Elliptic j-function and Borcherds Φ-function

(Joint work with Ken-Ichi Yoshikawa)

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Plan

1. Introduction – rough sketch of the result
2. Details
3. Ideas of proof

Part 1 Introduction

Elliptic j-function

Let $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \mathsf{Im}\,\tau > 0\}$ be the Siegel upper half plane, on which $\mathsf{SL}_2(\mathbb{Z})$ acts by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

For $\tau \in \mathfrak{H}$, let $E_\tau := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ be a 1-dimensional complex torus, i.e., an elliptic curve (with origin 0).

Then $E_\sigma \cong E_\tau \iff \sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$.

So, the quotient space $\mathsf{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ is the coarse moduli space of elliptic curves.

The elliptic j-function is a function on $\mathfrak{H}$ invariant under the action of $\mathsf{SL}_2(\mathbb{Z})$. In fact, it gives an isomorphism $j : \mathsf{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \xrightarrow{\sim} \mathbb{C}$.

Kummer surfaces and Enriques surfaces

For $\sigma, \tau \in \mathfrak{H}$, consider the 2-dimensional complex torus $E_\sigma \times E_\tau$.

The quotient $(E_\sigma \times E_\tau)/[-1]_{E_\sigma \times E_\tau}$ has 16 singular points. Blowing up at these points, we obtain a smooth complex surface $\mathsf{Km}(E_\sigma \times E_\tau)$.

It is called a Kummer surface of product type, and is denoted by $X_{\sigma,\tau} := \mathsf{Km}(E_\sigma \times E_\tau)$.

Let (σ, τ) be general. Up to conjugacy, there are 15 (fixed-point-free, anti-symplectic) involutions on $X_{\sigma,\tau}$.

- 9 of them are of even type (Lieberman involutions).
- 6 of them are of odd type (Kondo-Mukai involutions).

(I will explain in more detail later.)

For each involution ι, the quotient $X_{\sigma,\tau}/\iota$ is an Enriques surface.

Borcherds Φ-function

Let $\mathcal{M}^\circ := (\Omega_\Lambda^+ \setminus \mathcal{D})/O^+(\Lambda)$ denote the coarse moduli space of Enriques surfaces, where Ω_Λ^+ is a certain 10-dimensional symmetric bounded domain of type IV, $\mathcal{D}$ is the discriminant divisor, and $O^+(\Lambda)$ is the group of isometries.
(I will explain in more detail later.)

The Borcherds Φ-function is an automorphic form of weight 4 with some character on Ω_Λ^+. Thus, $\|\Phi\|$ is a function on $\mathcal{M} := \Omega_\Lambda^+/O^+(\Lambda)$, where $\|\cdot\|$ is the Petersson metric.

In particular, for an Enriques surface Y, we can consider the value $\|\Phi\|(Y) \in \mathbb{R}$.

Result

Let $\sigma, \tau \in \mathfrak{H}$.

We relate

- The difference of elliptic j-functions $j(\sigma) - j(\tau)$

  and

- Certain products of the values of the Borcherds Φ-function $\|\Phi\|(X_{\sigma,\tau}/\iota)$, where $X_{\sigma,\tau} := \mathsf{Km}(E_\sigma \times E_\tau)$ is a Kummer surface of product type, and ι runs over the 15 involutions on $X_{\sigma,\tau}$.

To be precise,

Main Theorem

$$\frac{\prod_{\iota:odd} \|\Phi\|(X_{\sigma,\tau}/\iota)^3}{\prod_{\iota:even} \|\Phi\|(X_{\sigma,\tau}/\iota)^2} = |j(\sigma) - j(\tau)|^6 .$$

Remark (1)

- The difference $j(\sigma) - j(\tau)$ is the denominator function of the monster Lie algebra. Borcherds gave the infinite product expansion for $j(\sigma) - j(\tau)$:

$$j(\sigma) - j(\tau) = \frac{1}{q_\sigma} \prod_{m>0, n\in\mathbb{Z}} (1 - q_\sigma^m {q_\tau}^n)^{c(mn)},$$

  where $q_\sigma = \exp(2\pi\sqrt{-1}\sigma)$ and $q_\tau = \exp(2\pi\sqrt{-1}\tau)$, and $c(n)$ is the coefficient of q^n in $j(\tau) = q^{-1} + 744 + 196884q + \cdots$.

- The Borcherds Φ function is the denominator function of the fake monster superalgebra. Borcherds also gave the infinite product expansion for Φ.

- **Main Theorem** relates these two infinite product expansions.

Remark (2)

- Yoshikawa showed that the Borcherds Φ-function is essentially the analytic torsion function on the coarse moduli space $\mathcal{M}^\circ$ of Enriques surfaces.

  (Here, for a compact Kähler manifold (Y, γ), Ray and Singer defined a quantity $\tau(Y, \gamma) \in \mathbb{R}_{>0}$ called analytic torsion, using spectral zeta functions of the Laplace operators acting on the space of differential forms on Y.)

  Theorem (Yoshikawa 2004) *Let Y be an Enriques surface, and γ be any Ricci-flat Kähler metric on Y. Then*

  $$T(Y) := \mathsf{Vol}(Y, \gamma)^{\frac{1}{2}} \tau(Y, \gamma)$$

  does not depend on γ. Further, there is a universal constant such that

  $$T(Y) = c \|\Phi\|(Y)^{\frac{1}{4}}.$$

Remark (2) (cont'd)

- **Main Theorem** says that $j(\sigma) - j(\tau)$ is related to the analytic torsion of the corresponding Kummer surface $X_{\sigma,\tau}$ and involutions on $X_{\sigma,\tau}$:

  Corollary

  $$\frac{\prod_{\iota:even} T\left(\mathsf{Km}(E_\sigma \times E_\tau)/\iota\right)^2}{\prod_{\iota:odd} T\left(\mathsf{Km}(E_\sigma \times E_\tau)/\iota\right)^3} = |j(\sigma) - j(\tau)|^{\frac{3}{2}}.$$

  Note: Vincent Maillot asked if $j(\sigma) - j(\tau)$ is expressed as analytic torsion. Main Theorem may be seen an answer to his question. (This was our original motivation.)

Remark (3)

- Gross and Zagier (1985) showed that $j(\sigma) - j(\tau)$ has some deep arithmetic properties:

  Let d, e be discriminants, i.e., integers with $d, e \equiv 0, 1 \pmod{4}$. Assume that d, e are relatively coprime and that if $d = d'D^2$ (resp. $e = e'E^2$) with an integer D (resp. E) and a discriminant d' (resp. e'), then $D = \pm 1$ (resp. $E = \pm 1$).

  Then $j(\sigma) - j(\tau)$ is an algebraic integer, and one has

$$\prod_{\sigma,\tau}(j(\sigma) - j(\tau)) = \pm \prod_{x \in \mathbb{Z},\ x^2 < de,\ x^2 \equiv de (\mathrm{mod}.4)} F\left(\frac{de - x^2}{4}\right),$$

  where σ and τ run over representatives of equivalence classes of imaginary quadratic irrationals with discriminant d and e, respectively, and $F(m) = \prod_{nn'=m, n, n'>0} n^{\varepsilon(n')}$ with $\varepsilon(n') = \pm 1$.

Remark (3) (cont'd)

- **Main Theorem** says that the Borcherds Φ-function also has some arithmetic properties (for Enriques surfaces associated to $X_{\sigma,\tau}$ with quadratic imaginary numbers σ, τ):

  Corollary *Let d, e be relatively coprime negative integers as above. Then*

$$\frac{\prod_{\iota:odd} \|\Phi\|(X_{\sigma,\tau}/\iota)^{\frac{1}{2}}}{\prod_{\iota:even} \|\Phi\|(X_{\sigma,\tau}/\iota)^{\frac{1}{3}}}$$

  is the absolute value of an algebraic integer, and we have

$$\prod_{\sigma,\tau} \frac{\prod_{\iota:odd} \|\Phi\|(X_{\sigma,\tau}/\iota)^{\frac{1}{2}}}{\prod_{\iota:even} \|\Phi\|(X_{\sigma,\tau}/\iota)^{\frac{1}{3}}} = \prod_{x^2<de,\, x^2\equiv de \bmod 4} F\left(\frac{de-x^2}{4}\right).$$

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Part 2 Details

Elliptic j-function

Let $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \mathsf{Im}\,\tau > 0\}$ be the Siegel upper half plane. The group $\mathsf{SL}_2(\mathbb{Z})$ acts on $\mathfrak{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

The elliptic j-function

$$j : \mathfrak{H} \to \mathbb{C},$$

is invariant under the action of $\mathsf{SL}_2(\mathbb{Z})$ and is given by

$$\begin{aligned} j(\tau) &= \frac{\left(1 + 240 \sum_{n>0} \sigma_3(n) q^n\right)^3}{q \prod_{n>0}(1-q^n)^{24}} \\ &= \frac{1}{q} + 744 + 196884q + \cdots, \end{aligned}$$

where $q = \exp(2\pi\sqrt{-1}\tau)$ and $\sigma_3(n) = \sum_{d|n} d^3$.

$K3$ surface

A smooth compact complex surface X is a $K3$ surface if

- $H^1(X, \mathcal{O}_X) = 0$, and
- $\Omega_X^2 \simeq \mathcal{O}_X$.

Then

$$H^2(X, \mathbb{Z}) \simeq \mathbb{L}_{K3} := U^{\oplus 3} \oplus E_8(2)^{\oplus 2} \quad \text{(isometry)},$$

where $H^2(X, \mathbb{Z})$ is endowed with the cup product, and $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8 is the negative Cartan matrix.

Example (Kummer surface) Let A be a 2-dimensional complex torus. Let $\mathrm{Km}(A)$ be the smooth surface obtained by blowing up the 16 singular points of the quotient $A/[-1]_A$.
Then $\mathrm{Km}(A)$ is a $K3$ surface.

Enriques surface

A smooth compact complex surface Y is an Enriques surface if

- $H^1(Y, \mathcal{O}_Y) = 0$,
- $\Omega_Y^2 \not\simeq \mathcal{O}_Y$, and
- $(\Omega_Y^2)^{\otimes 2} \simeq \mathcal{O}_Y$

Let $\pi : X \to Y$ be the universal covering. Then

- X is a $K3$ surface
- π is a degree 2 map.

The covering transformation induces a (fixed-point-free, anti-symplectic) involution

$$\iota : X \to X$$

with $Y \simeq X/\iota$.

Enriques surface (cont'd)

Let Y be an Enriques surface.
Let (X, ι) be as above.

Set
$$H^2_{\pm}(X, \mathbb{Z}) := \{\ell \in H^2(X, \mathbb{Z}) \mid \iota^*\ell = \pm\ell\}.$$

Set $\Lambda := U \oplus U(2) \oplus E_8(2)$, and fix a primitive embedding $\Lambda \subset \mathbb{L}_{K3}$.

Then there is an isometry $\alpha : H^2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ such that
$$\alpha(H^2_-(X, \mathbb{Z})) = \Lambda.$$

If $\eta \in H^0(X, \Omega^2_X) \setminus \{0\}$, then $\iota^*(\eta) = -\eta$.
Thus $\eta \in H^2_-(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$.

Moduli of Enriques surfaces

Set

$$\Omega_\Lambda := \{[\eta] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (\eta, \eta) = 0, (\eta, \overline{\eta}) > 0\} = \Omega_\Lambda^+ \amalg \Omega_\Lambda^-$$

$$\mathcal{D} := \bigcup_{d \in \Lambda, d^2 = -1} d^\perp \subset \Omega_\Lambda^+ \quad \text{(discriminant locus)}$$

$$O^+(\Lambda) := \{\varphi : \Lambda \to \Lambda \mid \varphi \text{ is an isometry preserving } \Omega_\Lambda^+\}$$

Then Ω_Λ^+ is a symmetric bounded domain of type IV of dimension 10. The group $O^+(\Lambda)$ acts on Ω_Λ^+ properly discontinuously.

Theorem (Horikawa 1978) *The coarse moduli space of Enriques surfaces is given by*

$$\mathcal{M}^\circ := (\Omega_\Lambda^+ \setminus \mathcal{D})/O^+(\Lambda).$$

For an Enriques surface Y, its point in $\varpi(Y) \in \mathcal{M}^\circ$ is given by $[\alpha(\eta_X)]$, where $\eta_X \in H^0(X, \Omega_X^2) \setminus \{0\}$.

Borcherds Φ-function

We set $L := U \oplus E_8(2)$. Then $\Lambda = U(2) \oplus L$.
Let $\mathcal{C}_L := \{v \in L \otimes \mathbb{R} \mid v^2 > 0\}$ be the positive cone. Then the tube domain $L \otimes \mathbb{R} + \sqrt{-1}\mathcal{C}_L$ is identified with Ω_Λ via the map

$$L \otimes \mathbb{R} + \sqrt{-1}\mathcal{C}_L \ni z \mapsto \left[(-z^2/2, 1/2, z)\right] \in \Omega_\Lambda.$$

We write $\mathcal{C}_L^+$ for the connected component of $\mathcal{C}_L$ such that $L \otimes \mathbb{R} + \sqrt{-1}\mathcal{C}_L^+$ is identified with Ω_Λ^+.

We set $\rho := ((0,1),0), \rho' := ((1,0),0) \in L$ and
$\Pi^+ := \mathbb{N}\rho \cup \{\lambda \in L; \langle \lambda, \rho \rangle_L > 0\}$.

Borcherds (1992) introduced the following infinite product on $L \otimes \mathbb{R} + \sqrt{-1}\mathcal{C}_L^+$:

$$\Phi(z) = e^{2\pi \langle \rho, z \rangle} \prod_{\lambda \in \Pi^+} \left(1 - e^{2\pi\sqrt{-1}\langle \lambda, z \rangle}\right)^{(-1)^{\langle \lambda, \rho - \rho' \rangle} c(\lambda^2/2)},$$

where $\{c(n)\}_{n \geq -1}$ is the generating function
$\sum_{n \geq -1} c(n) q^n = \eta(\tau)^{-8} \eta(2\tau^8) \eta(4\tau)^{-8}$.

Borcherds Φ-function (cont'd)

$\Phi(z)$ is called the Borcherds Φ-function. It is an automorphic form with some character on Ω_Λ^+ of weight 4 with zero divisor $\mathcal{D}$.

For $z \in L \otimes \mathbb{R} + \sqrt{-1}\mathcal{C}_L^+$, the Petersson norm of Φ is given by

$$\|\Phi(z)\|^2(z) := \langle \mathrm{Im}(z), \mathrm{Im}(z) \rangle^4 |\Phi(z)|^2.$$

Then $\|\Phi\|$ is a function on $\mathcal{M} := \Omega_\Lambda^+ / O^+(\Lambda)$.

Two types of involutions on $X_{\sigma,\tau}$

Let

$$e_1(\sigma) = \frac{1}{2}, \quad e_2(\sigma) = \frac{\sigma}{2}, \quad e_3(\sigma) = \frac{1+\sigma}{2}.$$

be the nonzero 2-torsion points on $E_\sigma = \mathbb{C}/\mathbb{Z} + \sigma\mathbb{Z}$. Similarly, let

$$e_1(\tau) = \frac{1}{2}, \quad e_2(\tau) = \frac{\tau}{2}, \quad e_3(\tau) = \frac{1+\tau}{2}.$$

be the nonzero 2-torsion points on $E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

We will explain two types of involutions on $X_{\sigma,\tau} := \mathsf{Km}(E_\sigma \times E_\tau)$:

- 9 even involutions (Lieberman involutions)
- 6 odd involutions (Kondo-Mukai involutions)

From now on, we assume that $(\sigma, \tau) \in \mathfrak{H} \times \mathfrak{H}$ is general.

Lieberman involutions (even involutions)

For $(i,j) \in \{1,2,3\}^2$, set

$$t_{ij} := (e_i(\sigma), e_j(\tau)) \in E_\sigma \times E_\tau.$$

Let $T_{t_{ij}} : E_\sigma \times E_\tau \to E_\sigma \times E_\tau$ be the translation by t_{ij}.

Then

$$([-1]_{E_\sigma}, \mathrm{id}_{E_\tau}) \circ T_{t_{ij}} : E_\sigma \times E_\tau \to E_\sigma \times E_\tau$$

induces a (fixed-point-free, anti-symplectic) involution on $X_{\sigma,\tau} := \mathrm{Km}(E_\sigma \times E_\tau)$.

These involutions are called Lieberman (even) involutions.
There are 9 of them (with respect to the choice of $(i,j) \in \{1,2,3\}^2$).

Kondo-Mukai involutions (odd involutions)

Odd involutions are more difficult to describe.

The Weierstrass $\wp$-function gives $2:1$ map $E_\sigma \to \mathbb{P}^1$ be the $2:1$ map ramified at $\{\wp(e_1(\sigma)), \wp(e_2(\sigma)), \wp(e_3(\sigma)), \infty\}$.

We identify $\mathbb{P}^1 \times \mathbb{P}^1$ with a quadratic in $\mathbb{P}^3$.
Let $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be projection from $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$.
Let $\varepsilon : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ be the the restriction of π to $\mathbb{P}^1 \times \mathbb{P}^1$.
Then ε is a birational map, and contracts two curves P and Q.

Let γ be a permutation of $\{1, 2, 3\}$. Set
$g_i = \varepsilon((\wp(e_i(\sigma)), \wp(e_{\gamma(i)}(\tau)))) \in \mathbb{P}^2$. Consider the Cremona transformation centered at g_1, g_2, g_3 interchanging P and Q.

Kondo-Mukai involutions (odd involutions) (cont'd)

$$X_{\sigma,\tau} \to X_{\sigma,\tau}/([-1]_{E_\sigma}, \mathrm{id}_{E_\tau}) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \overset{\varepsilon}{\dashrightarrow} \mathbb{P}^2$$

is a $2:1$ map.

The Cremona transformation on $\mathbb{P}^2$ induces two automorphisms $\nu, \mu : X_{\sigma,\tau} \to X_{\sigma,\tau}$. Then $\nu \circ \mu : X_{\sigma,\tau} \to X_{\sigma,\tau}$ is a (fixed-point-free, anti-symplectic) involution.

These involutions are called odd (Kondo-Mukai) involutions. There are 6 of them (with respect to the choice of $\gamma \in S_3$).

Remark Kondo-Mukai involutions are first found by Kondo, and then completed by Mukai. Ohashi (2007) showed that, if (σ, τ) is generic, then these $9 + 6 = 15$ involutions are the only involutions on $X_{\sigma,\tau}$ up to conjugacy. Further, he showed that there is a natural bijection between the conjugacy class of 15 involutions and $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4} \setminus \{0\}$.

Main Theorem (revisited)

For $(\sigma, \tau) \in \mathfrak{H} \times \mathfrak{H}$

$$\frac{\prod_{\iota:\text{odd}} \|\Phi\|(\overline{\varpi}(X_{\sigma,\tau}/\iota))^3}{\prod_{\iota:\text{even}} \|\Phi\|(\overline{\varpi}(X_{\sigma,\tau}/\iota))^2} = |j(\sigma) - j(\tau)|^6 .$$

(To be precise, the above formula holds for general (σ, τ), but we can interpret the formula for any (σ, τ).)

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Part 3 Ideas of proof

Setting

$$\begin{aligned}
Y(1) &:= \mathsf{SL}_2(\mathbb{Z})\backslash\mathfrak{H} && \cong \mathbb{C}\\
X(1) &:= Y(1) \cup \{\infty\} && \cong \mathbb{P}^1\\
\mathsf{Sym}^2(X(1)) &:= (X(1) \times X(1))\,/\Sigma_2 && \cong \mathbb{P}^2\\
B &:= \mathsf{Sym}^2(X(1)) \setminus \mathsf{Sym}^2(Y(1)) && \subset \mathbb{P}^2\\
\Delta &:= \text{diagonal locus of } \mathsf{Sym}^2(X(1)) && \subset \mathbb{P}^2
\end{aligned}$$

Given each of the 15 involutions ι on Kummer surfaces of product type, we have a holomorphic map $f_\iota : \mathfrak{H} \times \mathfrak{H} \to \mathcal{M}$. (Explained later.)

We are going to compare the pullback of $f_\iota^* \|\Phi\|$ (with various ι) and $|j(\sigma) - j(\tau)|^2$ on $\mathsf{Sym}^2(X(1)) \cong \mathbb{P}^2$.

Construction of $f_\iota : \mathfrak{H} \times \mathfrak{H} \to \mathcal{M}$

Assume that $(\sigma, \tau) \in \mathfrak{H} \times \mathfrak{H}$ is general. Let $X_{\sigma,\tau} = \mathsf{Km}(E_\sigma \times E_\tau)$ be a Kummer surface of product type, and
$\iota : X \to X$ be a (fixed-point-free, anti-symplectic) involution.

The rational map $E_\sigma \times E_\tau \dashrightarrow \mathsf{Km}(E_\sigma \times E_\tau)$ induces

$$H^1(E_\sigma, \mathbb{Z}) \otimes H^1(E_\sigma, \mathbb{Z}) \to H^2_-(X_{\sigma,\tau}, \mathbb{Z}).$$

Via the isometry $\alpha : H^2(X_{\sigma,\tau}, \mathbb{Z}) \to \mathbb{L}_{K3}$, we define

$$K_\iota := \alpha\left(\mathsf{Im}(H^1(E_\sigma, \mathbb{Z}) \otimes H^1(E_\sigma, \mathbb{Z}) \to H^2_-(X_{\sigma,\tau}, \mathbb{Z}))\right) \subset \Lambda \subset \mathbb{L}_{K3}.$$

Since $(\sigma, \tau) \in \mathfrak{H} \times \mathfrak{H}$ is general, we have

$$K_\iota \cong U(2) \oplus U(2).$$

Construction of $f_\iota : \mathfrak{H} \times \mathfrak{H} \to \mathcal{M}$ (cont'd)

The holomorphic $(2,0)$-form $dz_\sigma \otimes dz_\tau \in H^1(E_\sigma, \mathbb{C}) \otimes H^1(E_\sigma, \mathbb{C})$ gives an element $\eta_{X_{\sigma,\tau}} \in H^2_-(X_{\sigma,\tau}, \mathbb{Z}) \otimes \mathbb{C}$.

Via $\alpha : H^2_-(X_{\sigma,\tau}, \mathbb{Z}) \to \Lambda$, we obtain

$$[\alpha_{\mathbb{C}}(\eta_{X_{\sigma,\tau}})] \in \Omega^+_\Lambda \in \mathbb{P}(\Lambda \otimes \mathbb{C}).$$

Thus we have

$$f_\iota : \mathfrak{H} \times \mathfrak{H} \ni (\sigma, \tau) \dashrightarrow [\alpha_{\mathbb{C}}(\eta_{X_{\sigma,\tau}})] \in \mathcal{M} := \Omega^+_\Lambda / O^+(\Lambda)$$

In fact, f_ι is defined even if (σ, τ) is not general, and we have the holomorphic map

$$f_\iota : \mathfrak{H} \times \mathfrak{H} \to \mathcal{M}.$$

Image of f_ι

According to the parity of ι, $f_\iota : \mathfrak{H} \times \mathfrak{H} \to \mathcal{M}$ behaves differently.

Recall that $\mathcal{M}^\circ = \mathcal{M} \setminus \overline{\mathcal{D}}$, where $\overline{\mathcal{D}} := \mathcal{D}/O^+(\Lambda)$ is the discriminant locus of $\mathcal{M}$.

Proposition

(1) *Suppose ι is an even involution. Then* $\mathsf{Im}(f_\iota) \cap \overline{\mathcal{D}} = \emptyset$.

(2) *Suppose ι is an odd involution. Then* $\mathsf{Im}(f_\iota) \cap \overline{\mathcal{D}} \neq \emptyset$.

Proposition *Suppose ι is an odd involution, and $d, \delta \in \Lambda$ satisfy $d^2 = \delta^2 = -2$ and*

$$\mathsf{Im}(f_\iota) \cap d^\perp = \mathsf{Im}(f_\iota) \cap \delta^\perp.$$

Then $\delta = \pm d$ or $\delta = \pm\varsigma(d)$, where $\varsigma : \Lambda \to \Lambda$ is induced from $([-1]_{E_\sigma}, \mathsf{id}_{E_\tau}) : X_{\sigma,\tau} \to X_{\sigma,\tau}$.

These propositions can be proven by lattice theoretic arguments.

Pull-back by f_ι

We pull back $\|\Phi\|$ by $f_\iota : \mathfrak{H} \times \mathfrak{H} \to \mathcal{M}$.

We consider three functions $\prod_{\iota:\mathrm{odd}} f_\iota^* \|\Phi\|$, $\prod_{\iota:\mathrm{even}} f_\iota^* \|\Phi\|$, and $|j(\sigma) - j(\tau)|^2$ on $\mathfrak{H} \times \mathfrak{H}$.

These functions are all equivariant under the action of $\mathsf{SL}_2(\mathbb{Z}) \times \mathsf{SL}_2(\mathbb{Z})$ and the switching of the first and second factors of $\mathfrak{H} \times \mathfrak{H}$.

So, we consider these functions on $\mathsf{Sym}^2(X(1)) := ((\mathsf{SL}_2(\mathbb{Z})\backslash\mathfrak{H})^* \times (\mathsf{SL}_2(\mathbb{Z})\backslash\mathfrak{H})^*)/\Sigma_2$.

Poincaré metric

Recall that $Y(1) = \mathsf{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \cong \mathbb{C}$ and $X(1) = Y(1) \cup \{\infty\} \cong \mathbb{P}^1$

Let $\omega_{\mathfrak{H}\times\mathfrak{H}}$ be the Kähler form of the Poincaré metric on $\mathfrak{H} \times \mathfrak{H}$, i.e.,

$$\omega_{\mathfrak{H}\times\mathfrak{H}}(\sigma, \tau) := -dd^c \log \mathsf{Im}\, \sigma - dd^c \log \mathsf{Im}\, \tau.$$

Let $\omega_{\mathsf{Sym}^2(Y(1))}$ be the Kähler form on $\mathsf{Sym}^2(Y(1))$ in the sense of orbifolds induced from $\omega_{\mathfrak{H}\times\mathfrak{H}}$.

The $(1,1)$-from $\omega_{\mathsf{Sym}^2(Y(1))}$ extends trivially to a closed positive $(1,1)$-current on $\mathsf{Sym}^2(X(1))$, which we still denote by $\omega_{\mathsf{Sym}^2(Y(1))}$.

Pull-back by f_ι (cont'd)

Recall that $\mathsf{Sym}^2(X(1)) \cong \mathbb{P}^2$,
$B := \mathsf{Sym}^2(X(1)) \setminus \mathsf{Sym}^2(Y(1)) \subset \mathbb{P}^2$, and
$\Delta :=$ diagonal locus of $\mathsf{Sym}^2(X(1)) \subset \mathbb{P}^2$.

Theorem *As* $(1,1)$ *currents on* $\mathsf{Sym}^2(X(1))$, *we have the following.*

(1) $-dd^c \left[\log \prod_{\iota:odd} f_\iota^* \|\Phi\|^2\right] = 24\omega_{\mathsf{Sym}^2(Y(1))} - \delta_\Delta$.

(2) $-dd^c \left[\log \prod_{\iota:even} f_\iota^* \|\Phi\|^2\right] = 36\omega_{\mathsf{Sym}^2(Y(1))} - 3\delta_B$.

(3) $-dd^c \left[\log |(j(\sigma) - j(\tau))^2|^2\right] = -\delta_\Delta + 2\delta_B$.

Remark Since $j : X(1) \to \mathbb{C}$ is an isomorphism and j has a simple pole at $\infty \in \mathbb{P}^1$, we get Theorem(3).

(1) $-dd^c\left[\log\prod_{\iota:\mathrm{odd}} f_\iota^*\|\Phi\|^2\right] = 24\omega_{\mathrm{Sym}^2(Y(1))} - \delta_\Delta$.
(2) $-dd^c\left[\log\prod_{\iota:\mathrm{even}} f_\iota^*\|\Phi\|^2\right] = 36\omega_{\mathrm{Sym}^2(Y(1))} - 3\delta_B$.

Since Φ is an automorphic form of weight 4, as differential forms on some open sets of $\mathfrak{H}\times\mathfrak{H}$, we have

$$-dd^c f_\iota^*\|\Phi\|^2 = 4\omega_{\mathfrak{H}\times\mathfrak{H}},$$

which explains the coefficients 24 and 36 in Theorem(1)(2).

Using

$$\mathrm{Im}(f_\iota)\cap\overline{\mathcal{D}}\quad\begin{cases} =\emptyset & \text{if } \iota \text{ is even}\\ \neq\emptyset & \text{if } \iota \text{ is odd}\end{cases},$$

etc., we get the term $-\delta_\Delta$ and $-3\delta_B$ in Theorem(1)(2).

Proof of Main Theorem

As $(1,1)$ currents on $\mathsf{Sym}^2(X(1))$, Theorem(1)(2)(3) gives

$$dd^c \left[\log \left(\frac{\prod_{\iota:\mathrm{odd}} (\phi_\iota^* \|\Phi\|^2)^3}{\prod_{\iota:\mathrm{even}} (f_\iota^* \|\Phi\|^2)^2} \cdot |j(\sigma) - j(\tau)|^{-12} \right) \right] = 0.$$

Since $\mathsf{Sym}^2(X(1))$ is compact, we have

$$\frac{\prod_{\iota:\mathrm{odd}} (\phi_\iota^* \|\Phi\|^2)^3}{\prod_{\iota:\mathrm{even}} (f_\iota^* \|\Phi\|^2)^2} = C\, |j(\sigma) - j(\tau)|^{12}\,,$$

which proves **Main Theorem** up to a constant C.

Finally, comparing the infinite products of both hand-sides, we get $C = 1$.