Variation formulas for principal functions (I)

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1. Preliminary

Let $B = \{|t| < \rho\} \subset \mathbb{C}_t$. For $\forall t \in B$, $R(t)$ is a bordered Riemann surface in $\tilde{R}$ parameterized over $\mathbb{C}_z$ s.t. $0 \in R(t) \subset \tilde{R}$.

We identify the variation of $R(t) : t \in B \to R(t)$ with the total space of complex 2-dim manifold $\mathcal{R} = \bigcup_{t \in B}(t, R(t)) \subset B \times \tilde{R}$.

We say that $\mathcal{R}$ is a domain in $B \times \tilde{R}$, parametrized by $B \times \mathbb{C}_z$ with smooth boundary if $\mathcal{R}$ satisfies the following conditions:

- $\partial R(t)$ consists of a finite number of C^ω smooth closed curves
- $\partial R(t)$ varies C^ω smoothly with $t \in B$.

Assume that $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ is an unramified domain in $B \times \tilde{R}$, parametrized by $B \times \mathbb{C}_z$ with smooth boundary and $B \times \{0\} \subset \mathcal{R}$. Let $g(t, z)$ be the Green function for $(R(t), 0)$, and $\lambda(t)$ be the Robin constant for $(R(t), 0)$. Then, for $t \in B$,

Fact [04, Maitani-Yamaguchi]

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi}\int_{\partial R(t)} k_2(t, z)\left|\frac{\partial g(t, z)}{\partial z}\right|^2 ds_z - \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 g(t, z)}{\partial \bar{t}\partial z}\right|^2 dxdy.$$

Here the function

$$k_2(t, z) = \left(\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\mathrm{Re}\left\{\frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar{z}}\right\} + \left|\frac{\partial\varphi}{\partial t}\right|^2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right)\left|\frac{\partial\varphi}{\partial z}\right|^{-3}$$

on $\partial\mathcal{R}$, which does not depend on the choice of defining functions $\varphi(t, z)$ of $\partial\mathcal{R}$, and ds_z is the arc length element of $\partial R(t)$ at z. The fn $k_2(t, z)$ is due to N.Levenberg-H.Yamaguchi.

The proof of this variation formula for $\lambda(t)$ depends on the following fact : the Green function is a defining fn of $\partial\mathcal{R}$.

Motivation and Results

Motivation

In more general case than the Green functions, is there analogy to the variation formula for the Robin constants?

Results

- There exist analogue variation formulas for principal functions with logarithmic poles.

	L_1-principal fn	L_0-principal fn
one logarithmic pole and C_0	○	×
two logarithmic poles	○	○ (R(t) is planar)

- Application 1: The variation theorem for the harmonic span $s(t)$ of planar Riemann surfaces $R(t)$
- Application 2: The simultaneous uniformization of the Schottky covering of compact Riemann surfaces $(g \geq 2)$

For $\forall t \in B$, $R(t)$ is a bordered Riemann surface in $\tilde{R}$ over $\mathbb{C}_z$ s.t. $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$ is C^{ω}-class and $0 \in R(t) \subset \tilde{R}$.

Definition 1. (L_1-principal fn with one logarithmic pole)

$\exists 1\ u(t,z)$: a real-valued fn on $R(t) \setminus \{0\}$ s.t.

1. $u(t,z)$ is harmonic on $R(t) \setminus \{0\}$ and is C^0 on $\overline{R(t)}$;
2. $\exists$ a neighborhood $U(0)$ s.t. $u(t,z) = \log \frac{1}{|z|} + \gamma(t) + h(t,z)$, where $h(t,0) = 0$;
3. $u(t,z) = 0$ on $C_0(t)$;
4. for each $j = 1, \ldots, \nu$,

   $$\text{(i) } u(t,z) = \text{constant } a_j(t) \text{ on } C_j(t), \quad \text{(ii) } \int_{C_j(t)} *du(t,z) = 0.$$

In the case when $R(t)$ is a **planar** Riemann surface, $u(t,z)$ induces a circular slit mapping.

For $\forall t \in B$, $R(t)$ is a bordered Riemann surface in $\tilde{R}$ over $\mathbb{C}_z$ s.t. $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$ is C^ω-class and $0 \in R(t) \subset \tilde{R}$.

Definition 1^*. (L_0-principal fun with one logarithmic pole)

$\exists 1$ $u(t,z)$: a real-valued fn on $R(t) \setminus \{0\}$ s.t.

1. $u(t,z)$ is harmonic on $R(t) \setminus \{0\}$ and is C^0 on $\overline{R(t)}$;
2. $\exists$ a neighborhood $U(0)$ s.t. $u(t,z) = \log \frac{1}{|z|} + \Gamma(t) + h(t,z)$, where $h(t,0) = 0$;
3. $u(t,z) = 0$ on $C_0(t)$;
4. for each $j = 1, \ldots, \nu$, $\frac{\partial u}{\partial n_z}(t,z) = 0$ on $C_j(t)$.

In the case when $R(t)$ is a **planar** Riemann surface, $u(t,z)$ induces a radial slit mapping.

Lemma 1.

Assume that $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ is a domain in $B \times \tilde{R}$ over $B \times \mathbb{C}_z$ with smooth boundary s.t. $R(t) \ni 0$. Then, for $t \in B$,

$$\frac{\partial^2 \gamma(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left| \frac{\partial u(t,z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 u(t,z)}{\partial \bar{t} \partial z} \right|^2 dxdy.$$

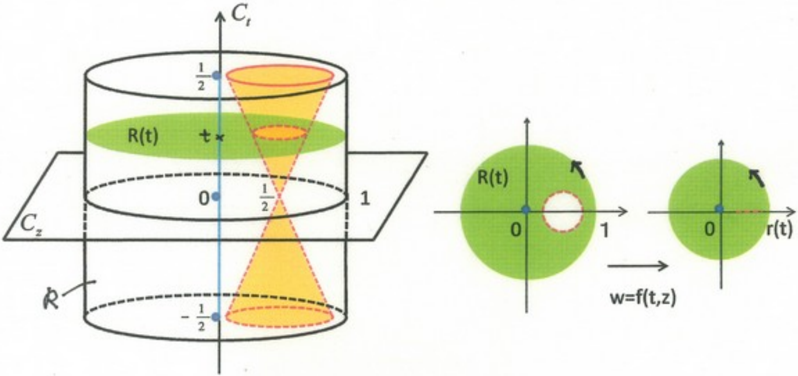
If $\mathcal{R}$ is a 2-dim pseudoconvex domain, then $k_2(t,z) \geq 0$ on $\partial \mathcal{R}$.

Theorem 1.

$\mathcal{R}$ is a **2-dim pseudoconvex domain** over $B \times \mathbb{C}_z$ with smooth boundary $\Longrightarrow$ $\gamma(t)$ is a **superharmonic fn** on B.

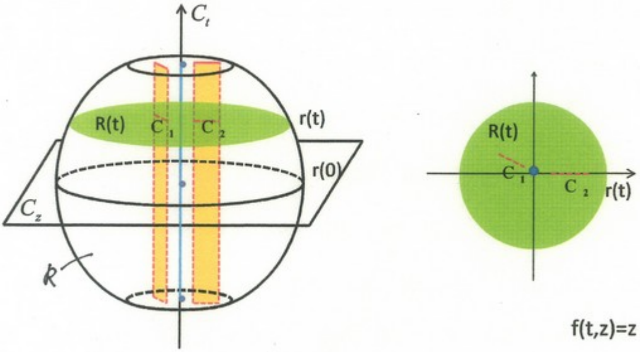
Cf. The radius of radial slit mapping

Remark The radius r(t) of radial slit mapping is **not** logarithmic superharmonic on B.



$$\mathcal{R} = \left\{|t| < \frac{1}{2}\right\} \times \{|z| < 1\} - \left\{\left|z - \frac{1}{2}\right| \le |t| < \frac{1}{2}\right\}$$

Remark The radius $r(t)$ of radial slit mapping is **not** logarithmic subharmonic on B.



$$\mathcal{R} = \bigcup_{t\in B}\{|z|<r(t)\} - \{B\times(C_1\cup C_2)\},$$

where $\log r(t)$ is superharmonic on B.

For $\forall t \in B$, $R(t)$ is a bordered Riemann surface in $\tilde{R}$ over $\mathbb{C}_z$ s.t. $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$ is C^ω-class and $R(t) \ni 0$, $\xi(t) (\neq 0$, holo for $t)$.

Definition 2. (L_1-principal fn with two logarithmic poles)

$\exists 1\ p(t,z)$: a real-valued fn on $R(t) \setminus \{0, \xi(t)\}$ s.t.

1. $p(t,z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and is C^0 on $\overline{R(t)}$;
2. $p(t,z) = \log \frac{1}{|z|} + h_0(t,z)$ on $U(0)$, where $h_0(t,0) = 0$;
3. $p(t,z) = \log |z - \xi(t)| + \alpha(t) + h_\xi(t,z)$ on $U(\xi(t))$, where $h_\xi(t, \xi(t)) = 0$;
4. for each $j = 0, 1, \ldots, \nu$,
   $\text{(i) } p(t,z) = \text{constant } a_j(t)$ on $C_j(t)$, $\quad \text{(ii) } \displaystyle\int_{C_j(t)} *dp(t,z) = 0.$

In the case when $R(t)$ is a **planar** Riemann surface, $u(t,z)$ induces a **circular** slit mapping.

For $\forall t \in B$, $R(t)$ is a bordered Riemann surface in $\tilde{R}$ over $\mathbb{C}_z$ s.t. $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$ is C^ω-class and $R(t) \ni 0$, $\xi(t)(\neq 0, \text{holo for } t)$.

Definition 2^*. (L_0-principal fn with two logarithmic poles)

$\exists 1\ q(t,z)$: a real-valued fn on $R(t) \setminus \{0, \xi(t)\}$ s.t.

1. $q(t,z)$ is harm on $R(t) \setminus \{0, \xi(t)\}$ and is C^0 on $\overline{R(t)}$;
2. $q(t,z) = \log \frac{1}{|z|} + h_0(t,z)$ on $U(0)$, where $h_0(t,0) = 0$;
3. $q(t,z) = \log|z - \xi(t)| + \beta(t) + h_\xi(t,z)$ on $U(\xi(t))$, where $h_\xi(t, \xi(t)) = 0$
4. for each $j = 0, 1, \ldots, \nu$,

$$\frac{\partial q(t,z)}{\partial n_z} = 0 \quad \text{on} \quad C_j(t).$$

In the case when $R(t)$ is a **planar** Riemann surface, $q(t,z)$ induces a **radial** slit mapping.

Assume that $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ is a domain in $B \times \tilde{R}$ over $B \times \mathbb{C}_z$ with smooth boundary s.t.
$R(t) \ni 0,\ \xi(t)(\neq 0,$ holo on $B)$. Then, for $t \in B$,

Lemma 2 [H]

$$\frac{\partial^2 \alpha(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial p(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 p(t,z)}{\partial \bar{t}\partial z}\right|^2 dxdy.$$

Lemma 2^* [H-M-Y]

$$\frac{\partial^2 \beta(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial q(t,z)}{\partial z}\right|^2 ds_z - \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 q(t,z)}{\partial \bar{t}\partial z}\right|^2 dxdy$$
$$-\frac{1}{\pi}\mathrm{Im}\left\{\sum_{k=1}^{g}\frac{\partial}{\partial t}\left(\int_{A_k(t)} *dq(t,z)\right)\frac{\partial}{\partial \bar{t}}\left(\int_{B_k(t)} *dq(t,z)\right)\right\}.$$

Here $R(t)$ is of genus g (≥ 0), $\{A_k(t), B_k(t)\}_{k=1}^{g}$ are A, B cycles on $R(t)$, and each $A_k(t),\ B_k(t)$ varies continuously with $t \in B$.

Theorem 2 [H]

$\mathcal{R}$ is a 2-dim pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary $\Longrightarrow$ L_1-const $\alpha(t)$ is a subharmonic fn on B.

Theorem 2^* [H-M-Y]

$\mathcal{R}$ is a 2-dim pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary and each $R(t)$, $t \in B$ is planar
$\Longrightarrow$ L_0-const $\beta(t)$ is a superharmonic fn on B.

Application 1 [H-M-Y]

$\mathcal{R}$ is a 2-dim pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary and each $R(t)$, $t \in B$ is planar
$\Longrightarrow$ $\boxed{s(t) := \alpha(t) - \beta(t)}$ is **subharmonic** on B.

$\uparrow$

harmonic span of $R(t)$

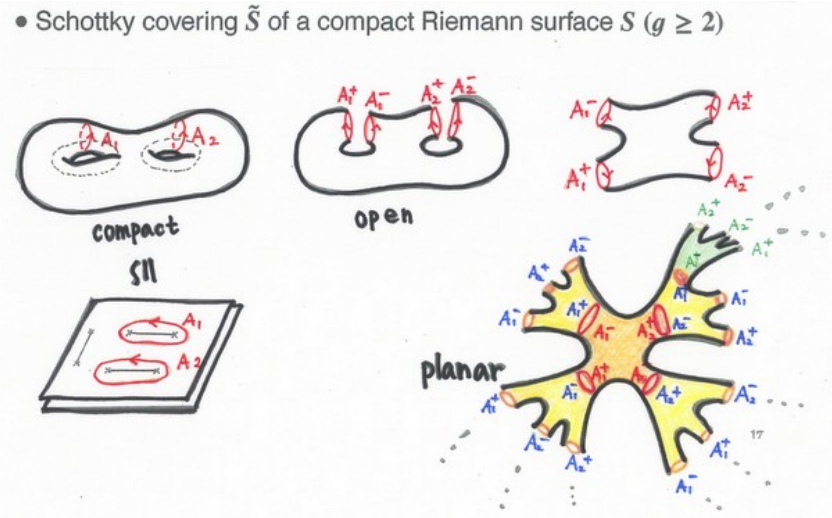


Figure: Schottky covering

Let B be a simply connected domain in $\mathbb{C}_t$. Let $\pi : \mathcal{S} \to B$ be a holomorphic family of compact Riemann surfaces $S(t) = \pi^{-1}(t)$ over B such that each fiber $S(t)$ is of genus ≥ 2 and non-singular in $\mathcal{S}$. For a fixed $t \in B$, we consider the Schottky covering $\widetilde{S}(t)$ of each $S(t)$. We denote by $\widetilde{\mathcal{S}}$ the total space of the variation: $t \in B \to \widetilde{S}(t)$, namely, $\widetilde{\mathcal{S}} = \bigcup_{t \in B}(t, \widetilde{S}(t))$. Then we have:

Application 2 [H]

The total space $\widetilde{\mathcal{S}}$ consisting of the Schottky covering $\widetilde{S}(t)$ of compact Riemann surfaces $S(t)$ with one complex parameter $t \in B$ is **holomorphically** uniformized to a univalent domain on $B \times \mathbb{P}^1$.

Cf.

Assume that $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ is a non-singular ramified pseudoconvex domain over $B \times \mathbb{C}_z$.

- [N]: $R(t)$, $t \in B$ is conformal equivalent to $\mathbb{C}^1$
$\Longrightarrow \mathcal{R} \cong B \times \mathbb{C}^1$

- [M-Y]: $R(t)$, $t \in B$ is planar and parabolic
$\Longrightarrow \mathcal{R} \cong$ a univalent domain in $B \times \mathbb{P}^1$

- [H]: $R(t)$, $t \in B$ is the Schottky covering of cpt Riemann surfs of $g \geq 2 \Longrightarrow \mathcal{R} \cong$ a univalent domain in $B \times \mathbb{P}^1$

Remark *In [Y], Yamaguchi wrote a resume about the Apprication 2 with a rough sketch of the proof. But his proof had a "gap". Then I bridge the gap by the variation formula for L_1-principal fn, and obtain it.*

References

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