Complex Geometric Structures on Lie Groups and Their Compact Quotient Spaces

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Part I. Locally conformal Kähler structures on compact homogeneous and locally homogeneous manifolds.

Part II. Skew-symmetric complex structures on Hermitian symmetric spaces of non-compact type.

* Part I is based on a joint work with **Y. Kamishima**; and Part II is based on a joint work with **V. Cortés**.

0. Introduction

A *locally conformal Kähler (LCK)* manifold M is a Hermitian manifold of which the fundamental form ω satisfies $d\omega = \omega \wedge \theta$ for some 1-form θ.

A Hopf manifold is known as the first example of a compact non-Kähler LCK manifold. A *generalized Hopf manifold* is a compact complex manifold of which the universal covering is $\mathbf{C}^n - \{0\}$. We have shown that any generalized Hopf manifold admits a LCK structure, using the technique of *Kähler potential*.

A classification of homogeneous Kähler manifolds is now well known; in particular a compact homogeneous Kähler manifold is biholomorphic to the product of a complex torus and a flag manifold. We have shown that a compact homogeneous LCK manifold (which is non-Kähler) is biholomorphic to a complex torus $T^1_{\mathbf{C}}$ bundle over a flag manifold.

We will consider a class of compact *locally homogeneous manifolds*, which are of the form $\Gamma\backslash G/K$, where G/K is a simply connected homogeneous manifold and Γ is a discrete subgroup of G. We have obtained some results on compact locally homogeneous (LCK) Kähler manifolds.

We know that an Hermitian symmetric space of non-compact type is a homogeneous Kähler manifold which is biholomorphic to a symmetric bounded domain; and the converse with the Bergman metric also holds. We consider a slightly generalized class of pseudo-Hermitian symmetric spaces of non-compact type, which can be constructed by defining a left-invariant complex structure compatible with the Kähler structure on a normal J-agebra associated to the Hermitian symmetric space. We found some examples of homogeneous domains which are pseudo-Hermitian but not Hermitian symmetric spaces of non-compact type. This contrasts with the fact [6] that an Hermitian symmetric space of compact type admits only the original J and $-J$ as compatible complex structures.

Part I

1. Preliminaries

Let M be a *homogeneous space* of Lie group G. We can express M as G/H, where G is a simply connected Lie group, H a closed subgroup of G. Let H_0 be the identity component of H.

Then, $\widetilde{M} = G/H_0$ is simply connected and a principal bundle over $M = G/H$ with structure group $\Gamma = H/H_0$ (the fundamental group of M) acting on $\widetilde{M}$ from the right.

We also consider the case when a discrete subgroup Γ of G is acting freely and properly discontinuously on $\widetilde{M}$ from the left. In this case M can be considered as $\Gamma\backslash G/H_0$ (double coset space), which defines a *locally homogeneous space.*

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebra of G and H_0 respectively. The tangent space of $\widetilde{M} = G/H_0$ is a G-bundle $G \times_{H_0} \mathfrak{g}/\mathfrak{h}$ over $\widetilde{M}$, where the action of H_0 on $\mathfrak{g}/\mathfrak{h}$ is given by the adjoint action $Ad(h)\,(h \in H_0)$.

Definition 1.

(1) A *homogeneous complex structure* on $\widetilde{M} = G/H_0$ is defined by an integrable complex structure J on $\mathfrak{g}/\mathfrak{h}$, which satisfies the condition $JAd(h) = Ad(h)J$ for $h \in H_0$, which is equivalent to the condition $Jad(X) = ad(X)J$ for $X \in \mathfrak{h}$ (since H_0 is connected).

(2) In the case when there exists a subspace $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{q} + \mathfrak{h}$ and $ad(X)(\mathfrak{q}) \subset \mathfrak{q}$ for $X \in \mathfrak{h}$, J may be defined on $\mathfrak{q}$ satisfying $Jad(X) = ad(X)J$ for $X \in \mathfrak{h}$.

(3) A homoegenous complex structure J on M is a homogeneous complex structure on $\widetilde{M}$ which is invariant by the right action of Γ. We may define it as an integrable complex structure on J on $\mathfrak{g}/\mathfrak{h}$, which satisfies the condition $JAd(h) = Ad(h)J$ for $h \in H$.

(4) If a discrete subgroup Γ of G is acting freely and properly discontinuously on $\widetilde{M}$ from the left, a homogeneous complex structure J on $\widetilde{M}$ defines a complex structure on $M = \Gamma\backslash G/H_0$, which we call a *locally homogeneous complex structure* or *left-invariant complex structure* on M.

Notes.

(1) A homogeneous complex structure on a simply connected Lie group G is nothing but a left-invariant complex structure on G; and it is both left and right invariant if and only if G is a complex Lie group.

(2) A homogeneous complex structure on a compact manifold M is said to be *complex-homogeneous*, that is, M can be written as $G_{\mathbf{C}}/D$, where $G_{\mathbf{C}}$ is a complex Lie group with closed complex subgroup D (since the automorphism group of M is a complex Lie group) [23].

(3) A homogeneous complex structure on $M = G/\Gamma$, where Γ is a discrete subgroup of a simply connected solvable Lie group G is a G-left and Γ-right invariant complex structure on G; but then G is actually a complex solvable Lie group.

(4) A discrete subgroup Γ of a simply connected G defines a locally homogeneous (or left-invariant) complex structure on $\Gamma\backslash G$.

Definition 2.

(1) M is a *homogeneous complex Kähler manifold*, if M is a homogeneous complex manifold G/H which admits a Kähler structure.

(2) M is a *homogeneous Kähler manifold*, if it is a homogeneous complex Kähler manifold G/H and the Kähler structure is invariant by the action of G from the left.

(3) If a discrete subgroup Γ of G acts freely and properly discontinuously on a simply connected homogeneous Kähler manifold G/K from the left, it defines a *locally homogeneous (or left-invariant) Kähler structure* on $M = \Gamma\backslash G/K$, where K is a compact subgroup of G.

We have the following fundamental result on compact homogeneous Kähler manifolds.

Theorem 1 (Matsushima, Borel-Remmert) ([16], [7]).

A compact homogeneous complex Kähler manifold is biholomorphic to a product of a complex torus and a flag manifold (which is a compact simply connected homogeneous algebraic manifold).

We have a class of compact locally homogeneous (left-invariant) Kähler manifolds which do not admit any homogeneous Kähler structures.

Example 1.

Let $G =: \mathbf{C}^l \rtimes \mathbf{R}^{2k}$, where the action $\phi : \mathbf{R}^{2k} \to \mathrm{Aut}(\mathbf{C}^l)$ is defined by

$$\phi(\bar{t}_i)((z_1, z_2, \dots, z_l)) = (e^{\sqrt{-1}\,\eta_1^i\,t_i} z_1, e^{\sqrt{-1}\,\eta_2^i\,t_i} z_2, ..., e^{\sqrt{-1}\,\eta_l^i\,t_i} z_l),$$

where $\bar{t}_i = t_i e_i$ (e_i: the i-th unit vector in $\mathbf{R}^{2k}$), and $e^{\sqrt{-1}\,\eta_j^i}$ is the s_i-th root of unity, $i = 1, \dots, 2k, j = 1, \dots, l$.

If an abelian lattice $\mathbf{Z}^{2l}$ of $\mathbf{C}^l$ is preserved by the action ϕ on $\mathbf{Z}^{2k}$, then $M = \Gamma\backslash G$ defines a solvmanifold, where $\Gamma = \mathbf{Z}^{2l} \rtimes \mathbf{Z}^{2k}$ is a lattice of G.

The Lie algebra $\mathfrak{g}$ of G is the following:

$$\mathfrak{g} = \{X_1, X_2, \dots, X_{2l}, X_{2l+1}, \dots, X_{2l+2k}\}_{\mathbf{R}},$$

where the bracket multiplications are defined by

$$[X_{2l+2i}, X_{2j-1}] = -X_{2j}, [X_{2l+2i}, X_{2j}] = X_{2j-1}$$

for $i = 1, \ldots, k, j = 1, \ldots, l$, and all other brackets vanish.

The canonical left-invariant complex structure is defined by

$$JX_{2j-1} = X_{2j}, JX_{2j} = -X_{2j-1},$$

$$JX_{2l+2i-1} = X_{2l+2i}, JX_{2l+2i} = -X_{2l+2i-1}$$

for $i = 1, \ldots, k, j = 1, \ldots, l$.

Notes.

(1) The class of complex surfaces with $l = k = 1$ in the above example coincides with the class of hyperelliptic surfaces [13].

(2) The above class of compact Kähler manifolds is exactly the class of compact locally homogeneous Kähler solvmanifolds ([13], [14]).

(3) It is well known that a simply connected homogeneous Kähler manifold is biholomorphic to $\mathbf{C}^k \times S \times D$, where S is a flag manifold, which is a projective manifold, D is a bounded domain.

We conjecture that a compact locally homogeneous Kähler manifold is, up to finite covering, biholomorphic to $T^k_{\mathbf{C}} \times S \times \Gamma\backslash D$, where $T^k_{\mathbf{C}}$ is a complex torus and D is a symmetric bounded domain. Remark that a a bounded domain admits a discrete automorphism group Γ such that $\Gamma\backslash D$ is compact if and only if it is a symmetric bounded domain.

2. Locally conformal Kähler structures

Definition 3.

(1) A *locally conformal Kähler structure* (*LCK structure* for short) on M is a Kähler structure ω on the universal covering $\widetilde{M}$ on which the the fundamental group Γ acts homothetically; that is, for every $\gamma \in \Gamma$, $\gamma^*\omega = \rho(\gamma)\omega$ holds for some positive costant $\rho(\gamma)$ (see [15]).

Let $\mathcal{H}(\widetilde{M})$ be the group of (holomorphic) homothetic transformations on M. We call $\rho : \mathcal{H}(\widetilde{M}) \to \mathbf{R}^+$ the monodromy map (which is a group homomorphism).

(2) A *locally conformal homogeneous Kähler structure* (or *homogeneous LCK*) on $M = G/H$ is defined by a homogeneous conformal Kähler structure ω on $\widetilde{M} = G/H_0$ on which Γ acts (holomorphically and) homothetically from the right. In other words, $G \subset \mathcal{H}(\widetilde{M})$, and Γ acts homothetically from the right.

(3) A *locally homogeneous LCK structure* on $M = \Gamma\backslash G/K$ is a homogeneous LCK structure on G/K with a discrete subgroup Γ of G which acts freely and properly discontinuously on G/K from the left.

Remark. A conformal Kähler structure can be defined by a Hermitian structure of which the fundamental form Ω satisfies $d\,\Omega = \theta \wedge \Omega$ for some closed 1-form θ (called *Lee form*).

Example 2.

Let $G = \mathbf{R} \times \mathrm{SU}(2) = \mathbf{R} \times \mathrm{U}(2)/\mathrm{U}(1)$ be a simply connected reductive Lie group, which is diffeomorphic to $\mathbf{R} \times S^3$. For $\lambda \in \mathbf{C}, 0 < |\lambda| < 1$, we have a diffeomorphism $\Phi : \mathbf{R} \times S^3 \to \mathbf{C}^2 - \{0\}$ defined by

$$\Phi : (t, z_1, z_2) \longrightarrow (\lambda^t\, z_1, \lambda^t\, z_2),$$

where $S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$, which induces a left-invariant complex structure on G.

The group homomorphism $(t, z_1, z_2) \to (t+1, z_1, z_2)$ corresponds to the holomorphic automorphism $(z_1, z_2) \to (\lambda z_1, \lambda z_2)$ on $\mathbf{C}^2 - \{0\}$; and $h \in \mathrm{SU}(2)$ corresponds the holomorphic automorphism $(z_1, z_2) \to h(z_1, z_2)$.

G acts as a group of homothetic transformations w.r.t. the standard Kähler structure $\omega = -\sqrt{-1} d\, z_1 \wedge d\,\overline{z_1} + d\, z_2 \wedge d\,\overline{z_2}$ on $\mathbf{C}^2 - \{0\}$.

The fundamental form and Lee form are given by $\Omega = \frac{1}{(|z_1|^2+|z_2|^2)}\omega$ and $\theta = -\frac{1}{(|z_1|^2+|z_2|^2)}\sum_{i=1}^{2}(z_i d\,\overline{z_i} + \overline{z_i}\,d\,z_i)$.
A uniform discrete subgroup Γ of $G = \mathbf{R} \times \mathrm{SU}(2)$ induces a locally homogeneous LCK structure on $\Gamma\backslash G$; and some Γ act homothetically on G from the right, defining a homogeneous LCK structure on G/Γ.

For instance, $\Gamma = \mathbf{Z} \subset \mathbf{R}$ clearly induces a homogeneous LCK structure on $S^1 \times \mathrm{SU}(2)$, which is biholomorphic to $\mathbf{C}^2 - \{0\}/\Gamma$. We also have a uniform discrete subgroup $\Gamma = \{(n, (-1)^n I_2) \in G \,|\, n \in \mathbf{Z}\}$, which induces a homogeneous complex structure and a homogeneous LCK structure on $U(2) = S^1 \rtimes SU(2)$.

Notes.

(1) A Hopf manifold is a compact complex manifold of which the universal covering is $\mathbf{C}^n - \{0\}$.

(2) For $G = \mathbf{R} \times \mathrm{U}(n)$ and $K = \mathrm{U}(n-1)$, G/K is diffeomorphic to $\mathbf{R} \times S^{2n-1}$. We can construct a homogeneous LCK manifold $(G/K)/\Gamma$ or a locally homogeneous LCK manifold $\Gamma\backslash G/K$ for some discrete subgroups Γ of G. Such a Hopf manifold is called of *linear type.*

(3) A (generalized) Hopf manifold is diffeomorphic to the linear one, which is a fiber bundle over S^1 with fiber $F\backslash S^{2n-1}$, where F is a finite subgroup of $\mathrm{U}(n)$ acting freely on S^{2n-1}.

Definition 4.

Let M be an LCK manifold. An *LCK potential* for M is a positive proper function ϕ on the universal covering $\{\widetilde{M}; \omega\}$, satisfying the following conditions:

(1) $-\sqrt{-1}\,\partial\overline{\partial}\phi = \omega$ (Kähler potential)

(2) For every $\gamma \in \Gamma$, $\gamma^*\phi = \rho(\gamma)\phi$ holds for some positive constant $\rho(\gamma)$

In Example 2, $\phi(z_1, z_2) = z_1\overline{z_1} + z_2\overline{z_2}$ is clearly defines a potential for any Hopf manifold M of linear type.

Theorem 2 (Ornea-Verbitsky) [19].

A small deformation of a compact LCK manifold with potential is also a LCK manifold with potential. In other words, LCK structure with potential is preserved under small deformations.

A generalized Hopf manifold M can be written as W/Γ, where W denotes $\mathbf{C}^n - \{0\}$, and Γ is the covering transformation group of M, which acts freely and properly discontinuously on W. Let $L(\Gamma)$ be a linear transformation group of W which consists of linear parts of each element of G. Then, we see that $L(\Gamma)$ and Γ is isomorphic as a group, and $L(\Gamma)$ acts on W freely and properly discontinuously.

We can then construct a complex analytic family

$$\{M_t | M_t = W/\Gamma(t) \ (t \in \mathbf{C}\},$$

where $\Gamma(t) = \{g_t | g \in \Gamma\}, g_t = L_t^{-1} g L_t$ for

$$L_t : (z_1, z_2, ..., z_n) \longrightarrow (tz_1, tz_2, ..., tz_n).$$

Note that $\Gamma(0) = L(\Gamma)$ and $M_0 = W/L(\Gamma)$; and M_{t_1} and M_{t_2} are biholomorphic for any $t_1, t_2 \neq 0$ (see [12]). Therefore, we have

Theorem 3.

Any generalized Hopf manifold admits a LCK structure with potential.

Remark. There exists an example of a compact complex surface which admits no LCK structures(due to Bergun [2]); it is an Inoue surface of type S^+ obtained by small deformations of the original Inoue surface of the same type S^+ which admits a LCK structure. This shows that small deformations do not preserve LCK structures.

3. Sasakian and LCK structures

Definition 5.

(1) A *contact metric structure* $\{\eta, \xi, \phi, g\}$ on M^{2n+1} is a contact structure $\eta\,, \eta \wedge (d\eta)^n \neq 0$ with the Reeb field $\xi\,, i(\xi)\eta = 1, i(\xi)d\eta = 0$, a $(1,1)$-tensor $\phi\,, \phi^2 = -I + \eta \otimes \xi$ and a Riemannian metric $g\,, g(X,Y) = \eta(X)\eta(Y) + d\,\eta(X, \phi Y)$ (see [3]).

(2) A *Sasaki structure* on M^{2n+1} is a contact metric structure $\{\eta, \xi, \phi, g\}$ satisfying $\mathcal{L}_\xi g = 0$ (Killing field) and the integrability of $J = \phi|\mathcal{D}$ on $\mathcal{D} = \ker \eta$ (CR-structure) (see [4]).

(3) The automorphism group $\mathcal{A}(M)$ of a Sasakian manifold M is the set of all diffeomorphisms ψ with $\psi^*\eta = \eta, J\psi_* = \psi_* J, \psi_*\mathcal{D} \subset \mathcal{D}$. M is a *homogeneous Sasakian manifold*, if $\mathcal{A}(M)$ acts transitively on M.

Notes.

(1) The automorphism group $\mathcal{A}(M)$ is a Lie group; and if M is compact, so is $\mathcal{A}(M)$.

(2) A locally homogeneous Sasakian manifold may be defined as in the case of Kähler or LCK structures.

Theorem 4 (Boothby-Wang).

Any compact homogeneous contact manifold $\{M, \eta\}$ admits a homogeneous Sasakian structure with contact form η, which is a S^1-bundle over a flag manifold.

We have a class of compact locally homogeneous Sasakian manifolds which do not admit any homogeneous Sasakian structures.

Example 3.

Let H_n be a Heisenberg Lie group, which can be expressed as a matrix form:

$$H_n = \left\{ \left(\begin{array}{ccc} 1 & \mathbf{x} & z \\ 0 & \mathrm{I}_n & \mathbf{y}^t \\ 0 & 0 & 1 \end{array} \right) \middle| \, \mathbf{x}, \mathbf{y} \in \mathbf{R}^n, z \in \mathbf{R} \right\},$$

of which the Lie algebra $\mathfrak{h}_n$ is generated by

$$\{X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z\}$$

with the bracket multiplications: $[X_i, Y_j] = \delta_{ij} Z$, and all other brackets are 0, where

$$X_i = \frac{\partial}{\partial x_i}, \; Y_j = \frac{\partial}{\partial y_j} + x_i \frac{\partial}{\partial z}, \; Z = \frac{\partial}{\partial z}.$$

H_n admits a family of uniform lattices

$$\Gamma_n^k = \left\{ \left(\begin{array}{ccc} 1 & \mathbf{a} & \frac{c}{k} \\ 0 & \mathrm{I}_n & \mathbf{b}^t \\ 0 & 0 & 1 \end{array} \right) \middle| \, \mathbf{a}, \mathbf{b} \in \mathbf{Z}^n, c \in \mathbf{Z} \right\},$$

where k is a fixed positive integer.

We can express Γ_n^k as a non-split short exact sequence

$$0 \to \mathbf{Z} \to \Gamma_n^k \to \mathbf{Z}^{2n} \to 0,$$

where $[\Gamma_n^k, \Gamma_n^k] = kZ$, and $\Gamma_n^k/[\Gamma_n^k, \Gamma_n^k] = \mathbf{Z}^{2n} \times \mathbf{Z}_k$.

$\mathcal{H}_n = \Gamma_n^k \backslash H_n$ is called a *Heisenberg manifold*, which is Sasakian with a contact metric structure $\{\eta, \xi, \phi, g\}$, where η is the dual form of Z, $\xi = Z$, and setting the dual forms of X_i, Y_j as ρ_i, σ_j, $i, j = 1, 2, ..., n$,

$$\phi = \sum_i Y_i \otimes \rho_i - X_i \otimes \sigma_i,$$

$$g = \sum_i \rho_i \otimes \sigma_i + \eta^2.$$

The complex structure J is simply defined as

$$JX_i = Y_i, JY_i = -X_i, i = 1, 2, ..., n.$$

Example 4.

A (generalized) Hopf manifold M of linear type can be written as $\Gamma\backslash G/K$ or $(G/K)/\Gamma$, where $G = \mathbf{R} \times \mathrm{U}(n), K = \mathrm{U}(n-1)$ and Γ is a discrete subgroup of G. We have $\mathrm{U}(n) = S^1 \times S$, where $S^1 = \{\alpha I_n || \alpha| = 1\}$ and S is a compact semi-simple Lie group. The diffeomorphism $\Phi : G/K = \mathbf{R} \times S^{2n-1} \to W = \mathbf{C}^n - \{0\}$ defined by

$$\Phi : (t, z_1, z_2, ..., z_n) \longrightarrow (\lambda^t z_1, \lambda^t z_2, ..., \lambda^t z_n)$$

induces a homogeneous complex structure on G/K. For the case where $\Gamma = \mathbf{Z}$, $\pi \cdot \Phi$ induces a holomorphic map from M onto $\mathbf{C}P^n$ where $\pi : W \to \mathbf{C}P^n$ is a canonical projection, which turns out to be a holomorphic $T^1_{\mathbf{C}}$ bundle over $\mathbf{C}P^n$.

We have shown that a compact homogeneous LCK manifold has a similar structure.

Theorem 5.

A compact homogeneous LCK manifold (which is non-Kähler) is biholomorphic to a complex torus $T^1_{\mathbf{C}}$ bundle over a flag manifold.

To be more precise, let $M = G/H$ be a homogeneous LCK manifold; then $G = \mathbf{R} \times (S^1 \times S)$ where S is a compact semi-simple Lie group, and $\mathbf{R} \times S^1$ is the center of G, which induces a complex torus action on M. M is diffeomorphic to a product of S^1 and a compact homogeneous Sasaki manifold, which is a S^1- bundle over a flag manifold S/K with a parabolic subgroup K of S.

A LCK manifold $\{M, \Omega\}$ with the Lee form $\theta\,(d\,\Omega = \theta \wedge \Omega)$ is *of Vaisman type* if the Lee form θ is parallel w.r.t. the Hermitian metric.

As expected, there is a close relation between LCK and Sasaki structures.

Theorem 6.

Let M be a compact LCK manifold of Vaisman type. Then M is a fiber bundle over S^1 with fiber a compact Sasakian manifold.

Note that as a consequence of Theorem 5, a compact homogeneous LCK manifold is of Vaisman type
Concerning compact locally homogeneous LCK manifolds, we have some partial results.

Theorem 7 (Sawai [24]).

A compact locally homogeneous LCK nilmanifold is biholomorphic to $\mathcal{H}_n \times S^1$.

We have a following slightly generalized result.

Theorem 8.

A homogeneous Sasakian (LCK) structure on a nilpotent Lie group is a Heisenberg Lie group H_n ($H_n \times \mathbf{R}$) with a standard Sasakian (LCK) structure.

Accordingly, a compact locally homogeneous Sasakian (LCK) nilmanifold is $\mathcal{H}_n$ ($\mathcal{H}_n \times S^1$). Furthermore, there is no homogeneous Sasakian (LCK) nilmanifolds.

Proof (of Theorem 8 for LCK).

Let $\mathfrak{g}$ be a nilpotent Lie algebra with LCK form Ω. Ω is a non-degenerate 2-form such that $d\Omega = \alpha \wedge \Omega$ for some closed 1-form α. By Theorem of Dixmier, there exists a 1-form β such that $\Omega = -\alpha \wedge \beta + d\beta$.

Let A, B be the dual element of $\mathfrak{g}$ corresponding to α, β. Let $\mathfrak{h}$ be the vector subspace of $\mathfrak{g}$ generated by A, B, and $\mathfrak{n}$ the orthogonal complement of $\mathfrak{h}$ w.r.t. Ω. Since Ω is non-degenerate, there exist $X_i, Y_j \in \mathfrak{n}$, $i, j = 1, ..., m$ such that $\mathfrak{n}$ is generated by X_i, Y_j, and $d\beta = \sum \rho_i \wedge \sigma_i$, where ρ_i, σ_i is the dual forms corresponding to X_i, Y_i.

Since $\Omega(X_i, Y_i) = d\beta(X_i, Y_i) = \beta([X_i, Y_i]) = 1$, we must have $[X_i, Y_i] = B \mod \mathfrak{n}$, $i, j = 1, ..., m$. Since $d\alpha(U, V) = \alpha([U, V]) = 0$ and $d\beta(U, V) = \beta([U, V]) = 0$ except for the case $U = X_i, V = Y_i$, We have that $[U, V] \in \mathfrak{n} = \{X_i, Y_j\}_{\mathbf{R}}$.

Let $\mathfrak{g}^(i) = [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$, $\mathfrak{g}^(0) = \mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, there exists some positive integer k such that $\mathfrak{g}^{(k)} \neq \{0\}$ and $\mathfrak{g}^{(k+1)} = \{0\}$.

We will show that $\mathfrak{g}^{(k)} = \{B\}$. In fact, any element Z of $\mathfrak{g}^{(k)}$ can be written as $Z = bB + \sum x_i X_i + y_j Y_j$ $(b, x_i, y_j \in \mathbf{R})$. Then $[Z, Y_i] = x_i B = 0 \mod \mathfrak{n}$, so $x_i = 0$. In the same way we get $y_j = 0$; and thus $\mathfrak{g}^{(k)} = \{B\}$.

The associated metric $g(U, V) = \Omega(U, JV)$ is positive definite. Since $g(A, A) = \Omega(A, JA)$ is non-zero, $JA = B$ (up to constant); and we may actually put $JA = B$. In fact, we can set $g(A, A) = g(B, B) = 1$, $JA = B + Z$, and $JB = -A + Z'$ for $Z \in \{A, X_i, Y_j\}, Z' \in \{B, X_i, Y_j\}$; and thus we have $Z' = -JZ$. Then we have

$$\Omega(A, JA) = \Omega(B + Z, JB + JZ) = \Omega(B, JB) + \Omega(Z, JZ),$$

from which we get $g(Z, Z) = \Omega(Z, JZ) = 0$. We also have $JX_i = Y_i$, $i, j = 1, ..., m$.

We can consider $\mathfrak{g}$ as an extension of $\mathfrak{n}'$ by A, where $\mathfrak{n}'$ is an extension of B by $\mathfrak{n}$:

$$0 \to \mathfrak{n}' \to \mathfrak{g} \to A \to 0$$

$$0 \to B \to \mathfrak{n}' \to \mathfrak{n} \to 0$$

Since $\{\mathfrak{n}, J\}$ is a nilpotent Kähler algebra, $\mathfrak{n}$ must be abelian (due to Hano). Since $\mathrm{ad(A)}(\mathfrak{n}') \subset \mathfrak{n}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{n}', \mathfrak{n}'] = \{B\}$. In particular, $\mathrm{ad(A)}(\mathfrak{n}') \subset \mathfrak{n} \cap \mathfrak{g}^{(1)} = \{0\}$.

We have shown that $\mathfrak{g}$ is an extension of $\mathfrak{h}$, which is an abelian ideal generated by A, B, by the abelian algebra $\mathfrak{n}$.

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{n} \to 0$$

Conjectures.

(1) A compact locally Homogeneous LCK solvmanifold is diffeomorphic to a "generalized Inoue manifold", which is a fiber bundle over a T^k with fiber T^l or a Heisenberg manifold.

(2) A compact nilmanifold (solvmanifold) admits a LCK structure if and only if it is a product of S^1 and Heisenberg manifold (a generalized Inoue manifold respectively).

(3) A compact nilmanifold (solvmanifold) admits a Sasakian structure if and only if it is a Heisenberg manifold.

Part II

4. Hermitian symmetric spaces and skew-symmetric complex structures

Definition 6

A Hermitian manifold M is *Hermitian symmetric* if each point $p \in M$ is an isolated fixed point of an involutive holomorphic isometry s_p of M.

(1) A Hermitian symmetric space M is a Riemannian symmetric space $\{M; g\}$ with a Hermitian complex structure J, defining a Kähler structure on M. It is a simply connected homogeneous Kähler manifold.

(2) A Hermitian symmetric space M is *irreducible* if it is irreducible as a Riemannian symmetric space (i.e. the holonomy representation is irreducible).

There are two types, *non-compact type* and *compact type*, of irreducible Hermitian symmetric spaces.

(a) If M is *of non-compact type*, then it can be written as G/H (effectively), where G is a connected non-compact simple Lie group with center $\{e\}$ and H is a maximal compact subgroup of G which has non-discrete center Z_H.

(b) If M is *of compact type*, then it can be written as G/H (effectively), where G is a connected compact simple Lie group with center $\{e\}$ and H is a maximal connected proper subgroup of G which has non-discrete center Z_H.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ that of H. Then we have the standard decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \text{ (as a vector space)},$$

where $\mathfrak{h} = \{X \in \mathfrak{g} | \sigma X = X\}$, $\mathfrak{m} = \{X \in \mathfrak{g} | \sigma X = -X\}$, and $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$ is isomorphic to the holonomy algebra $ad_{\mathfrak{m}}[\mathfrak{m}, \mathfrak{m}]$.

We know that any G-invariant complex structure is defined by $J \in GL(\mathfrak{m})$, satisfying the following conditions:

(1) $J^2 = -1$.

(2) $J \cdot ad_{\mathfrak{m}} X = ad_{\mathfrak{m}} X \cdot J$ for every $X \in \mathfrak{h}$.

(3) $[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$ for all $X, Y \in \mathfrak{m}$.

We have the Hermitian complex structure $J \in GL(\mathfrak{m})$ as the form $J = ad_{\mathfrak{m}} Z$ for some $Z \in \mathfrak{z}_{\mathfrak{h}}$, defining a G-invariant complex structure on M. We know that Z_H is actually a cyclic group with the Lie algebra $\mathfrak{z}_{\mathfrak{h}}$ of dimension 1. Hence, we have only two G-invariant complex structure J and $-J$, which are compatible with the Riemannian metric (see [11]).

Definition 7

A complex structure J on a Riemannian manifold $\{M; g\}$ is *orthogonal* or (*skew-symmetric*) if J is compatible with respect to the Riemannian metric g, defining a Hermitian structure $\{g, J\}$ on M.

Burstall-Rawnsley Conjecture [7]

Orthogonal complex structures on an irreducible Hermitian symmetric space $\{M; g, J\}$ are unique up to sign, namely $J, -J$.

They showed that the conjecture holds for Hermitian symmetric spaces of compact type. The proof is based on Twistor theory of symmetric spaces they have developed. They also show the following related result.

Theorem 9 (Burstall et al) [6]

An inner symmetric space of compact type admits an orthogonal complex structure if and only if it is a Hermitian symmetric space.

In particular, any sphere $S^{2m}, m \geq 2$ or their products admit no orthogonal complex structures. Recall that $S^{2m} = SO_{2m+1}/SO_{2m}$ with an involution σ defined by

$$\sigma(g) = sgs^{-1} \text{ for } g \in SO_{2m+1},$$

where

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -I_{2m} \end{pmatrix} \in SO_{2m+1}.$$

For a non-compact simple Lie group G, we have Iwasawa decomposition: $G = SH$, where S is a simply connected solvable Lie group (called the *Iwasawa group*).

S acts simply-transitively on the Hermitian symmetric space $M = G/H$. Hence, M can be considered as a homogeneous Kähler solvable Lie group.

Let $\mathfrak{s}$ be the Lie algebra of S. Then $\mathfrak{s}$ is a *non-unimodular* and *split* solvable Lie algebra, and has a so-called *normal J-algebra* structure, which is defined as follows:

Definition 8

A normal J-algebra is a solvable Lie algebra with an inner product $<,>$ and a complex structure $J \in GL(\mathfrak{s})$ ($J^2 = -1$), satisfying the following conditions:

(i) $< JX, JY >=< X, Y >$ for all $X, Y \in \mathfrak{s}$.

(ii) $< [X, Y], JZ > + < [Y, Z], JX > + < [Z, X], JY >$
$= 0$ for all $X, Y, Z \in \mathfrak{s}$.

(iii) $[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$
for all $X, Y, Z \in \mathfrak{s}$.

(iv) $ad_{\mathfrak{s}}X$ has only real eigenvalues for all $X \in \mathfrak{s}$.

(v) there is a linear form ω such that $< X, Y >= \omega[JX, Y]$.

A solvable Lie algebra satisfying (i), (ii), (iii) is called a *solvable Kähler algebra.* A solvable Lie algebra satisfying (iv) is of *split* (or *completely solvable*) type.

Theorem 10 (Gindikin-Vinberg [22], Pyatetskii-Shariro [20])

A split solvable Kähler algebra $\mathfrak{s}$ is decomposed into the semi-direct sum of an abelian J-invariant ideal and a normal J-algebra.

The corresponding Lie group S is a homogeneous Kähler solvmanifold which is biholomorphic to a direct product of $\mathbf{C}^k$ and a bounded homogeneous domain D.

It is known [9] that any homogeneous Kähler-Einstein metric on a bounded homogeneous domain is a positive multiple of the Bergman metric; and [17] that a bounded homogeneous domain with non-positive sectional curvature in Bergman metric is biholomorphic to a bounded symmetric domain.
Therefore, we see from [8] the following.

Theorem 11

A simply connected irreducible homogeneous Kähler-Einstein solvmanifold with non-positive sectional curvature is biholomorphic to an irreducible bounded symmetric domain; and the converse also holds for the Bergman metric.

Definition 9

J-algebras $\{\mathfrak{s}; J\}$ and $\{\mathfrak{s}'; J'\}$ are *isomorphic* if there exists a Lie algebra isomorphism $\phi : \mathfrak{s} \rightarrow \mathfrak{s}'$ such that $\phi J = J'\phi$.

(1) It is known (due to Pyatetskii-Shapiro) that there exists one to one correspondence between isomorphism classes of normal J-algebras and biholomorphic equivalence classes of bounded homogeneous domains ([20]).

(2) It is known (due to Dotti-Miatello [17]) that irreducible normal J-algebras $\{\mathfrak{s}; J\}$ and $\{\mathfrak{s}'; J'\}$ are *isomorphic* up to sign if and only if solvable Lie algebras $\mathfrak{s}$ and $\mathfrak{s}'$ are isomorphic as Lie algebras.

Let G be a connected simply connected (solvable) Lie group of dimension $2m$, and $\mathfrak{g}$ the Lie algebra of G

Lemma 1

An almost complex structure J on $\mathfrak{g}$ is integrable if and only if the subspace $\mathfrak{h}_J$ of $\mathfrak{g}_{\mathbf{C}}$ generated by $X + \sqrt{-1}JX$ $(X \in \mathfrak{g})$ is a complex subalgebra of $\mathfrak{g}_{\mathbf{C}}$ which satisfy $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_J \oplus \overline{\mathfrak{h}_J}$.

Lemma 2

Let V be a real vector space of dimension $2m$. Then, for a complex subspace W of $V \otimes \mathbf{C}$ such that $V \otimes \mathbf{C} = W \oplus \overline{W}$, there exists a unique $J_W \in \mathrm{GL}(V, \mathbf{R})$, ${J_W}^2 = -I$ such that $W = \{X + \sqrt{-1}J_W X | X \in V\}_{\mathbf{C}}$.

There exists one to one correspondence between complex structures J on $\mathfrak{g}$ and complex Lie subalgebras $\mathfrak{h}$ which satisfy $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h} \oplus \overline{\mathfrak{h}}$, given by $J \to \mathfrak{h}_J$ and $\mathfrak{h} \to J_{\mathfrak{h}}$.

For a complex structure J, the complex Lie subgroup H_J of $G_{\mathbf{C}}$ corresponding to $\mathfrak{h}_J$ is closed, simply connected, and $G_{\mathbf{C}}/H_J$ is biholomorphic to $\mathbf{C}^m$.

The canonical inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbf{C}}$ induces an inclusion $G \hookrightarrow G_{\mathbf{C}}$, and $\Gamma = G \cap H_J$ is a discrete subgroup of G. We have the following canonical map $g = i \circ \pi$:

$$G \xrightarrow{\pi} G/\Gamma \xhookrightarrow{i} G_{\mathbf{C}}/H_J,$$

where π is a covering map, and i is an inclusion. The left-invariant complex structure J on G is the one induced by g from an open set $U = \operatorname{Im} g \subset \mathbf{C}^m$ (see [21]).

Example 5

Let $\mathfrak{s}_{m+1}$ be a solvable Lie algebra of dimension $2m+2$ with a basis $\beta = \{X_i, Y_j, Z, W\}$ for which the bracket multiplications are defined by

$$[X_i, Y_i] = -Z,\ [W, X_j] = \frac{1}{2}X_j,\ [W, Y_k] = \frac{1}{2}Y_k,\ [W, Z] = Z,$$

where $i, j, k = 1, ..., m$, and all other brackets are 0.

We can express $\mathfrak{s}_{m+1}$ as the semi-direct sum of a nilpotent ideal $\mathfrak{n}_m$ generated by $X_i, Y_j, Z, i, j = 1, ..., m$ and an abelian Lie algebra $\mathfrak{w}$ generated by $\{W\}$.

The inner product $<,>$ is defined with respect to which β is an orthonormal basis.

The complex structure J is defined by

$$JW = Z, JZ = -W, JX_i = Y_i, JY_j = -X_j,$$

where $i, j = 1, ..., m$.

It is easy to check that J is integrable, and a linear form ω defined by

$$\omega(Z) = 1, \omega(X_i) = \omega(Y_j) = \omega(W) = 0,$$

satisfies $< A, B >= \omega([JA, B])$ for any $A, B \in \mathfrak{s}_{m+1}$; and thus $\{\mathfrak{s}_{m+1}; J\}$ is a (irreducible) normal J-algebra.

We now take another complex structure J_k on $\mathfrak{s}_{m+1}$. The complex structure J_k, $k = 1, 2, ..., m$ is defined by

$$J_kW = Z, J_kZ = -W, J_kX_i = Y_i, J_kY_i = -X_i, i = 1, 2, ..., k$$

and

$$J_kX_j = -Y_j, J_kY_j = X_j, j = k + 1, 2, ..., m,$$

then J_k is compatible with the inner product and integrable, but the condition (ii) of normal J-algebra does not hold (Kähler form is not closed).

We see that the complex subalgebra $\mathfrak{h}$ and $\mathfrak{h}_k$ of $\mathfrak{s}_{\mathbf{C}}$ corresponding to J and J_k is given by,

$$\mathfrak{h} = \{W + \sqrt{-1}Z, X_1 + \sqrt{-1}Y_1, X_2 + \sqrt{-1}Y_2, ..., X_m + \sqrt{-1}Y_m\}_{\mathbf{C}},$$

$$\mathfrak{h}_k = \{W+\sqrt{-1}Z, X_1+\sqrt{-1}Y_1, ..., X_k+\sqrt{-1}Y_k, X_{k+1}-\sqrt{-1}Y_{k+1}, ..., X_m-\sqrt{-1}Y_m\}_{\mathbf{C}}$$

where $[W + \sqrt{-1}Z, X_i \pm \sqrt{-1}Y_i] = \frac{1}{2}(X_i \pm \sqrt{-1}Y_i)$, $i = 1, 2, ..., m$.

The corresponding Lie group S_{m+1} is expressed as

$$S_{m+1} = H_m \rtimes \mathbf{R},$$

where H_m is the Heisenberg group in Example 3 and the action $\phi : \mathbf{R} \to \mathrm{Aut}\,(H_k)$ is defined by

$$\phi(s) : \begin{pmatrix} 1 & \mathbf{x} & z \\ 0 & \mathrm{I}_m & \mathbf{y}^t \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & e^{\frac{1}{2}s}\mathbf{x} & e^s z \\ 0 & I_m & e^{\frac{1}{2}s}\mathbf{y}^t \\ 0 & 0 & 1 \end{pmatrix}.$$

The complex subgroup $\mathcal{H}_k$ of $S_{\mathbf{C}}$ corresponding to $\mathfrak{h}_k$ is expressed as a semi-direct product $\mathcal{H}_k = \mathcal{U}_k \rtimes \mathcal{V}$, where

$$\mathcal{U}_k = \begin{pmatrix} 1 & \mathbf{u} & \frac{1}{2}\sqrt{-1}\|\mathbf{u}\|_k \\ 0 & \mathrm{I}_m & \sqrt{-1}\varepsilon_k \mathbf{u}^t \\ 0 & 0 & 1 \end{pmatrix}, k = 1, 2, ..., m,$$

$$\mathcal{V} = \Big(\begin{pmatrix} 1 & 0 & \sqrt{-1}(e^s - 1) \\ 0 & \mathrm{I}_m & 0 \\ 0 & 0 & 1 \end{pmatrix}, s \Big),$$

$\mathbf{u} \in \mathbf{C}^m, s \in \mathbf{C}$, $\|\mathbf{u}\|_k = \mathbf{u}\epsilon_k\mathbf{u}^t$ ($\epsilon_k = \begin{pmatrix} \mathrm{I}_{m-k} & 0 \\ 0 & -\mathrm{I}_k \end{pmatrix}$). Note that $\mathcal{U}_k$ is an abelian subgroup of $S_{\mathbf{C}}$ and $\mathcal{V}$ is a 1-parameter subgroup of $S_{\mathbf{C}}$ corresponding to $W + \sqrt{-1}V$.

Define $\phi_k : S_{\mathbf{C}} \to \mathbf{C}^{m+1}$ by

$$\Big(\begin{pmatrix} 1 & \mathbf{u} & z \\ 0 & \mathrm{I}_m & \mathbf{v}^t \\ 0 & 0 & 1 \end{pmatrix}, s \Big) \to (\mathbf{u} + \sqrt{-1}\epsilon_k \mathbf{v}, (< \mathbf{u}, \mathbf{v} > -2z) + \sqrt{-1}\,(\frac{1}{2}(\|\mathbf{u}\|_k^2 + \|\mathbf{v}\|_k^2) + 2e^s)).$$

Then, ϕ_k induces a biholomorphic map $\overline{\phi}_k : S_{\mathbf{C}}/\mathcal{H}_k \to \mathbf{C}^{m+1}$, and the image of S_{m+1} is the open subset of $\mathbf{C}^{m+1}$:

$$\mathcal{S}_k = \overline{\phi}_k(S_{m+1}) = \{(\mathbf{z}, w) \in \mathbf{C}^{m+1} \,|\, \mathrm{Im}\, w > \frac{1}{2}\|\mathbf{z}\|_k^2\}.$$

We know that $\mathcal{S}_0$ is biholomorphic to $D_{m+1} = \{(\mathbf{z}, w) |\, \|\mathbf{z}\|^2 + |w|^2 < 1\}$, which is a complex hyperbolic $(m+1)$-space (or a Siegel domain of type II). And we can see that $\mathcal{S}_m$ is biholomorphic to $D'_{m+1} = \{(\mathbf{z}, w) \in \mathbf{C}^{m+1} \,|\, \mathrm{Im}\, w < \frac{1}{2}\|\mathbf{z}\|^2\}$, which can be considered as $\mathbf{CP}^{m+1} - \overline{D}_{m+1} \cup \mathcal{P}$, where $\mathcal{P}$ is a projective m-plane tangent to the boundary of D_{m+1} (cf. [18] for $m = 1$).

Remark. The homogeneous complex solvmanifold $\mathcal{S}_k = \{S_{m+1}; J_k\}$ is non-Kähler in any S_{m+1}-invariant metric: Suppose it admits a S_{m+1}-invariant Kähler metric. Then $\{s_{m+1}; J_k\}$ defines an irreducible split solvable Kähler algebra. Since $\mathfrak{s}_{m+1}$ has no J_k-invariant abelian ideal, it is an irreducible normal J-algebra. But then, according to the above result of Dotti-Miatello, we must have $J_k = J, \mathrm{or} - J$. In particular, $\mathcal{S}_k$ is not biholomorphic to $\mathcal{S}_0 = \{S_{m+1}; \pm J\}$.

We now consider irreducible Hermitian symmetric spaces of non-compact type in general, that is irreducible bounded homogeneous symmetric domain. The following theorem was obtained based on Twistor theory of symmetric spaces.

Theorem 12 (Burstall and Rawnsley) [7]

Let $\{M; g, J\}$ be a Hermitian symmetric space of non-compact type. Then, any orthogonal complex structure J' on M commutes with J at each point of M.

Let D be an irreducible bounded homogeneous space on which a simply connected solvable Lie group S act simply-transitively, and $\{\mathfrak{s}, g, J\}$ be the corresponding irreducible normal J-algebra.

We have the structure theorem (due to Gindikin, Vinberg and Pyatetskii-Shapiro).

Theorem 13 (Gindikin-Vinberg and Pyatetskii-Shariro)

Let $\{\mathfrak{s}; g, J\}$ be a normal J-algebra. Then $\mathfrak{s}$ can be decomposed as

$$\mathfrak{s} = \mathfrak{f} \oplus \mathfrak{n},$$

where $\mathfrak{f}$ is abelian, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$. Furthermore, $\mathfrak{n}$ can be decomposed as an orthogonal direct sum of root spaces,

$$\mathfrak{n} = \oplus\, \mathfrak{n}_\alpha,$$

where $\mathfrak{n}_\alpha = \{X \in \mathfrak{n} \,|\, [A, X] = \alpha(A)X, A \in \mathfrak{f}\}$.

There are roots $\alpha_1, \alpha_2, ...\alpha_f$ ($f = \dim \mathfrak{f}$), $\dim \mathfrak{n}_{\alpha_i} = 1$ for $i = 1, ..., f$, and all other roots are expressed as

$$\frac{1}{2}\alpha_i, \frac{1}{2}(\alpha_j \pm \alpha_k)$$

for $j, k\, (1 \leq j < k \leq f)$. We also have

$$J\,\mathfrak{n}_{\alpha_i} = \mathfrak{f}, J\,\mathfrak{n}_{\frac{1}{2}\alpha_i} = \mathfrak{n}_{\frac{1}{2}\alpha_i}, J\,\mathfrak{n}_{\frac{1}{2}(\alpha_j \pm \alpha_k)} = \mathfrak{n}_{\frac{1}{2}(\alpha_j \mp \alpha_k)}.$$

In Example 5, $\mathfrak{f}$ is generated by W, and

$$\alpha(W) = 1,\ \mathfrak{n}_\alpha = \{Z\},\ \mathfrak{n}_{\frac{1}{2}\alpha} = \{X_i, Y_j\}.$$

In particular, if J and J' commute, we can take an orthonormal basis β, preserving the decomposition, with respect to which J and J' are expressed by the skew-symmetric matrices

$$J = J_1 \oplus J_1 \oplus \cdots \oplus J_1,$$

and

$$J' = \pm J_1 \oplus \pm J_1 \oplus \cdots \oplus \pm J_1,$$

where

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, we have a finite number of S-invariant skew-symmetric complex structures on Hermitian symmetric spaces of no-compact type $\{S; g, J\}$.

We still have to check which of those skew-symmetric complex structures are integrable, and whether some of them define biholomorphic complex structures.

Remark. The corresponding homogeneous domain $\{D; J'\}$ is non-Kähler in any S-invariant metric: Suppose it admits a S-invariant Kähler metric. Then $\{s; J'\}$ defines an irreducible split solvable Kähler algebra. Since $\mathfrak{s}$ has no J'-invariant abelian ideal, it is an irreducible normal J-algebra. But then we must have $J' = J, \text{or} - J$.

Example 6 (Apostolov et al) [1]

Let X_i be a generator of $\mathfrak{n}_{\alpha_i}$, $i = 1, ..., f$. Then we have $JX_i \in \mathfrak{f}$, and $X_1, JX_1, X_2, JX_2, ..., X_f, JX_f$ can be extended to an orthonormal basis of $\mathfrak{s}$.

Assume that $\mathfrak{n}_{\frac{1}{2}\alpha_i} = 0$, $i = 1, ..., f$, and let η be the dual of $X_f \wedge JX_f$ with respect to $<,>$. Then η is closed and J is compatible with η:

$$\eta(JX, JY) = \eta(X, Y)$$

$$\eta([X, Y], Z) + \eta([Y, Z], X) + \eta([Z, X], Y) = 0$$

If we define a new complex structure J' by $J' = -J$ on the J-invariant subalgebra $\mathfrak{x}_f$ generated by $\{X_f, JX_f\}$ and $J' = J$ on the orthogonal complement of $\mathfrak{x}_f$, then J' is compatible with $<,>$ and the fundamental form $\Omega - 2\eta$ is closed, where Ω is the fundamental from for J. Remark that J' is not integrable (due to the result of Dotti-Miatello).

It is known that the normal J-algebra associated to a Siegel domain of type I satisfies the condition $\mathfrak{n}_{\frac{1}{2}\alpha_i} = 0$, $i = 1, ..., f$.

We now consider skew-symmetric complex structures on a Lie algebra $\mathfrak{g}$ with an inner product $<,>$. Note that J is skew-symmetric with respect to $<,>$ if and only if $\mathfrak{h}_J$ is (*maximal*) *isotropic* with respect to $<,>_{\mathbf{C}}$, that is $< \mathfrak{h}_J, \mathfrak{h}_J >_{\mathbf{C}} = 0$.

Lemma 3

Let J be a left invariant complex structure on $\mathfrak{g}$ with its corresponding subalgebra $\mathfrak{h}_J$. Then, there is a one to one correspondence between left invariant complex structures J' commuting with J and decompositions

$$\mathfrak{h}_J = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are subalgebras of $\mathfrak{h}$ which satisfies

$$[\mathfrak{p}^+, \overline{\mathfrak{p}^-}] \subset \mathfrak{p}^+ \oplus \overline{\mathfrak{p}^-}.$$

In this case the subalgebra $\mathfrak{h}_{J'}$ corresponding to J' satisfies $\mathfrak{h}_{J'} = \mathfrak{p}^+ \oplus \overline{\mathfrak{p}^-}$ with $\mathfrak{p}^+ = \mathfrak{h}_J \cap \mathfrak{h}_{J'}$ and $\mathfrak{p}^- = \mathfrak{h}_J \cap \overline{\mathfrak{h}_{J'}}$. And J' is skew-symmetric with respect to $<,>$ if and only if $< \mathfrak{p}^+, \overline{\mathfrak{p}^-} >= 0$.

We consider again Example 5 of normal J-algebra $\mathfrak{s}_{m+1}$ (which is called an *elementary* J-algebra). Note that the corresponding homogeneous symmetric domain is a complex hyperbolic space $\mathbf{C}H^{m+1}$.

The corresponding subalgebra $\mathfrak{h}_J$ is a simi-direct sum of $\{W + \sqrt{-1}Z\}_{\mathbf{C}}$ and $\mathfrak{z}$, where $\mathfrak{z} = \{X_i + \sqrt{-1}Y_i\}_{\mathbf{C}}$ $(i = 1, 2, ..., m)$ is an abelian ideal of $\mathfrak{h}_J$ with

$$[W + \sqrt{-1}Z, X_i + \sqrt{-1}Y_i] = \frac{1}{2}(X_i + \sqrt{-1}Y_i).$$

Lemma 4

Let $\mathfrak{p}^+$ be a subalgebra of $\mathfrak{h}_J$ which corresponds to an orthogonal complex structure J' commuting with J, satisfying the condition in Lemma 3. Then $\mathfrak{p}^+$ is, up to isomorphism, one of the following types.

(1) $\mathfrak{p}^+$ is an abelian subalgebra of $\mathfrak{z}$

(2) $\mathfrak{p}^+$ is a semi direct sum of $\{W + \sqrt{-1}Z\}_{\mathbf{C}}$ and an abelian subalgebra $\mathfrak{z}'$ of $\mathfrak{z}$ with $[W + \sqrt{-1}Z, S] = \frac{1}{2}S$ for $S \in \mathfrak{z}'$.

We may assume, taking a suitable basis $\{U_i, V_i\}$ of $\mathfrak{z}$ if necessary, that $\mathfrak{p}^+ = \{W + \sqrt{-1}Z, U_i + \sqrt{-1}V_i\}_{\mathbf{C}}$ for $i = 1, 2, ..., k\,(1 \le k \le m)$, and $\mathfrak{p}^- = \{U_i + \sqrt{-1}V_i\}_{\mathbf{C}}$ for $i = k+1, 2, ..., m$, or the other way around, where J is defined by $JW = Z, JZ = -W, JU_i = V_i, JV_i = -U_i, i = 1, 2, ..., m$.

We have $\mathfrak{h}_{J_k} = \mathfrak{p}^+ \oplus \overline{\mathfrak{p}^-}$, where J_k is defined by

$$J_kW = Z, J_kZ = -W, J_kU_i = V_i, J_kV_i = -U_i, i = 1, 2, ..., k$$

and

$$J_kU_j = -V_j, J_kV_j = U_j, j = k+1, 2, ..., m,$$

or the other way around.

Therefore, there exist, up to sign, $m+1$ skew-symmetric complex structures on $\mathfrak{s}_{m+1}$. As remarked before, we still have to check that none of them define biholomorphic homogeneous complex structures ($\{\mathfrak{s}_{m+1}; J_k\}$ is not a normal J-algebra anymore).

Remark. For each J_k, $< X, Y >_k = < JX, J_kY >$ defines a pseudo-Kähler structure on $\mathfrak{s}_{m+1}$, and the corresponding solvable Lie group with the invariant pseudo-Kähler structure defines a pseudo-Hermitian symmetric space of non-compact type.

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