det(G_n^{\mu_n, w_t}) = \operatorname{trace} \log(G_n^{\mu_n, w_t})$,

$$\frac{2dnN}{d+1} f_n'(t) = -\frac{d}{dt} \operatorname{trace} \left( \log(G_n^{\mu_n, w_t}) \right) =$$

$$-\operatorname{trace} \left( \frac{d}{dt} \log(G_n^{\mu_n, w_t}) \right)$$

$$-\operatorname{trace} \left( (G_n^{\mu_n, w_t})^{-1} \frac{d}{dt} G_n^{\mu_n, w_t} \right)$$

$$= 2n \operatorname{trace}\big( (G_n^{\mu_n, w_t})^{-1} \big[ \int_K p_i(z) \overline{p_j(z)} u(z) w(z)^{2n} \cdot$$

$$\exp(-2ntu(z)) d\mu_n \big] \big).$$

Now use

$$\operatorname{trace}(ABC) = \operatorname{trace}(CAB) = CAB$$

and some linear algebra.

**15. Proof of Theorem 2. (cont'd)**

First do the $L^2-$case with $E = E' = \mathbf{C}^d$ and $Q, Q'$ as in Theorem 1 with $d\mu = d\mu' = \omega_d$. Take $u = Q' - Q$ and for $0 \leq t \leq 1$ let $w_t(z) := w(z)\exp(-tu(z))$. By Theorem 1,

$$\frac{1}{N} B_n^{\omega_d, w_t} \omega_d \to \frac{1}{(2\pi)^d} (dd^c P(Q + tu))^d \text{ weak} - *.$$

Define $f_n(t) := -\frac{(d+1)}{2dnN} \log \det(G_n^{\mu_n, w_t})$ where we take $\mu_n := \omega_d$ for all $n$ and the basis $\beta_n := \{p_1, ..., p_N\}$ of $\mathcal{P}_n$ is chosen to be an orthonormal basis with respect to $||w^n p||_{L^2(\mu)}$. Then $f_n(0) = 0$. Using (II)

$$\lim_{n\to\infty} \frac{d}{d+1} f_n'(t) = \int u \frac{1}{(2\pi)^d} (dd^c P(Q + tu))^d.$$

Integrate the expression for $f_n'(t)$ from $t = 0$ to $t = 1$:

$$\frac{d}{d+1}[f_n(1)] = \frac{-1}{2nN} \log \det(G_n^{\mu_n, w'})$$

$$= \frac{-1}{2nN}[\mathcal{B}^2(\mathbf{C}^d, \omega_d, nQ) : \mathcal{B}^2(\mathbf{C}^d, \omega_d, nQ')]$$

$$= \frac{1}{N} \int_{t=0}^1 dt \int B_n^{\omega_d, w_t}(Q - Q')\omega_d.$$

Apply Theorem 1. again to conclude that

$$\frac{-1}{2nN}[\mathcal{B}^2(\mathbf{C}^d, \omega_d, nQ) : \mathcal{B}^2(\mathbf{C}^d, \omega_d, nQ')]$$

$$= \frac{1}{N}\int_{t=0}^{1} dt \int B_n^{\omega_d, w_t}(Q - Q')\omega_d.$$

$$\to \int_{t=0}^{1} dt \int (Q - Q')\frac{1}{(2\pi)^d}(dd^c P(Q + tu))^d.$$

But by (I), the integrated version of differentiability of $\mathcal{E} \circ P$, we have

$$(d+1)\int_{t=0}^{1} dt \int (Q - Q')\frac{1}{(2\pi)^d}(dd^c P(Q + tu))^d =$$

$$\frac{1}{(2\pi)^d}\mathcal{E}(P(Q'), P(Q))$$

which proves Theorem 2. in the $L^2-$case when $E = E' = \mathbf{C}^d$ with $Q, Q' \in C^2(\mathbf{C}^d)$ as in Theorem 1. with $d\mu = d\mu' = \omega_d$. By the weighted Bernstein-Markov property this also proves the $L^\infty-$case when $E = E' = \mathbf{C}^d$ and $Q, Q' \in C^2(\mathbf{C}^d)$ as in Theorem 1.

The remaining cases follow from approximation and the cocycle properties of $\mathcal{E}$ and the ball volume ratios.

## 16. Applications.

Theorem 2. implies several important results:

(1) a proof of *Rumely's formula* relating the transfinite diameter $\delta(K)$ of a compact set $K$ in $\mathbf{C}^d$ with certain integrals involving the Robin function $\rho_K(z) := \limsup_{|\lambda|\to\infty}[V_K^*(\lambda z) - \log|\lambda|]$ of $K$, as well as weighted versions of the formula;

(2) a proof of *the weighted Fekete conjecture*: for each $n$, take an $n-$th weighted Fekete set $x_1^{(n)}, ..., x_N^{(n)}$ for $K$ and $w$ and let $\mu_n := \frac{1}{N}\sum_{j=1}^N \delta_{x_j^{(n)}}$. Then

$$\mu_n \to \mu_{K,Q} := \frac{1}{(2\pi)^d}(dd^c V_{K,Q}^*)^d \text{ weak} - *;$$

(3) analogous results to (2) for (weighted) *optimal measures* (defined by maximizing certain Gram determinants);

(4) general results on *strong Bergman asymptotics* for BM pairs $(K,\mu)$ where $\mu$ is a measure on $K$, as well as for weighted BM triples $(K,\mu,Q)$; i.e., we have

$$\frac{1}{N}B_n^{\mu,w}d\mu \to \mu_{K,Q} \text{ weak-} * . \qquad (SBA)$$