Quasiplurisubharmonic Green functions

joint work with Vincent Guedj

Local setting: pluricomplex Green functions

$D \subset\subset \mathbb{C}^n$, $u \in PSH(D)$, $x \in D$,
Lelong number $\nu(u,x)$ = largest ν for which $u(z) \leq \nu \log|z-x| + O(1)$ holds for z near x.

Say u has an *isotropic pole* at x with $\nu(u,x) = \nu$ if $u(z) = \nu \log|z-x| + O(1)$ for z near x.

Given $p_j \in D$, $\nu_j > 0$, $1 \leq j \leq k$,

$$g(z) := \sup\{u(z) : u \in PSH(D), u < 0, \nu(u,p_j) \geq \nu_j\}.$$

Then $g \in PSH(D)$ has isotropic poles at each p_j with $\nu(g,p_j) = \nu_j$ and

$$(dd^c g)^n = \sum_{j=1}^k \nu_j^n \, \delta_{p_j}, \quad d^c := \frac{1}{2\pi i}(\partial - \overline{\partial}).$$

(Lempert, Klimek, Demailly, Lelong)

(X,ω) compact Kähler mfd., $n = \dim X$, $V_\omega = \int_X \omega^n$

$$PSH(X,\omega) = \{\varphi \in L^1(X,\mathbb{R}) \text{ usc}, \ \omega + dd^c\varphi \geq 0\}$$

$$\mathcal{P}(\omega) = \{\text{positive closed } (1,1) \text{ currents } T \sim \omega\}$$

$$T \in \mathcal{P}(\omega) \Longleftrightarrow T = \omega + dd^c\varphi, \ \varphi \in PSH(X,\omega)$$

Definition. *$\varphi \in PSH(X,\omega)$ is called a ω-psh Green function with (isolated) poles at $p_1,\ldots,p_k \in X$ if it is locally bounded in $X \setminus \{p_1,\ldots,p_k\}$ and*

$$(\omega + dd^c\varphi)^n = V_\omega \sum_{j=1}^{k} m_j \delta_{p_j}, \ m_j > 0, \ \sum_{j=1}^{k} m_j = 1.$$

Local positivity indicators for ω (Demailly):

$\nu(\{\omega\},x) := \sup\{\nu(\varphi,x) : \ \varphi \in PSH(X,\omega)\}$ (max)

$\varepsilon(\{\omega\},x) :=$
$\sup\{\gamma : \exists\, \varphi\ \omega-\text{psh}, \ \|\varphi - \gamma \log dist(\cdot,x)\|_{L^\infty(X)} < +\infty\} =$
$\sup\{\gamma : \exists\, \varphi\ \omega-\text{psh}, \ \nu(\varphi,x) = \gamma, \ \varphi \in L^\infty_{loc}(U \setminus \{x\})\}$,

where U is a neighborhood of x depending on φ.

$$\varepsilon(\{\omega\},x) = \min_V \left(\frac{\int_V \omega^{\dim V}}{\text{mult}_x V} \right)^{\frac{1}{\dim V}},$$

where $V \subseteq X$ irred. subvariety, $\dim V \geq 1$, $x \in V$, is the *Seshadri constant* of $\{\omega\}$ at x (Demailly-Paun).

Have $\quad 0 < \varepsilon(\{\omega\},x) \leq V_\omega^{1/n} \leq \nu(\{\omega\},x)$, $\forall\, x \in X$.

If φ is ω-psh Green function with one isotropic pole at x then $\quad \nu(\varphi,x)^n \delta_x = (\omega + dd^c\varphi)^n = V_\omega \delta_x$, so

$$\nu(\varphi,x) = \varepsilon(\{\omega\},x) = V_\omega^{1/n}.$$

Will see that in general $\varepsilon(\{\omega\},x) < V_\omega^{1/n}$ $(X = \mathbb{P}^1 \times \mathbb{P}^1)$.

Proposition. *$x \to \nu(\{\omega\},x)$ is usc, $x \to \varepsilon(\{\omega\},x)$ is lsc (neither is in general continuous).*

Notations: $PSH^-(X,\omega) = \{\varphi \in PSH(X,\omega),\ \varphi \leq 0\}$,
$M(\varphi)$ is the *unbounded locus* of $\varphi \in PSH(X,\omega)$.

For $p \in X$, $\mathcal{G}_p(V_\omega)$ is the set of germs of functions u at p so that $u \in PSH(U) \cap L^\infty_{loc}(U \setminus \{p\})$ for some open set $p \in U \subset X$, $u(p) = -\infty$, $(dd^c u)^n = V_\omega \delta_p$ on U.

Theorem. *Let $p \in X$ and $u \in \mathcal{G}_p(V_\omega)$. There exists a unique function $g = g_{u,p} \in PSH^-(X, \omega)$ such that*
(i) $g \leq u + C$ holds near p, for some constant C.
(ii) If $\varphi \in PSH^-(X, \omega)$ and $\liminf_{q \to p} \varphi(q)/u(q) \geq 1$ then $\varphi \leq g$ on X.
In addition, $(\omega + dd^c g)^n = 0$ on $X \setminus (M(g) \cup \{g = 0\})$. If p is an isolated point of $M(g)$ then $M(g) = \{p\}$ and g is a ω-psh Green function on X with pole at p.

Remark. Same can be done for any k points $p_j \in X$, with $u_j \in \mathcal{G}_{p_j}(V_\omega)$, $m_j > 0$, $\sum_{j=1}^k m_j = 1$. Now
(i) $g \leq m_j^{1/n} u_j + C$ holds near each p_j.
(ii) If $\varphi \in PSH^-(X, \omega)$, $\liminf_{q \to p_j} \varphi(q)/u_j(q) \geq m_j^{1/n}$, then $\varphi \leq g$ on X.

Remark. $u = V_\omega^{1/n} \log dist(\cdot, p) \Longrightarrow \nu(\{\omega\}, p) \geq V_\omega^{1/n}$.

Proof. Fix $U \subset X$ an open coordinate ball around p, so that u has the above properties on U.

Step 1. Technique of Demailly $\Rightarrow \exists\, \varphi$ ω-psh with $\varphi \le u$ near p.

Fix $p \in W \subset\subset W' \subset\subset U$, $\chi \in C_0^\infty(W')$, $0 \le \chi \le 1$, $\chi = 1$ on W. Let $\rho \le 0$ psh on W', $dd^c\rho = \omega$. May assume $u \ge 0$ on ∂W.

Let $u_j \searrow u$ be smooth psh on W', so $(dd^c u_j)^n \to V_\omega \delta_p$.

Let $\mu_j = C_j \chi\,(dd^c u_j)^n$, $C_j > 0$ s.t. $\mu_j(X) = V_\omega$. Note $\int_X \chi\,(dd^c u_j)^n \to V_\omega \chi(p) = V_\omega$, so $C_j \to 1$.

Yau's theorem (& Kolodziej) $\Rightarrow \exists\, \varphi_j$ continuous ω-psh with $(\omega + dd^c \varphi_j)^n = \mu_j$, $\max_X \varphi_j = 0$.

Wlog $\varphi_j \to \varphi \in PSH^-(X, \omega)$, ω^n-a.e. on X.

Choose $a_j \ge 1$, $a_j^n C_j > 1$, $a_j \to 1$. We have

$$a_j(\varphi_j + \rho) \le 0 \le u_j \text{ on } \partial W,$$

$$a_j^n (dd^c(\varphi_j + \rho))^n = a_j^n C_j \chi\,(dd^c u_j)^n \ge (dd^c u_j)^n \text{ on } W,$$

as $\chi = 1$ on W. By the minimum principle of Bedford and Taylor, $a_j(\varphi_j + \rho) \le u_j$ on W, so $\varphi + \rho \le u$.

Step 2. An upper envelope method. Consider

$$\mathcal{F} = \left\{ \varphi \in PSH^{-}(X,\omega) : \liminf_{q \to p} \frac{\varphi(q)}{u(q)} \geq 1 \right\} \neq \emptyset.$$

(Rashkovskii: the relative type of φ w.r.t. u is ≥ 1).

$g := \sup\{\varphi : \varphi \in \mathcal{F}\}$, $g^{\star} \in PSH^{-}(X,\omega)$. Will show $g^{\star} \leq u + C$ near p. So $g = g^{\star} \in \mathcal{F}$, g verifies $(i), (ii)$.

Pick $M > 0$ s.t. the conn. comp. D of $\{u < -M\}$ which contains p is relatively compact in U. Let $\rho < 0$ on U with $dd^c\rho = \omega$. Given $\varphi \in \mathcal{F}$ there exist domains $D_j \subset\subset D$, with $D_j \searrow \{p\}$ and

$$\varphi \leq (1 - j^{-1})u \text{ on } \overline{D}_j.$$

We have

$$\rho + \varphi \leq 0 \leq (1 - j^{-1})(u + M) \text{ on } \partial D,$$

$$\rho + \varphi \leq (1 - j^{-1})(u + M) \text{ on } \partial D_j.$$

Since u is maximal psh on $U \setminus \{p\}$, it follows

$$\rho + \varphi \leq (1 - j^{-1})(u + M) \text{ on } D \setminus D_j,$$

$$\rho + \varphi \leq u + M \text{ on } D,$$

$$g^{\star} \leq u + (M - \min_D \rho) \text{ on } D.$$

The remaining properties of g:

Let $q \in \Omega = X \setminus (M(g) \cup \{g = 0\})$ (open), let ρ be defined near of q with $dd^c\rho = \omega$, $\rho(q) = 0$.

$\exists\, \varepsilon > 0$ and G open, $q \in G \subset \Omega$ s.t. $g < -\varepsilon$, $|\rho| < \varepsilon/2$ on G. Let $W \subset\subset G$, v psh on W, $v^\star \leq \rho + g$ on ∂W.

$$\varphi := \begin{cases} g, \text{ on } X \setminus W, \\ \max\{\rho + g, v\} - \rho, \text{ on } W, \end{cases} , \quad \varphi \in PSH^-(X, \omega).$$

Since $\varphi = g$ near p, have $\varphi \in \mathcal{F}$. So $v \leq \rho + g$ on W, and the psh function $\rho + g$ is maximal on G. Conclude $(\omega + dd^c g)^n = 0$ in G, and hence on Ω.

Assume $p \in M(g)$ is isolated: $\exists\, K$ closed ball centered at p with $K \cap M(g) = \{p\}$, so $g > -C$ on ∂K.

$$\varphi := \begin{cases} g, \text{ on } K, \\ \max\{g, -C\}, \text{ on } X \setminus K, \end{cases} , \quad \varphi \in \mathcal{F}.$$

Thus $\varphi \leq g$, so $M(g) = \{p\}$ and

$$(\omega + dd^c g)^n(\{p\}) \geq (dd^c u)^n(\{p\}) = V_\omega.$$

Conclude $\quad (\omega + dd^c g)^n = V_\omega \delta_p. \quad \diamondsuit$

Case $X = \mathbb{P}^n$.

Let $[z_0 : \ldots : z_n]$, $\pi_n : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, ω_n be the Fubini-Study form, so $\pi_n^\star \omega_n = dd^c \log \|z\|$ and $V_{\omega_n} = 1$.

Wlog $p = [1 : 0 : \ldots : 0] \in \mathbb{C}^n \subset \mathbb{P}^n$

ω_n-psh functions φ correpsond to

$$u(z) = \log \|z\| + \varphi(\pi_n(z)) \in PSH(\mathbb{C}^{n+1}),$$

u log homogeneous. So $u(1, z_1, \ldots, z_n) \in \mathcal{L}(\mathbb{C}^n)$, and $\nu(\varphi, p) \leq 1$. Conclude

$$\nu(\{\omega_n\}, p) = \varepsilon(\{\omega_n\}, p) = 1.$$

Have $\nu(\varphi, p) = 1$ if and only if

$$\varphi(\pi_n(z)) = \frac{1}{2} \log \frac{|z_1|^2 + \ldots + |z_n|^2}{|z_0|^2 + \ldots + |z_n|^2} + h[z_1 : \ldots : z_n],$$

where $h \in PSH(\mathbb{P}^{n-1}, \omega_{n-1})$.

$\varphi \in L^\infty_{loc}(\mathbb{P}^n \setminus \{p\})$ iff $h \in L^\infty_{loc}(\mathbb{P}^{n-1})$ iff φ has isotropic pole at p. In this case $(\omega_n + dd^c \varphi)^n = \delta_p$.

Thus ω_n-psh Green functions with one pole of maximal Lelong number 1 must have an isotropic pole.

Case $X = \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^1_z \times \mathbb{P}^1_w$.

Let $\pi_z : X \to \mathbb{P}^1_z$, $\pi_w : X \to \mathbb{P}^1_w$, $\omega_z = \pi_z^\star \omega_1$, $\omega_w = \pi_w^\star \omega_1$

$$\omega_{a,b} := a\omega_z + b\omega_w,\ a, b > 0.$$

Let $\pi : (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}) \to X$,

$$\pi(z_0, z_1, w_0, w_1) = ([z_0 : z_1], [w_0 : w_1]),$$

and identify $(z_1, w_1) \in \mathbb{C}^2$ to $\pi(1, z_1, 1, w_1) \in X$.

$T \in \mathcal{P}(\omega_{a,b}) \iff \pi^\star T = dd^c u$, where u is bihomogeneous psh on $\mathbb{C}^4$:

$$u(\lambda z_0, \lambda z_1, \mu w_0, \mu w_1) = \\ a \log|\lambda| + b \log|\mu| + u(z_0, z_1, w_0, w_1),\ \lambda, \mu \in \mathbb{C}.$$

Proposition. *For all $p = (x, y) \in X$, we have*

$$\nu(\{\omega_{a,b}\}, p) = a + b,\ \varepsilon(\{\omega_{a,b}\}, p) = \min\{a, b\}.$$

If $T \in \mathcal{P}(\omega_{a,b})$ and $\nu(T, p) = a + b$ then

$$T = a[z = x] + b[w = y].$$

If T does not charge $\{z = x\}$, $\{w = y\}$ then

$$\nu(T, p) \leq \min\{a, b\}.$$

Proof. Wlog $a \geq b$, $p = (0,0) \in \mathbb{C}^2$. Let $T \in \mathcal{P}(\omega_{a,b})$.

Consider $R_{a,b} \in \mathcal{P}(\omega_{a,b})$ defined by $\pi^\star R_{a,b} = dd^c u_{a,b}$,

$$u_{a,b}(z_0, z_1, w_0, w_1) := \\ b \log \sqrt{|z_1 w_0|^2 + |w_1 z_0|^2} + (a-b) \log |z_0|.$$

It shows $\varepsilon(\{\omega_{a,b}\}, p) \geq b$. As $T \wedge R_{1,1}$ is well defined,

$$\nu(T,p) = T \wedge R_{1,1}(\{p\}) \leq \\ \int_X T \wedge R_{1,1} = \int_X \omega_{a,b} \wedge \omega_{1,1} = a + b.$$

Assume T does not charge $\{z = x\}$. Demailly's regularization: $\exists\, \epsilon_j \searrow 0$, $T_j \in \mathcal{P}(\omega_{a,b} + \epsilon_j \omega_{1,1})$ with analytic singularities, s.t. $0 \leq \nu(T,q) - \nu(T_j,q) \leq \epsilon_j$, $\forall\, q \in X$.

So $T_j \wedge [z = x]$ is well defined. If $T_j = dd^c v_j$ near p,

$$\nu(T_j,p) \leq \nu(v_j|_{\{z=x\}}, p) = \\ T_j \wedge [z = x](\{p\}) \leq \int_X T_j \wedge [z = x] = b + \epsilon_j.$$

Conclude that $\nu(T,p) \leq b$, so $\varepsilon(\{\omega_{a,b}\}, p) \leq b$.

Assume finally $\nu(T,p) = a+b$. Then

$$T = a'[z=x] + b'[w=y] + T',\ T' \in \mathcal{P}(\omega_{a-a',b-b'}),$$

where T' does not charge $\{z=x\}$ and $\{w=y\}$. So

$$a+b = \nu(T,p) \le a' + b' + \min\{a-a', b-b'\}.$$

This implies that $a'=a$, $b'=b$, and $T'=0$. $\diamondsuit$

Note $V_{\omega_{a,b}}^{1/2} = \sqrt{2ab} > \min\{a,b\} = \varepsilon(\{\omega_{a,b}\},p)$, hence:

Corollary. *There is no Green function with one isotropic pole on $\mathbb{P}^1 \times \mathbb{P}^1$.*

There are however Green functions g on X with one pole p and with many different types of singularities at p, even when $\nu(g,p)$ is maximal. E.g., let

$$p = (0,0) \in \mathbb{C}^2,\ a=b=1,\ \pi^\star(\omega_{1,1} + dd^c g) = dd^c u,$$

$$u(1,z_1,1,w_1) = \frac{1}{2k}\log\left(|z_1^k w_1^k|^2 + |z_1^k + w_1^k + z_1 w_1 Q(z_1,w_1)|^2\right),$$

where $k \ge 1$, $Q(z_1,w_1) = \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-1} c_{i_1 i_2} z_1^{i_1} w_1^{i_2}$. Have

$$\nu(g,p) = 1 \iff ord(Q,p) \ge k-2.$$