PROCEEDINGS OF HAYAMA SYMPOSIUM ON COMPLEX ANALYSIS IN SEVERAL VARIABLES 2004

December 18–21

SHONAN VILLAGE

Preface

This set of proceedings consists of contributions of the speakers at Hayama Symposium on Complex Analysis in Several Variables 2004, held at Shonan Village Center from Dec. 18 to 21.

Hayama Symposium has been held annually since 1995 and has been promoted communication among the researchers in this field. This year we had 63 participants and 18 lectures. It is our pleasure to leave on record these stimulating lectures.

We are grateful to all the participants and the speakers for their active discussions which made this symposium fruitful. This symposium was made possible by the financial support by Grant-in-Aid for Scientific Research in 2004:

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Yasuichiro Nishimura
Junjiro Noguchi
Shigeharu Takayama
Tetsuo Ueda

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Hayama Symposium on Several Complex Variables 2004
@Shonan Village Center, December 18-21

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COMPACTNESS IN THE $\overline{\partial}$-NEUMANN PROBLEM.

FRIEDRICH HASLINGER

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. We consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0,$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J{}' a_J d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J.$$

The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. Then $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$.
The complex Laplacian

$$\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$$

acts as an unbounded selfadjoint operator on

$$L^2_{(0,q)}(\Omega), \ 1 \le q \le n,$$

it is surjective and therefore has a continuous inverse, the $\overline{\partial}$-Neumann operator N_q. If v is a closed $(0,q+1)$-form, then $\overline{\partial}^*\, N_{q+1}v$ provides the canonical solution to $\overline{\partial}u = v$, which is orthogonal to the kernel of $\overline{\partial}$ and so has minimal norm.

A survey of the L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem is given in [BS].

The question of compactness of N_q is of interest for various reasons. For example, compactness of N_q implies global regularity in the sense of preservation of Sobolev spaces [KN]. Also, the Fredholm theory of Toeplitz operators is an immediate consequence of compactness in the $\overline{\partial}$-Neumann problem [V], [HI], [CD]. There are additional ramifications for certain C^*-algebras naturally associated to a domain in $\mathbb{C}^n$ [SSU]. Finally, compactness is a more robust property than global regularity - for example, it localizes, whereas global regularity does not - and it is generally believed to be more tractable than global regularity.
A thourough discussion of compactness in the $\overline{\partial}$-Neumann problem can be found in [FS2].

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Catlin [Ca1] showed that for sufficiently smooth bounded pseudoconvex domains satisfying what he called property(P), the $\overline{\partial}$ - Neumann problem is compact, and that all domains of finite type in the sense of D'Angelo [D'A] satisfy property(P). In this connection earlier work of Diederich and Fornæss [DF] and Diederich and Pflug [DP] is of importance.
The condition on boundary smoothness was removed by Straube in [S].
Property(P) for a domain Ω is the condition that for all $M > 0$ there should exist a function $\lambda_M \in \mathcal{C}^\infty(\overline{\Omega})$ with $0 \leq \lambda_M \leq 1$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda_M}{\partial z_j \partial \overline{z}_k}(z)\; t_j \overline{t}_k \geq M|t|^2,$$

for all $z \in b\Omega$ and for all complex vectors $t \in \mathbb{C}^n$. A systematic and very useful study of property(P) (under the name of B-regularity) in the context of arbitrary compact sets in $\mathbb{C}^n$ is in [Si]. More recently, McNeal [Mc] introduced an intriguing variant of property(P) which still implies compactness.

Compactness is completely understood on (bounded) locally convexifiable domains. On such domains, the following are equivalent [FS1], [FS2]:
(i)N_q is compact,
(ii) the boundary of the domain satisfies (an analogue of) property(P) (for q-forms),
(iii) the boundary contains no q-dimensional analytic variety.

In general, however, the situation is not understood at all.

Discs in the boundary are the most obvious violation of property(P). It is therefore natural to ask whether there is a corresponding failure of compactness in this case. In $\mathbb{C}^2$, this has been known for many years to be the case (assuming the boundary of the domain is at least Lipschitz) [Ca2], [FS2] . In higher dimensions, this is open.

There are interesting connections between $\overline{\partial}$ and Schrödinger operators, see for example the discussion in [B] and between compactness in the $\overline{\partial}$-Neumann problem and property(P) on the one hand, and the asymptotic behavior, in a semi-classical limit, of the lowest eigenvalues of certain magnetic Schrödinger operators and of their non-magnetic counterparts, respectively, on the other ([FS3]).

The study of the $\overline{\partial}$-Neumann problem is essentially equivalent (in a sense that can be made precise) to the study of the canonical solution operator to $\overline{\partial}$. Interestingly, in many situations, the restriction of the canonical solution operator to forms with *holomorphic* coefficients arises naturally [SSU], [FS1].

The restriction of the canonical solution operator to forms with holomorphic coefficients has many interesting aspects, which in most cases correspond to certain growth properties of the Bergman kernel. It is also of great interest to clarify to what extent compactness of the restriction already implies compactness of the original solution operator to $\overline{\partial}$. This is the case for convex domains, see [FS1]. There are many other examples of non-compactness where the obstruction already occurs for forms with holomorphic coefficients (see [L], [Kr]).

In [Has1] the canonical solution operator S_1 to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients is investigated. Let $A^2_{(0,1)}(\Omega)$ denote the space of all $(0,1)$-forms with holomorphic coefficients belonging to $L^2(\Omega)$. It is shown that the canonical solution operator $S_1 : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$ has the form

$$S_1(g)(z) = \int_\Omega B(z,w) < g(w), z - w > \, d\lambda(w),$$

where B denotes the Bergman kernel of Ω and

$$< g(w), z - w > = \sum_{j=1}^{n} g_j(w)(\overline{z}_j - \overline{w}_j),$$

for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. It follows that the canonical solution operator is a Hilbert Schmidt operator for the unit disc $\mathbb{D}$ in $\mathbb{C}$, but fails to be Hilbert Schmidt for the unit ball in $\mathbb{C}^n, \ n \geq 2$.

Not very much is known in the case of unbounded domains.

In [Has2] it is shown that the canonical solution operator to $\overline{\partial}$ restricted to the Fock space A^2_φ, where $\varphi(z) = |z|^2$ fails to be compact, whereas in the case $\varphi(z) = |z|^m \ , \ m > 2$ the canonical solution operator to $\overline{\partial}$ restricted to A^2_φ is compact but fails to be Hilbert Schmidt. See also [Sch], where in this context the situation in several variables is investigated.
These results were generalized to Schatten p-classes by Lovera and Youssfi [LY].

The connection between $\overline{\partial}$ and Schrödinger operators [B], [Ch] , [FS3], [Has3], is of considerable interest from the point of view of complex analysis as well as from that of the theory of Schrödinger operators. For example, the main result in [FS3] implies that (for certain Hartogs domains in $\mathbb{C}^2$) the implication property(P) $\Rightarrow$ compactness may be interpreted as a consequence of well known diamagnetic inequalities in the theory of Schrödinger operators [He]. Another example arises in [Has3], where it is shown how known properties of magnetic Schrödinger operators easily give compactness results for the canonical solution operator restricted to certain weighted L^2 - spaces on $\mathbb{C}$.
For this purpose let $\varphi : \mathbb{C} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-weight function and consider the Hilbert spaces

$$L^2_\varphi = \{ f : \mathbb{C} \longrightarrow \mathbb{C} \text{ measureable } : \ \|f\|^2_\varphi := \int_\mathbb{C} |f(z)|^2 \, e^{-2\varphi(z)} \, d\lambda(z) < \infty \}.$$

It is essentially due to L. Hörmander [H] that for a suitable weight function φ and for every $f \in L^2_\varphi$ there exists $u \in L^2_\varphi$ satisfying

$$\overline{\partial} u = f.$$

In fact there exists a continuous solution operator $\tilde{S} : L^2_\varphi \longrightarrow L^2_\varphi$ for $\overline{\partial}$, i.e.

$$\|\tilde{S}(f)\|_\varphi \leq C \|f\|_\varphi$$

and $\overline{\partial}\tilde{S}(f) = f$, see also [Ch] .
Let $P_\varphi : L^2_\varphi \longrightarrow A^2_\varphi$ denote the Bergman projection. Then $S = (I - P_\varphi)\tilde{S}$ is the uniquely determined canonical solution operator to $\overline{\partial}$, i.e.
$\overline{\partial} S(f) = f$ and $S(f) \perp A^2_\varphi$.

A nonnegative Borel measure ν defined on $\mathbb{C}$ is said to be doubling if there exists a constant C such that for all $z \in \mathbb{C}$ and $r \in \mathbb{R}^+$,

$$\nu(B(z, 2r)) \leq C \nu(B(z, r)).$$

$\mathcal{D}$ denotes the set of all doubling measures ν for which there exists a constant δ such that for all $z \in \mathbb{C}$,

$$\nu(B(z,1)) \geq \delta.$$

Let $\varphi : \mathbb{C} \longrightarrow \mathbb{R}$ be a subharmonic function. Then $\Delta\varphi$ defines a nonnegative Borel measure, which is finite on compact sets.
Let $\mathcal{W}$ denote the set of all subharmonic functions $\varphi : \mathbb{C} \longrightarrow \mathbb{R}$ such that $\Delta\varphi \in \mathcal{D}$.
Let φ be a $\mathcal{C}^2$-function. We want to solve $\overline{\partial} u = f$ for $f \in L^2_\varphi$. The canonical solution operator to $\overline{\partial}$ gives a solution with minimal L^2_φ-norm. Following [Ch] we substitute $v = u\, e^{-\varphi}$ and $g = f\, e^{-\varphi}$ and the equation becomes

$$\overline{D} v = g \ , \ \text{where } \overline{D} = e^{-\varphi}\, \frac{\partial}{\partial \overline{z}}\, e^{\varphi}.$$

u is the minimal solution to the $\overline{\partial}$-equation in L^2_φ if and only if v is the solution to $\overline{D} v = g$ which is minimal in $L^2(\mathbb{C})$.
The formal adjoint of $\overline{D}$ is

$$D = -e^{\varphi} \frac{\partial}{\partial z} e^{-\varphi}.$$

As in [Ch] we define

$$\mathrm{Dom}(\overline{D}) = \{ f \in L^2(\mathbb{C}) \ : \ \overline{D} f \in L^2(\mathbb{C}) \}$$

and likewise for D. Then $\overline{D}$ and D are closed unbounded linear operators from $L^2(\mathbb{C})$ to itself.
Further we define

$$\mathrm{Dom}(\overline{D} D) = \{ u \in \mathrm{Dom}(D) \ : \ Du \in \mathrm{Dom}(\overline{D}) \}$$

and we define $\overline{D} D$ as $\overline{D} \circ D$ on this domain. Any function of the form $e^{\varphi}\, g$, with $g \in \mathcal{C}^2_0$ belongs to $\mathrm{Dom}(\overline{D} D)$ and hence $\mathrm{Dom}(\overline{D} D)$ is dense in $L^2(\mathbb{C})$.
Since

$$\overline{D} = \frac{\partial}{\partial \overline{z}} + \frac{\partial \varphi}{\partial \overline{z}}$$

and

$$D = -\frac{\partial}{\partial z} + \frac{\partial \varphi}{\partial z}$$

we see that

$$\begin{aligned} \overline{D} D &= -\frac{\partial^2}{\partial z \partial \overline{z}} - \frac{\partial \varphi}{\partial \overline{z}} \frac{\partial}{\partial z} + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \overline{z}} + \left| \frac{\partial \varphi}{\partial z} \right|^2 + \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} \\ &= -\frac{1}{4} \left((d - iA)^2 - \Delta\varphi \right), \end{aligned}$$

where $A = A_1\, dx + A_2\, dy = -\varphi_y\, dx + \varphi_x\, dy$.
Hence $\overline{D} D$ is a Schrödinger operator with electric potential $\Delta\varphi$ and with magnetic field $B = dA$.

Now let $\|u\|^2 = \int_{\mathbb{C}} |u(z)|^2 \, d\lambda(z)$ for $u \in L^2(\mathbb{C})$ and

$$(u,v) = \int_{\mathbb{C}} u(z) \overline{v(z)} \, d\lambda(z)$$

denote the inner product of $L^2(\mathbb{C})$.

For $\phi, \psi \in \mathcal{C}_0^\infty(\mathbb{C})$ we set

$$h_{A,\varphi}(\phi,\psi) = (-\frac{1}{4}((d-iA)^2 - \Delta\varphi)\phi, \psi) =$$

$$\sum_{j=1}^{2}(\frac{1}{4}(\Pi_j(A)\phi, \Pi_j(A)\psi) + ((\Delta\varphi)\phi, \psi),$$

where

$$\Pi_j(A) = \frac{1}{i}(\frac{\partial}{\partial x_j} - A_j)\ ,\ j = 1, 2,$$

hence $h_{A,\varphi}$ is a nonnegative symmetric form on $\mathcal{C}_0^\infty(\mathbb{C})$.

In [Ch] the following results are proved

Lemma 1. *Let $\varphi \in \mathcal{W}$. If $u \in Dom(D)$ and $Du \in Dom(\overline{D})$, then*

$$\|Du\|^2 = (\overline{D}(Du), u).$$

$\overline{D}D$ is a closed operator and

$$\|u\| \leq C\|\overline{D}Du\|$$

for all $u \in Dom(\overline{D}D)$. Moreover, for any $f \in L^2(\mathbb{C})$ there exists a unique $u \in Dom(\overline{D}D)$ satisfying

$$\overline{D}Du = f.$$

Hence $(\overline{D}D)^{-1}$ is a bounded operator on $L^2(\mathbb{C})$.

The closure $\overline{h}_{A,\varphi}$ of the form $h_{A,\varphi}$ is a nonnegative symmetric form. The selfadjoint operator associated with $\overline{h}_{A,\varphi}$ is the operator $\overline{D}D$ from the above Lemma 1 (see [I] and [CFKS]).
The next lemma follows from the fact that a bounded operator T is compact if and only if T^*T is compact (see [W]).

Lemma 2. *Let $\varphi \in \mathcal{W}$. The canonical solution operator $S : L^2_\varphi \longrightarrow L^2_\varphi$ to $\overline{\partial}$ is compact if and only if $(\overline{D}D)^{-1} : L^2(\mathbb{C}) \longrightarrow L^2(\mathbb{C})$ is compact.*

Using the main theorem in [I] we get

Theorem 1. *Let $\varphi \in \mathcal{W}$. The canonical solution operator $S : L^2_\varphi \longrightarrow L^2_\varphi$ to $\overline{\partial}$ is compact if and only if there exists a real valued continuous function μ on $\mathbb{C}$ such that $\mu(z) \to \infty$ as $|z| \to \infty$ and*

$$h_{A,\varphi}(\phi,\phi) \geq \int_{\mathbb{C}} \mu(z)\, |\phi(z)|^2\, d\lambda(z)$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{C})$.

Theorem 2. *If $\varphi(z) = |z|^2$, then the canonical solution operator $S : L^2_\varphi \longrightarrow L^2_\varphi$ to $\overline{\partial}$ fails to be compact.*

Theorem 3. *Let $\varphi \in \mathcal{W}$ and suppose that $\Delta\varphi(z) \to \infty$ as $|z| \to \infty$. Then the canonical solution operator $S : L^2_\varphi \longrightarrow L^2_\varphi$ to $\overline{\partial}$ is compact.*

Remark. In [Has2] it shown that for $\varphi(z) = |z|^2$ even the restriction of the canonical solution operator S to the Fock space A^2_φ fails to be compact and that for $\varphi(z) = |z|^m$, $m > 2$ the restriction of S to A^2_φ fails to be Hilbert Schmidt.

Several complex variables.

In [Sch] it is shown that the restriction of the canonical solution operator to the Fock space A^2_φ fails to be compact, where

$$\varphi(z) = |z_1|^m + \cdots + |z_n|^m,$$

for $m \geq 2$ and $n \geq 2$. Hence the canonical solution operator cannot be compact on the corresponding L^2-spaces.
Here we investigate the solution operator on L^2-spaces and try to generalize the method from above for several complex variables.
Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-weight function and consider the space

$$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2 \, e^{-2\varphi} \, d\lambda < \infty\}$$

and the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$.
For $v \in L^2(\mathbb{C}^n)$ let

$$\overline{D} v = \sum_{k=1}^n \left(\frac{\partial v}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k} \, v \right) \, d\overline{z}_k$$

and for $g = \sum_{j=1}^n g_j \, d\overline{z}_j \in L^2_{(0,1)}(\mathbb{C}^n)$ let

$$\overline{D}^* g = \sum_{j=1}^n \left(\frac{\partial \varphi}{\partial z_j} \, g_j - \frac{\partial g_j}{\partial z_j} \right),$$

where the derivatives are taken in the sense of distributions. It is easy to see that $\overline{\partial} u = f$ for $u \in L^2(\mathbb{C}^n, \varphi)$ and $f \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ if and only if $\overline{D} v = g$, where $v = u \, e^{-\varphi}$ and $g = f \, e^{-\varphi}$. It is also clear that the necessary condition $\overline{\partial} f = 0$ for solvability holds if and only if $\overline{D} g = 0$ holds. Here

$$\overline{D} g = \sum_{j,k=1}^n \left(\frac{\partial g_j}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k} \, g_j \right) \, d\overline{z}_k \wedge d\overline{z}_j.$$

Then

$$\overline{D} \, \overline{D}^* g = \overline{D} \left(\sum_{j=1}^n \left(\frac{\partial \varphi}{\partial z_j} \, g_j - \frac{\partial g_j}{\partial z_j} \right) \right)$$

$$= \sum_{k=1}^n \left[\sum_{j=1}^n \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \, g_j - \frac{\partial^2 g_j}{\partial z_j \partial \overline{z}_k} + \frac{\partial g_j}{\partial \overline{z}_k} \, \frac{\partial \varphi}{\partial z_j} \ - \frac{\partial g_j}{\partial z_j} \, \frac{\partial \varphi}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial z_j} \, \frac{\partial \varphi}{\partial \overline{z}_k} \, g_j \right) \right] \, d\overline{z}_k.$$

Proposition 1. *The operator $\overline{D}\,\overline{D}^*$ defined on*

$$Dom\overline{D}^* \cap ker\overline{D}$$

has the form

$$\sum_{k=1}^n \left[\sum_{j=1}^n \left(2\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}\, g_j - \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_j}\, g_k - \frac{\partial^2 g_k}{\partial z_j \partial \overline{z}_j} \;+\frac{\partial g_k}{\partial \overline{z}_j}\,\frac{\partial \varphi}{\partial z_j} - \frac{\partial g_k}{\partial z_j}\,\frac{\partial \varphi}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial z_j}\,\frac{\partial \varphi}{\partial \overline{z}_j}\, g_k\right)\right] d\overline{z}_k.$$

Remark. The only term where g_j appears in the last line is

$$2\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}\, g_j,$$

and we will get a diagonal system if we restrict to weight functions of a special form, for instance

$$\varphi(z) = |z_1|^2 + \cdots + |z_n|^2,$$

the case of the Fock space.

Proposition 2. *Suppose that the weight function φ is of the form*

$$\varphi(z_1, \ldots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n),$$

where $\varphi_j : \mathbb{C} \longrightarrow \mathbb{R}$ are $\mathcal{C}^2$-functions for $j = 1, \ldots, n$.
Then the equation $\overline{D}\,\overline{D}^* g = h$, for $h = \sum_{k=1}^n h_k\, d\overline{z}_k$, splits into the n-equations

$$2\,\frac{\partial^2\varphi}{\partial z_k \partial \overline{z}_k}\, g_k + \sum_{j=1}^n \left(-\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_j}\, g_k - \frac{\partial^2 g_k}{\partial z_j \partial \overline{z}_j} \;+\frac{\partial g_k}{\partial \overline{z}_j}\,\frac{\partial \varphi}{\partial z_j} - \frac{\partial g_k}{\partial z_j}\,\frac{\partial \varphi}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial z_j}\,\frac{\partial \varphi}{\partial \overline{z}_j}\, g_k\right) = h_k,$$

for $k = 1, \ldots, n$. These equations can be represented as Schrödinger operators $\mathcal{S}_k$ with magnetic fields, where

$$\mathcal{S}_k v = 2\,\frac{\partial^2\varphi}{\partial z_k \partial \overline{z}_k}\, v + \sum_{j=1}^n \left(-\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_j}\, v - \frac{\partial^2 v}{\partial z_j \partial \overline{z}_j} \;+\frac{\partial v}{\partial \overline{z}_j}\,\frac{\partial \varphi}{\partial z_j} - \frac{\partial v}{\partial z_j}\,\frac{\partial \varphi}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial z_j}\,\frac{\partial \varphi}{\partial \overline{z}_j}\, v\right)$$

and v is a $\mathcal{C}^2$-function. The operators $\mathcal{S}_k$ can be written in the form

$$\mathcal{S}_k = \frac{1}{4}\left[-\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} - ia_j\right)^2 - \sum_{j=1}^n \left(\frac{\partial}{\partial y_j} - ib_j\right)^2\right] + V_k,$$

where $z_j = x_j + iy_j$ and $a_j = -\frac{\partial \varphi}{\partial y_j}$, $b_j = \frac{\partial \varphi}{\partial x_j}$, for $j = 1, \ldots, n$ and

$$V_k = 2\frac{\partial^2\varphi}{\partial z_k \partial \overline{z}_k} - \sum_{j=1}^n \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_j},$$

for $k = 1, \ldots, n$.

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Institut für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: `friedrich.haslinger@univie.ac.at`

A group-theoretic characterization of the direct product of a ball and a Euclidean space

Jisoo BYUN, Akio KODAMA and Satoru SHIMIZU

Introduction. This is the outgrouth of the talk given by the second author at the Hayama Symposium on Complex Analysis in Several Variables 2004.

Let M be a connected complex manifold and let $\mathrm{Aut}(M)$ be the group of all holomorphic automorphisms of M equipped with the compact-open topology. Then one of the fundamental problems in complex geometric analysis is to determine the complex analytic structure of M by its holomorphic automorphism group $\mathrm{Aut}(M)$. Of course, in many cases, this is a very difficult problem. One reason may be that $\mathrm{Aut}(M)$ cannot have the structure of a Lie group, in general. However, even when $\mathrm{Aut}(M)$ is not a Lie group, one can sometimes use techniques developed in the Lie group theory. For instance, consider the space $\mathbf{C}^k \times (\mathbf{C}^*)^\ell$. Then $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$ is terribly big when $k + \ell \geq 2$, and cannot have the structure of a Lie group with respect to the compact-open topology. In the previous paper [3], by looking at some topological subgroups with Lie group structures of $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^\ell)$, we obtained an interesting theorem on characterization of $\mathbf{C}^k \times (\mathbf{C}^*)^\ell$ by its holomorphic automorphism group. And, as an application of the method used in the proof of this, we proved that *if a connected Stein manifold M of dimension $n \geq 2$ admits an effective continuous action of $U(n)$ by holomorphic automorphisms, then M is biholomorphically equivalent to either B^n or $\mathbf{C}^n$*, where B^n is the open unit ball in $\mathbf{C}^n$ and $U(n)$ is the unitary group of degree n. In view of this, it would be natural to ask *what happens when M admits an effective continuous action of the direct product $U(n_1)\times\cdots\times U(n_s)$ of unitary groups by holomorphic automorphisms*, where $n_1 + \cdots + n_s = \dim M$. As a typical example of such a manifold M, we have the direct product space $B^{n_1} \times \cdots \times B^{n_{s-1}} \times \mathbf{C}^{n_s}$. Notice that the holomorphic automorphism group of this model space does not have the structure of a Lie group with respect to the compact-open topology in the case where $n_s \geq 2$, or $n_s \geq 1$ and $n_1 + \cdots + n_{s-1} \geq 1$.

The purpose of this article is to study exclusively the product space $B^k \times \mathbf{C}^\ell$ in connection with the question above and give the following group-theoretic characterization of it. The details can be found in [2]:

Theorem. *Let M be a connected Stein manifold of dimension n. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(B^k \times \mathbf{C}^{n-k})$ *as topological groups for some integer k with $0 \leq k \leq n$. Then M itself is biholomorphically equivalent to $B^k \times \mathbf{C}^{n-k}$.*

As a consequence of this theorem, we can obtain the following fundamental result on the topological group structure of $\mathrm{Aut}(B^k \times \mathbf{C}^\ell)$.

Corollary. *If two pairs (k,ℓ) and (k',ℓ') of nonnegative integers do not coincide, then the groups* $\mathrm{Aut}(B^k \times \mathbf{C}^\ell)$ *and* $\mathrm{Aut}(B^{k'} \times \mathbf{C}^{\ell'})$ *are not isomorphic as topological groups.*

After some preliminaries in Section 1, we will give only an outline of the proof of our Theorem in Section 2.

1. Preliminaries. In this Section, we shall recall basic concepts and results on Reinhardt domains (cf. [4], [5]). For an element $\alpha = (\alpha_1, \dots, \alpha_n)$ of $(\mathbf{C}^*)^n$, we denote by π_α an element of $\mathrm{Aut}(\mathbf{C}^n)$ given by

$$\pi_\alpha(z_1, \dots, z_n) = (\alpha_1 z_1, \dots, \alpha_n z_n), \quad (z_i) \in \mathbf{C}^n.$$

We may regard the multiplicative group $(\mathbf{C}^*)^n$ as a closed subgroup of $\mathrm{Aut}(\mathbf{C}^n)$. Let D be a Reinhardt domain in $\mathbf{C}^n$. Then, π_α maps D onto itself and induces a holomorphic automorphism of D for every element α of the n-dimensional torus $T^n = (U(1))^n$. The mapping ρ_D sending α to π_α is an injective continuous group homomorphism of T^n into the topological group $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$. Furthermore, we denote by $\Pi(D)$ the topological subgroup of $\mathrm{Aut}(D)$ consisting of all elements φ of $\mathrm{Aut}(D)$ such that φ has the form $\varphi = \pi_\gamma$, where γ is an element of $(\mathbf{C}^*)^n$. And we denote by $\mathrm{Aut_{alg}}(D)$ the topological subgroup of $\mathrm{Aut}(D)$ consisting of all elements φ of $\mathrm{Aut}(D)$ such that each component of φ is given by a Laurent monomial, that is, φ has the form

$$(1.1) \qquad \varphi(z_1, \dots, z_n) = (w_1, \dots, w_n) \quad \text{with} \quad w_i = \alpha_i z_1^{a_{i1}} \cdots z_n^{a_{in}},\ i = 1, \dots, n,$$

where $(a_{ij}) \in GL(n, \mathbf{Z})$ and $(\alpha_i) \in (\mathbf{C}^*)^n$. Each element of $\mathrm{Aut_{alg}}(D)$ is called an *algebraic automorphism of* D. The groups $\Pi(D)$ and $\mathrm{Aut_{alg}}(D)$ can be characterized group-theoretically as follows:

Lemma 1.1. *The centralizer of the torus $T(D)$ in* $\mathrm{Aut}(D)$ *is given by $\Pi(D)$, while the normalizer of $T(D)$ in* $\mathrm{Aut}(D)$ *is given by* $\mathrm{Aut_{alg}}(D)$.

Here consider the mapping $\varpi\colon \mathrm{Aut_{alg}}(D) \to GL(n, \mathbf{Z})$ that sends an element φ of $\mathrm{Aut_{alg}}(D)$ written in the form (1.1) into the element $(a_{ij}) \in GL(n, \mathbf{Z})$. Then ϖ is a group homomorphism with $\ker \varpi = \Pi(D)$; and so it induces a group isomorphism

$$(1.2) \qquad \mathrm{Aut_{alg}}(D)/\Pi(D) \xrightarrow{\cong} \mathcal{G}(D) := \varpi(\mathrm{Aut_{alg}}(D)) \subset GL(n, \mathbf{Z}).$$

Now let us consider the special case where $D = B^k \times \mathbf{C}^\ell$. Then, denoting by $(z_1, \dots, z_k, w_1, \dots, w_\ell)$ the coordinate system of $\mathbf{C}^k \times \mathbf{C}^\ell$ and putting $z = (z_1, \dots, z_k)$, $w = (w_1, \dots, w_\ell)$ for simplicity, we can show the following fact:

Proposition 1.1. (1) *Each element F of* $\mathrm{Aut}(B^k \times \mathbf{C}^\ell)$ *has the form* $F(z,w) = (f(z),\ g(z,w)+h(z))$, *where* $f \in \mathrm{Aut}(B^k)$, $h\colon B^k \to \mathbf{C}^\ell$ *is a holomorphic mapping and* $g\colon B^k \times \mathbf{C}^\ell \to \mathbf{C}^\ell$ *is a holomorphic mapping such that* $g(z,\cdot\,) \in \mathrm{Aut}(\mathbf{C}^\ell)$ *and* $g(z,0) = 0$ *for each fixed point* $z \in B^k$.

(2) *Each element F of* $\mathrm{Aut_{alg}}(B^k \times \mathbf{C}^\ell)$ *has the form*

$$F(z_1,\dots,z_k,w_1,\dots,w_\ell) = \big(\alpha_1 z_{\sigma(1)},\dots,\alpha_k z_{\sigma(k)},\beta_1 w_{\tau(1)},\dots,\beta_\ell w_{\tau(\ell)}\big),$$

where $(\alpha_i) \in T^k$, $(\beta_j) \in (\mathbf{C}^*)^\ell$ *and* σ, τ *are permutations of* $\{1,\dots,k\}$, $\{1,\dots,\ell\}$, *respectively. In particular, the group* $\mathcal{G}(B^k \times \mathbf{C}^\ell)$ *is isomorphic to the direct product* $S_k \times S_\ell$ *of the symmetric groups* S_k *and* S_ℓ *of degrees* k *and* ℓ*; so that its cardinality is equal to* $k!\ell!$.

2. Proof of the Theorem. Throughout this Section, we use the following notation: For the given integer k and a point $(z_1,\dots,z_n) \in \mathbf{C}^n$, we set

$$\ell = n-k,\quad \Omega_{k,\ell} = B^k \times \mathbf{C}^\ell,\quad z = (z_1,\dots,z_k) \in \mathbf{C}^k$$
$$\text{and}\quad w = (w_1,\dots,w_\ell) = (z_{k+1},\dots,z_n) \in \mathbf{C}^\ell.$$

For a set S, we denote by $\sharp S$ the cardinality of S. Let W be a domain in $\mathbf{C}^n$ and Γ a subgroup of $\mathrm{Aut}(W)$. Then we denote by

$C_W(\Gamma)$ the centralizer of Γ in $\mathrm{Aut}(W)$, and

$Z_W(\Gamma)$ the commutator group of $C_W(\Gamma)$ in $\mathrm{Aut}(W)$.

Now let M be a connected Stein manifold of dimension n, and assume that there exists a topological group isomorphism $\Phi\colon \mathrm{Aut}(\Omega_{k,\ell}) \to \mathrm{Aut}(M)$. Since $\Omega_{k,\ell}$ is a Reinhardt domain in $\mathbf{C}^n$, we have the injective continuous group homomorphism $\rho_{\Omega_{k,\ell}}\colon T^n \to \mathrm{Aut}(\Omega_{k,\ell})$. Thus we obtain an injective continuous group homomorphism $\Phi \circ \rho_{\Omega_{k,\ell}}\colon T^n \to \mathrm{Aut}(M)$. Then, by a result of Barrett-Bedford-Dadok [1, Theorem 1], there exists a biholomorphic mapping F of M into $\mathbf{C}^n$ such that $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$ and $F(\Phi \circ \rho_{\Omega_{k,\ell}})(T^n)F^{-1} = T(D)$. Therefore we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and we have an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_{k,\ell}) \to \mathrm{Aut}(D)$ such that $\Phi(T(\Omega_{k,\ell})) = T(D)$. In particular, $\Phi(\Pi(\Omega_{k,\ell})) = \Pi(D)$ and $\Phi(\mathrm{Aut_{alg}}(\Omega_{k,\ell})) = \mathrm{Aut_{alg}}(D)$ by Lemma 1.1; and accordingly, we see that the groups $\mathcal{G}(\Omega_{k,\ell})$ and $\mathcal{G}(D)$ defined in (1.2) are isomorphic.

When $n = 1$, it is easy to prove our Theorem. So, let us consider the case where $n \geq 2$. Assume that $k = 0$ or $\ell = 0$. Then D admits an effective continuous action of $U(n)$ by holomorphic automorphisms. Consequently, D is biholomorphically equivalent to B^n or $\mathbf{C}^n$, as stated in the introduction. Since $\mathrm{Aut}(B^n)$ is not isomorphic to $\mathrm{Aut}(\mathbf{C}^n)$ as topological groups, this shows our Theorem. Therefore, we have only to prove the Theorem under the assumption that $n \geq 2$ and k, $\ell \geq 1$.

2.1. The case $n = 2$. Throughout this Subsection, we put $\Omega = \Omega_{1,1}$.

Recall that $\Phi(T(\Omega)) = T(D)$ and $\Phi(\mathit{\Pi}(\Omega)) = \mathit{\Pi}(D)$. Our first task is to determine the form of the isomorphism $\Phi\colon \mathit{\Pi}(\Omega) \to \mathit{\Pi}(D)$. To this end, notice that $\mathit{\Pi}(\Omega)$ consists of all elements φ of $\mathrm{Aut}(\Omega)$ having the form $\varphi(z,w) = (\alpha z, \beta w)$ with $|\alpha| = 1$, $\beta \in \mathbf{C}^*$. Hence $\mathit{\Pi}(\Omega)$ is a real Lie group of dimension 3 that can be identified with the subgroup $T^1 \times \mathbf{C}^*$ of $\mathbf{C}^* \times \mathbf{C}^*$. Also, one may identify $\mathit{\Pi}(D)$ with a closed subgroup of $\mathbf{C}^* \times \mathbf{C}^*$. Accordingly, by considering the new coordinates $(\tilde{z}, \tilde{w}) = (w, z)$ in the space $\mathbf{C}^2$ containing D, if necessary, $\Phi\colon \mathit{\Pi}(\Omega) \to \mathit{\Pi}(D)$ can be written in the form

$$
\begin{aligned}
(2.1)\qquad \Phi\big(e^{2\pi i\theta}, e^{2\pi i(\phi+i\psi)}\big) \;=\;& \\
&\big(e^{2\pi i(a_{11}\theta+a_{12}\phi+a_{13}\psi+ia\psi)},\; e^{2\pi i(a_{21}\theta+a_{22}\phi+a_{23}\psi+ia\lambda\psi)}\big)
\end{aligned}
$$

for all $\theta, \phi, \psi \in \mathbf{R}$, where

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2,\mathbf{Z}), \quad a_{13},\ a_{23},\ a,\ \lambda \in \mathbf{R} \ \text{ and } \ a \neq 0.
$$

Here let us consider a one-parameter subgroup

$$
\mathit{\Gamma} = \{(1, e^{-2\pi\psi}) \mid \psi \in \mathbf{R}\}
$$

of $\mathit{\Pi}(\Omega)$ and put $\mathit{\Lambda} = \Phi(\mathit{\Gamma})$. Then, since $\Phi\colon \mathrm{Aut}(\Omega) \to \mathrm{Aut}(D)$ is a group isomorphism, it induces a group isomorphism $C_\Omega(\mathit{\Gamma}) \cong C_D(\mathit{\Lambda})$; and accordingly $Z_\Omega(\mathit{\Gamma}) \cong Z_D(\mathit{\Lambda})$. By (2.1) we have

$$
\mathit{\Lambda} \;=\; \big\{\big(e^{2\pi i(a_{13}+ia)\psi},\; e^{2\pi i(a_{23}+ia\lambda)\psi}\big) \;\big|\; \psi \in \mathbf{R}\big\}.
$$

With this notation, we can prove the following:

Lemma 2.1. (1) *Let λ be the real number appearing in* (2.1). *Then $\lambda \in \mathbf{Q}$.*

(2) *The group $C_\Omega(\mathit{\Gamma})$ consists af all elements F in* $\mathrm{Aut}(\Omega)$ *of the form $F(z,w) = (\alpha(z), \beta(z)w)$, where $\alpha \in \mathrm{Aut}(B^1)$ and β is a nowhere vanishing holomorphic function on B^1. In particular, $Z_\Omega(\mathit{\Gamma})$ as well as $C_\Omega(\mathit{\Gamma})$ is not an abelian group.*

Moreover, noting that D is a pseudoconvex Reinhardt domain, we obtain that:

Lemma 2.2. *There exists a boundary point (z_o, w_o) of D with $z_o w_o \neq 0$.*

Therefore, for every point $(z_o, w_o) \in \partial D$ with $z_o w_o \neq 0$, the orbit $\mathit{\Lambda}\cdot(z_o, w_o)$ of $\mathit{\Lambda}$ passing through the point (z_o, w_o) must lie in the boundary ∂D; and accordingly

$$
(2.2)\qquad \partial|D| \supset \big\{\big(e^{-2\pi a\psi}|z_o|,\; e^{-2\pi a\lambda\psi}|w_o|\big) \;\big|\; \psi \in \mathbf{R}\big\},
$$

where $|D|$ denotes the real representative domain of the Reinhardt domain D.

In the following, we wish to list up all the possible cases where D satisfies the condition (2.2), and then we will eliminate all the possibilities except in the case where D is biholomorphically equivalent to Ω.

Case I. $\lambda = 0$.

Since D is a pseudoconvex Reinhardt domain in $\mathbf{C}^2$ satisfying the condition (2.2), one can see that there are eight possibilities as follows:

(I.1) $D = \{(z,w) \mid z \in \mathbf{C}, \lvert w\rvert < R\}$	(I.5) $D = \{(z,w) \mid z \in \mathbf{C}, \lvert w\rvert > R\}$
(I.2) $D = \{(z,w) \mid z \in \mathbf{C}^*, \lvert w\rvert < R\}$	(I.6) $D = \{(z,w) \mid z \in \mathbf{C}^*, \lvert w\rvert > R\}$
(I.3) $D = \{(z,w) \mid z \in \mathbf{C}, 0 < \lvert w\rvert < R\}$	(I.7) $D = \{(z,w) \mid z \in \mathbf{C}, r < \lvert w\rvert < R\}$
(I.4) $D = \{(z,w) \mid z \in \mathbf{C}^*, 0 < \lvert w\rvert < R\}$	(I.8) $D = \{(z,w) \mid z \in \mathbf{C}^*, r < \lvert w\rvert < R\}$

where r and R are some positive real numbers.

Lemma 2.3. *In Case* I, *D is biholomorphically equivalent to Ω.*

Proof. Clearly, D is biholomorphically equivalent to Ω in the case (I.1). We here assert that all the others do not occur. Indeed, assume that one of them occurs. Then, defining $\varphi(z,w) := (z^{-1}, w)$, $(z,w) \in \mathbf{C}^* \times \mathbf{C}$, and $\psi(z,w) := (zw^{\mu}, w)$, $(z,w) \in \mathbf{C} \times \mathbf{C}^*$, for $\mu \in \mathbf{Z}$, we have $\varphi \in \mathrm{Aut}_{\mathrm{alg}}(D)$ in the case (I.2) and $\psi \in \mathrm{Aut}_{\mathrm{alg}}(D)$ in the cases (I.3)−(I.8). Thus $\sharp\mathcal{G}(D) \geq 2$. On the other hand, $\sharp\mathcal{G}(\Omega) = 1$ by Proposition 1.1. Since $\mathcal{G}(D)$ is isomorphic to $\mathcal{G}(\Omega)$, this is a contradiction. Therefore we conclude that D is biholomorphically equivalent to Ω in Case I. □

Case II. $\lambda < 0$.

By Lemma 2.1, the number λ can be written uniquely in the form $\lambda = -p/q$ with $p, q \in \mathbf{N}$, $(p,q) = 1$. We wish to prove that Case II does not occur. For this purpose, let us set

$$D_{p,q} = \{(z,w) \mid |z|^p|w|^q < 1\}, \quad D^z_{p,q} = \{(z,w) \mid |z|^p|w|^q < 1,\ z \neq 0\},$$
$$\text{and } D^w_{p,q} = \{(z,w) \mid |z|^p|w|^q < 1,\ w \neq 0\}.$$

Then, after a suitable change of coordinates of the form $(\tilde{z}, \tilde{w}) = (z, \beta w)$ with $\beta \in \mathbf{C}^*$, if necessary, we have the following six possibilities by (2.2):

(II.1) $D = D_{p,q}$	(II.4) $D = \{(z,w) \mid 0 < \lvert z\rvert^p\lvert w\rvert^q < 1\}$
(II.2) $D = D^z_{p,q}$	(II.5) $D = \{(z,w) \mid \lvert z\rvert^p\lvert w\rvert^q > 1\}$
(II.3) $D = D^w_{p,q}$	(II.6) $D = \{(z,w) \mid r < \lvert z\rvert^p\lvert w\rvert^q < 1\}$

where r is a real number with $0 < r < 1$. Noting that $D \subset \mathbf{C}^* \times \mathbf{C}^*$ in the cases (II.4)−(II.6), we can first prove the following:

Lemma 2.4. *The cases* (II.4) – (II.6) *do not occur.*

Now we wish to eliminate the possibility of Case II by showing the following:

Lemma 2.5. *The cases* (II.1) – (II.3) *also do not occur.*

Proof. We assume contrarily that one of these cases occurs, and we would like to derive a contradiction. The proof will be divided into two steps.

Step 1. Every element F of $C_D(\mathit{\Lambda})$ has the form

$$F(z,w) = \big(\alpha(z^p w^q)z,\ \beta(z^p w^q)w\big), \quad (z,w) \in D,$$

where $\alpha(u), \beta(u)$ are nowhere vanishing holomorphic functions on B^1. In particular, in the cases (II.2) *and* (II.3)*, every element $F \in C_D(\mathit{\Lambda})$ extends to a holomorphic automorphism of $D_{p,q}$.*

Step 2. The cases (II.1), (II.2) *and* (II.3) *do not occur*: Assume that one of these cases occurs. Since $Z_\Omega(\mathit{\Gamma})$ is not an abelian group by Lemma 2.1 and $Z_D(\mathit{\Lambda})$ is isomorphic to $Z_\Omega(\mathit{\Gamma})$, it is enough to show that $Z_D(\mathit{\Lambda})$ is abelian. For this purpose, take an arbitrary element F of $C_D(\mathit{\Lambda})$. Then F can be expressed in the form $F(z,w) = (\alpha(u)z,\ \beta(u)w)$, $u = z^p w^q$, as in Step 1; and moreover, F can be regarded as a holomorphic automorphism of $D_{p,q}$. Here we assert that

$$(2.3) \qquad \alpha(u)^p\beta(u)^q \ = \ \alpha(0)^p\beta(0)^q \quad \text{for all } \ u \in B^1 \ \text{ and } \ |\alpha(0)^p\beta(0)^q| = 1.$$

Indeed, define a holomorphic function f on B^1 by setting $f(u) = \alpha(u)^p\beta(u)^q u$, $u \in B^1$. Then we can prove that $f(B^1) \subset B^1$ and f gives rise to an automorphism of B^1 with $f(0) = 0$. Consequently, f has to be of the form $f(u) = Au$ with $|A| = 1$, showing our assertion (2.3). Thanks to (2.3), if we set $A = \alpha(0)^p\beta(0)^q$, the inverse element F^{-1} of F is given by $F^{-1}(z,w) = \big(\alpha(A^{-1}u)^{-1}z,\ \beta(A^{-1}u)^{-1}w\big)$, $u = z^p w^q$. Then, direct computations show that $Z_D(\Lambda)$ is, in fact, abelian; completing the proof of the assertion in Step 2. Therefore the proof of Lemma 2.5 is completed. □

By Lemmas 2.4 and 2.5, we have shown that Case II does not occur.

Case III. $\lambda > 0$.

We want to show that this case also does not occur. As before, we can write $\lambda = p/q$ with $p, q \in \mathbf{N}$, $(p,q) = 1$. Since the group $\mathit{\Lambda}$ leaves ∂D invariant, we now have five possibilities as follows:

(III.1) $D = \{(z,w) \mid |w| > k|z|^\lambda\}$

(III.2) $D = \{(z,w) \mid |w| < k|z|^\lambda\}$

(III.3) $D = \{(z,w) \mid |w| > k|z|^\lambda > 0\}$

(III.4) $D = \{(z,w) \mid 0 < |w| < k|z|^\lambda\}$

(III.5) $D = \{(z,w) \mid k|z|^\lambda < |w| < K|z|^\lambda\}$

where $0 < k,\ K \in \mathbf{R}$. Note that in the cases (III.1), (III.2), and (III.3)$-$(III.5), D is contained in $\mathbf{C} \times \mathbf{C}^*$, $\mathbf{C}^* \times \mathbf{C}$, and $\mathbf{C}^* \times \mathbf{C}^*$, respectively.

Lemma 2.6. *Case* III *does not occur.*

Proof. Assuming that this case does occur, we define the algebraic automorphisms φ of $\mathbf{C}\times\mathbf{C}^*$ and ψ of $\mathbf{C}^*\times\mathbf{C}$ by $\varphi(z,w) := (z, w^{-1})$ and $\psi(z,w) := (z^{-1}, w)$, respectively. Then, after a change of coordinates of the form $(\tilde{z}, \tilde{w}) = (z, \beta w)$, $\beta \in \mathbf{C}^*$, if necessary, it can be seen that the images of the domains of the form (III.1)$-$(III.5) under the mappings φ or ψ coincide with the domains

$$D^w_{p,q},\ D^z_{p,q},\ \{(z,w) \mid 0 < |z|^p|w|^q < 1\}\ \text{ or }\ \{(z,w) \mid r < |z|^p|w|^q < 1\},$$

where $0 < r < 1$ and $D^w_{p,q}$, $D^z_{p,q}$ are the same domains defined in Case II. But, as we have already shown in Case II, none of these domains can arise as D. Therefore we arrive at a contradiction; thereby completing the proof. $\square$

Therefore, we have proved the Theorem in the case $n = 2$.

2.2. The case $n \geq 3$. Throughout this Subsection, we write $\Omega = \Omega_{k,\ell}$.

As stated before, we have only to prove the Theorem under the assumption that $k, \ell \geq 1$. Note that the direct product $U(k)\times U(\ell)$ of the unitary groups is contained in $\mathrm{Aut}(\Omega)$. In the following part, denoting by E_d the identity matrix of degree d and $SU(k)$ the special unitary group of degree k, we use the natural identification given by $SU(k) = SU(k) \times \{E_\ell\} \subset U(k) \times U(\ell)$. By using a result on the normal form of some compact group action on a Reinhardt domain due to Shimizu (cf. [3, Proposition 1.1]), we can first show the following:

Lemma 2.7. *By considering the image of D under a suitable algebraic automorphism, if necessary, one may assume that $\Phi(U(k) \times U(\ell)) = U(k) \times U(\ell)$; and consequently, D is invariant under the standard action on $\mathbf{C}^n$ of $U(k) \times U(\ell)$.*

Thus the isomorphism $\Phi\colon \mathrm{Aut}(\Omega) \to \mathrm{Aut}(D)$ induces an isomorphism between the centralizers $C_\Omega(U(k) \times U(\ell))$ and $C_D(U(k) \times U(\ell))$ of $U(k) \times U(\ell)$ in $\mathrm{Aut}(\Omega)$ and in $\mathrm{Aut}(D)$. Here we wish to determine the form of this isomorphism $\Phi\colon C_\Omega(U(k) \times U(\ell)) \to C_D(U(k) \times U(\ell))$. For this purpose, we need the following:

Lemma 2.8. *The group $C_\Omega(U(k)\times U(\ell))$ consists of all elements φ having the form $\varphi(z,w) = (\alpha z, \beta w)$ with $|\alpha| = 1$, $\beta \in \mathbf{C}^*$. Also, the group $C_D(U(k)\times U(\ell))$ consists of all elements $\varphi \in \mathrm{Aut}(D)$ having the form $\varphi(z,w) = (\alpha z, \beta w)$ with $\alpha, \beta \in \mathbf{C}^*$.*

Hence, both groups $C_\Omega(U(k) \times U(\ell))$ and $C_D(U(k) \times U(\ell))$ are real Lie groups of dimension 3 that can be naturally identified with subgroups of $\{(\alpha E_k, \beta E_\ell) \mid \alpha, \beta \in \mathbf{C}^*\} \subset \mathrm{Aut}(\mathbf{C}^n)$. Just as in the case $n = 2$, this together with the fact that

$\Phi(T(\Omega)) = T(D)$ implies the following: The isomorphism $\Phi\colon C_\Omega(U(k) \times U(\ell)) \to C_D(U(k) \times U(\ell))$ can be written in the form

$$\Phi\big(e^{2\pi i\theta}E_k,\, e^{2\pi i(\phi+i\psi)}E_\ell\big) = \begin{cases} \big(e^{2\pi i(a_{11}\theta+a_{12}\phi+a_{13}\psi+ia\psi)}E_k, \\ \qquad e^{2\pi i(a_{21}\theta+a_{22}\phi+a_{23}\psi+ia\lambda\psi)}E_\ell\big) \quad \cdots \quad \text{(Type A)} \\ \text{or} \\ \big(e^{2\pi i(a_{11}\theta+a_{12}\phi+a_{13}\psi+ia\lambda\psi)}E_k, \\ \qquad e^{2\pi i(a_{21}\theta+a_{22}\phi+a_{23}\psi+ia\psi)}E_\ell\big) \quad \cdots \quad \text{(Type B)} \end{cases}$$

for all $\theta, \phi, \psi \in \mathbf{R}$, where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2, \mathbf{Z}), \quad a_{13},\, a_{23},\, a,\, \lambda \in \mathbf{R} \ \text{ and } \ a \neq 0.$$

It should be remarked that, when $k = \ell$, we may assume that Φ is of Type A. However, in the case when $k \neq \ell$, the situation is quite different. So, for the proof of the Theorem, we need to treat two cases where Φ is of Type A and of Type B separately. Now, as in the case of $n = 2$, it can be seen easily the following:

Lemma 2.9. *There exists a boundary point (z_o, w_o) of D with $\|z_o\|\|w_o\| \neq 0$.*

We now consider the subgroup $\varLambda$ of $\mathrm{Aut}(D)$ defined by

$$\varLambda = \begin{cases} \big\{\big(e^{-2\pi a\psi}A,\ e^{-2\pi a\lambda\psi}B\big) \bigm| \psi \in \mathbf{R},\ A \in U(k),\ B \in U(\ell)\big\} \\ \text{or} \\ \big\{\big(e^{-2\pi a\lambda\psi}A,\ e^{-2\pi a\psi}B\big) \bigm| \psi \in \mathbf{R},\ A \in U(k),\ B \in U(\ell)\big\} \end{cases}$$

according to Φ is of Type A or of Type B.

Since $\varLambda$ can be regarded as a subgroup of $\mathrm{Aut}(\mathbf{C}^n)$, it leaves ∂D invariant. Thus, for every point $(z_o, w_o) \in \partial D$ with $\|z_o\|\|w_o\| \neq 0$, we have $\varLambda\cdot(z_o, w_o) \subset \partial D$. As in the case $n = 2$, by making use of this fact, we shall prove our Theorem in the general case $n \geq 3$. Again we have three cases to consider:

Case I. $\lambda = 0$.

We want to prove that D is biholomorphically equivalent to Ω in this case. The proof of this will be divided into two cases where Φ is of Type A or of Type B.

Case (I.A). *Φ is of Type A*: In this case, having the boundary invariant under the action of $\varLambda$, D may coincide with one of the following domains:

(I.A.1) $\{(z, w) \mid z \in \mathbf{C}^k, \|w\| < R\}$	(I.A.5) $\{(z, w) \mid z \in \mathbf{C}^k, \|w\| > R\}$
(I.A.2) $\{(z, w) \mid \|z\| > 0, \|w\| < R\}$	(I.A.6) $\{(z, w) \mid \|z\| > 0, \|w\| > R\}$
(I.A.3) $\{(z, w) \mid z \in \mathbf{C}^k, 0 < \|w\| < R\}$	(I.A.7) $\{(z, w) \mid z \in \mathbf{C}^k, r < \|w\| < R\}$
(I.A.4) $\{(z, w) \mid \|z\| > 0, 0 < \|w\| < R\}$	(I.A.8) $\{(z, w) \mid \|z\| > 0, r < \|w\| < R\}$

where r and R are some positive real numbers. Then, by considering the structure of the group $\mathcal{G}(D)$ defined in (1.2), one can prove the following:

Lemma 2.10. *The case* (I.A.1) *occurs only when $k = \ell$, and D is biholomorphically equivalent to Ω in this case. And, the cases* (I.A.2) – (I.A.8) *do not occur.*

Case (I.B). *Φ is of Type B*: In this case, interchanging the role of z and w in Case (I.A), we have also eight possible cases. However, in exactly the same way as in Case (I.A), it can be shown that the domain $\{(z, w) \mid w \in \mathbf{C}^\ell,\ \|z\| < R\}$ has only the possibility for D, where R is a positive real number. Of course, this domain is actually biholomorphically equivalent to Ω.
Therefore, we conclude that D is biholomorphically equivalent to Ω in Case I.

Case II. $\lambda < 0$.

We want to show this case does not occur. The proof will be divided into two cases.

Case (II.A). *Φ is of Type A*: First we set $D_A(k,\ell) = \big\{(z,w) \bigm| \|z\|^{-\lambda}\|w\| < 1\big\}$, $D_A^z(k,\ell) = \big\{(z,w) \bigm| \|z\|^{-\lambda}\|w\| < 1,\ z \neq 0\big\}$ and $D_A^w(k,\ell) = \big\{(z,w) \bigm| \|z\|^{-\lambda}\|w\| < 1,\ w \neq 0\big\}$ in $\mathbf{C}^k \times \mathbf{C}^\ell$. Then, recalling that D is pseudoconvex and $\mathit{\Lambda}\cdot\partial D = \partial D$ and $(U(k)\times U(\ell))\cdot D = D$, one can see that there exist only three cases as follows among them (after a change of coordinates of the form $(\tilde{z}, \tilde{w}) = (z, \beta w)$ with $\beta \in \mathbf{C}^*$, if necessary):

$$D = D_A(k,\ell), \quad D = D_A^z(1,\ell)\ (\ell \geq 2) \quad \text{or} \quad D = D_A^w(k,1)\ (k \geq 2).$$

Assuming that one of these cases does occur, we fix an arbitrary point $(z_o, w_o) \in \partial D$ with $\|z_o\|^{-\lambda}\|w_o\| = 1$. Take an arbitrary element $\varphi \in \mathit{\Pi}(D)$ and write $\varphi(z,w) = (\zeta_1 z_1, \dots, \zeta_k z_k, \eta_1 w_1, \dots, \eta_\ell w_\ell)$ with $\zeta_i,\ \eta_j \in \mathbf{C}^*$. Then, since $(\|z_o\|u,\ \|w_o\|v) \in \partial D$ and $\varphi(\|z_o\|u,\ \|w_o\|v) \in \partial D$ for each $u = (u_i) \in \mathbf{C}^k$, $v = (v_j) \in \mathbf{C}^\ell$ with $\|u\| = 1,\ \|v\| = 1$, a simple computation shows that

$$\big(|\zeta_1 u_1|^2 + \cdots + |\zeta_k u_k|^2\big)^{-\lambda}\big(|\eta_1 v_1|^2 + \cdots + |\eta_\ell v_\ell|^2\big) \;=\; 1.$$

Thus $|\zeta_1| = \cdots = |\zeta_k| =: r,\ |\eta_1| = \cdots = |\eta_\ell| =: R$ and $r^{-\lambda}R = 1$. Therefore, $\mathit{\Pi}(D)$ is a real Lie group of dimension $k + \ell + 1 = n + 1$. On the other hand, we know that $\mathit{\Pi}(D)$ is isomorphic to the real Lie group $\mathit{\Pi}(\Omega)$ of dimension $k + 2\ell = n + \ell$; and hence, $\ell = 1$. Therefore we have obtained the following:

Lemma 2.11. *In Case* (II.A)*, we have only two possibilities of $D = D_A(k,1)$ or $D = D_A^w(k,1)$ for $k = n - 1 \geq 2$.*

In the both cases in Lemma 2.11, $\mathit{\Phi}(SU(k)) = SU(k)$ by Lemma 2.7; and so $C_\Omega(SU(k))$ is isomorphic to $C_D(SU(k))$. On the other hand, $C_\Omega(SU(k))$ is a real

Lie group of dimension 5, while $C_D(SU(k))$ is of dimension 3 by the following lemmas. Therefore we can eliminate the possibility of Case (II.A):

Lemma 2.12. *For $k = n-1 \geq 2$, we have*

$$C_\Omega(SU(k)) \;=\; \bigl\{(z,w) \mapsto (\alpha z, \beta w + \gamma) \bigm| |\alpha| = 1,\, \beta \in \mathbf{C}^*,\, \gamma \in \mathbf{C}\bigr\}.$$

Hence, $C_\Omega(SU(k))$ is a real Lie group of dimension 5.

Lemma 2.13. (1) *In the case $D = D_A(k,1)$ for $k = n-1 \geq 2$, we have*

$$C_D(SU(k)) \;=\; \bigl\{(z,w) \mapsto (\alpha z, \beta w) \bigm| |\alpha|^{-\lambda}|\beta| = 1\bigr\}.$$

(2) *In the case $D = D_A^w(k,1)$ for $k = n-1 \geq 2$, we have the following:*
(i) *If $-2/\lambda \notin \mathbf{N}$, then* $C_D(SU(k)) = \bigl\{(z,w) \mapsto (\alpha z, \beta w) \bigm| |\alpha|^{-\lambda}|\beta| = 1\bigr\}$.
(ii) *If $-2/\lambda \in \mathbf{N}$, then*

$$\begin{aligned} C_D(SU(k)) = \bigl\{(z,w) \mapsto (\alpha z, \beta w) \bigm| |\alpha|^{-\lambda}|\beta| = 1\bigr\} \\ \bigcup \bigl\{(z,w) \mapsto (\alpha w^{-2/\lambda} z,\, \beta w^{-1}) \bigm| |\alpha|^{-\lambda}|\beta| = 1\bigr\}. \end{aligned}$$

Hence, $C_D(SU(k))$ is a real Lie group of dimension 3 *in any case.*

Case (II.B). *Φ is of Type B*: We assert that this case also does not occur. Indeed, putting

$$D_B(k,\ell) = \bigl\{(z,w) \bigm| \|z\|\|w\|^{-\lambda} < 1\bigr\},\; D_B^w(k,\ell) = \bigl\{(z,w) \bigm| \|z\|\|w\|^{-\lambda} < 1,\; w \neq 0\bigr\}$$

in $\mathbf{C}^k \times \mathbf{C}^\ell$, we can show that, after a change of coordinates of the form $(\tilde{z}, \tilde{w}) = (\alpha z, w)$ with $\alpha \in \mathbf{C}^*$, if necessary, there are only two possibilities for D:

$$D = D_B(k,1) \quad \text{or} \quad D = D_B^w(k,1) \quad \text{for } \; k = n-1 \geq 2.$$

Assume that one of these cases occurs. Then we have $\Phi(SU(k)) = SU(k)$ in each case. Thus the groups $C_\Omega(SU(k))$ and $C_D(SU(k))$ are isomorphic. Recall that $C_\Omega(SU(k))$ is a real Lie group of dimension 5 by Lemma 2.12. However, with exactly the same argument as in Case (II.A), one can verify the following:

Lemma 2.15. (1) *In the case $D = D_B(k,1)$ for $k = n-1 \geq 2$, we have*

$$C_D(SU(k)) \;=\; \bigl\{(z,w) \mapsto (\alpha z, \beta w) \bigm| |\alpha||\beta|^{-\lambda} = 1\bigr\}.$$

(2) *In the case $D = D_B^w(k,1)$ for $k = n-1 \geq 2$, we have the following:*
(i) *If $-2\lambda \notin \mathbf{N}$, then* $C_D(SU(k)) = \bigl\{(z,w) \mapsto (\alpha z, \beta w) \bigm| |\alpha||\beta|^{-\lambda} = 1\bigr\}$.

(ii) *If* $-2\lambda \in \mathbf{N}$, *then*

$$C_D(SU(k)) = \{(z,w) \mapsto (\alpha z, \beta w) \mid |\alpha||\beta|^{-\lambda} = 1\} \\ \bigcup \{(z,w) \mapsto (\alpha w^{-2\lambda} z,\ \beta w^{-1}) \mid |\alpha||\beta|^{-\lambda} = 1\}.$$

Hence, $C_D(SU(k))$ *is a real Lie group of dimension* 3 *in any case.*

By this contradiction, we conclude that CASE II does not occur, as required.

CASE III. $\lambda > 0$.

We claim that this case is impossible. To this end, we first consider the following:

Case (III.A). Φ *is of Type A*: By the same reasoning as in Case (II.A), we may have the following two possible cases for D, after a change of coordinates of the form $(\tilde{z}, \tilde{w}) = (z, \beta w)$, if necessary:

(III.A.1) $D = \{(z,w) \in \mathbf{C} \times \mathbf{C}^{n-1} \mid \|w\| < |z|^\lambda\} \subset \mathbf{C}^* \times \mathbf{C}^{n-1}$, and
(III.A.2) $D = \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} \mid |w| > \|z\|^\lambda\} \subset \mathbf{C}^{n-1} \times \mathbf{C}^*$.

Here, consider the algebraic automorphisms φ of $\mathbf{C}^* \times \mathbf{C}^{n-1}$ and ψ of $\mathbf{C}^{n-1} \times \mathbf{C}^*$ defined by $\varphi(z,w) := (z^{-1}, w)$ and $\psi(z,w) := (z, w^{-1})$. Then, for the domains D in (III.A.1) and (III.A.2), we see that $\varphi(D)$ and $\psi(D)$ coincide with the domains

$$\{(u,v) \mid |u|^\lambda \|v\| < 1,\ u \neq 0\} \quad \text{and} \quad \{(u,v) \mid \|u\|^\lambda |v| < 1,\ v \neq 0\},$$

respectively. However, we have already shown in Case (II.A) that these domains do not arise as D; and consequently, Case (III.A) does not occur, as asserted.

Case (III.B). Φ *is of Type B*: In this case, after a suitable change of coordinates if necessary, we again have two possibilities as follows:

(III.B.1) $D = \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} \mid \|z\| < |w|^\lambda\} \subset \mathbf{C}^{n-1} \times \mathbf{C}^*$, and
(III.B.2) $D = \{(z,w) \in \mathbf{C} \times \mathbf{C}^{n-1} \mid |z| > \|w\|^\lambda\} \subset \mathbf{C}^* \times \mathbf{C}^{n-1}$.

Notice that, by the algebraic automorphisms ψ or φ defined in Case (III.A), these domains are transformed onto the domains

$$\{(u,v) \mid \|u\||v|^\lambda < 1,\ v \neq 0\} \quad \text{and} \quad \{(u,v) \mid |u|\|v\|^\lambda < 1,\ u \neq 0\}$$

which were already studied in Case (II.B). Then the conclusion in Case (II.B) implies that Case (III.B) is also impossible.
Therefore CASE III is impossible, as claimed.

Summarizing our results obtained in the above, we have shown that only CASE I occurs and the domain D is biholomorphically equivalent to the model space Ω; thereby completing the proof of our Theorem in the case $n \geq 3$. $\square$

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Jisoo BYUN
Université de Provence
Marseille, France
jisoo@cmi.univ-mrs.fr

Akio KODAMA
Division of Mathematical and Physical Sciences
Graduate School of Natural Science and Technology
Kanazawa University
Kanazawa 920-1192, Japan
kodama@kenroku.kanazawa-u.ac.jp

Satoru SHIMIZU
Mathematical Institute
Tohoku University
Sendai 980-8578, Japan
shimizu@math.tohoku.ac.jp

A statement of Weierstrass

Yukitaka Abe

(Toyama University, Japan)

1 Weierstrass' statement and the main theorem

Weierstrass frequently stated the following in his lectures at Berlin:

Every system of n (independent) functions with n variables which admits an addition theorem is algebraic combination of n abelian (or degenerate) functions with the same periods.

But his proof was meither published nor taught (see [13], other episodes are stated there).

In the case $n = 1$, a degenerate elliptic function is a rational function or a rational function of an exponential function (see [19]), and the statement is true. We can see its proof due to Osgood in [5]. I gave another proof in the previous paper [2].

On the other hand, it is not clear what are degenerate abelian functions or quasi-abelian functions when $n \geqq 2$. If we consider the field of meromorphic functions on $\mathbb{C}^n$ with period Γ of rank $\Gamma < 2n$, its transcendence degree over $\mathbb{C}$ is not always finite even when $\mathbb{C}^n/\Gamma$ does not contain $\mathbb{C}$ or $\mathbb{C}^*$ as a direct summand. Then, we can not consider degenerate abelian functions merely as meromorphic functions with degenerate periods.

Our main theorem is the following.

Theorem 1.1 *Let $K \subset \mathfrak{M}(\mathbb{C}^n)$ be a non-degenerate algebraic function field of n variables over $\mathbb{C}$ which admits an algebraic addition theorem. Then K is considered as a subfield of $\mathbb{C}(z_1, \ldots, z_p, w_1, \ldots, w_q, g_0, \ldots, g_r)$, where $z_1, \ldots, z_p$ are coordinate functions of $\mathbb{C}^p$, $w_1, \ldots, w_q$ are those of $(\mathbb{C}^*)^q$ and $g_0, \ldots, g_r$ are generators of an abelian function field of dimension r, $p + q + r = n$.*

We state the outline of its proof. The details will appear in [3].

2 Definitions

Let $\mathfrak{M}(\mathbb{C}^n)$ be the field of meromorphic functions on $\mathbb{C}^n$. We consider a subfield K of $\mathfrak{M}(\mathbb{C}^n)$. We assume that K is finitely generated over $\mathbb{C}$ and Trans $K = n$, where Trans K is the transcendence degree of K over $\mathbb{C}$. Such a field K is called an algebraic function field of n variables over $\mathbb{C}$. Let $f_0, \ldots, f_n$ be generators of K.

Definition 2.1 *We say that $f_0, \ldots, f_n$ admit an algebraic addition theorem (we abbreviate it (AAT)) if for any $j = 0, \ldots, n$ there exists a rational function R_j such that*

$$f_j(x + y) = R_j(f_0(x), \ldots, f_n(x), f_0(y), \ldots, f_n(y)) \tag{1}$$

for all $x, y \in \mathbb{C}^n$. An algebraic function field K of n variables over $\mathbb{C}$ admits the (AAT) if it has generators $f_0, \ldots, f_n$ which admit the (AAT).

We note that if K admits the (AAT), then any generators $g_0, \ldots, g_n$ of K admit the (AAT).

Definition 2.2 *An algebraic function field K of n variables over $\mathbb{C}$ admits another addition theorem (AAT*) if there exist algebraically independent $f_1, \dots, f_n \in K$ such that for any $j = 1, \dots, n$ we have a non-zero polynomial P_j with*

$$P_j(f_j(x+y), f_1(x), \dots, f_n(x), f_1(y), \dots, f_n(y)) = 0 \tag{2}$$

for all $x, y \in \mathbb{C}^n$.

By an elementary algebraic argument, we see that K admits the (AAT) if and only if it admits the (AAT*).

A function $f \in \mathfrak{M}(\mathbb{C}^n)$ is degenerate if there exist an invertible linear transformation $\mathcal{L} : \mathbb{C}^n \longrightarrow \mathbb{C}^n,\ x = \mathcal{L}(y)$ and a non-negative integer r with $r < n$ such that $f(\mathcal{L}(y))$ does not depend on $y_{r+1}, \dots, y_n$. We say that f is non-degenerate if it is not degenerate.

Definition 2.3 *A subfield K of $\mathfrak{M}(\mathbb{C}^n)$ is said to be non-degenerate if there exists a non-degenerate function in K.*

We assume that K is a non-degenerate algebraic function field of n variables over $\mathbb{C}$ which admits the (AAT).

3 Picard varieties

Let $f \in K$ be a non-degenerate function. Since $\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}$ are linearly independent, there exist $a^{(1)}, \dots, a^{(n)} \in \mathbb{C}^n$ such that

$$\operatorname{rank} \begin{pmatrix} \frac{\partial f}{\partial z_1}(a^{(1)}) & \cdots & \frac{\partial f}{\partial z_n}(a^{(1)}) \\ \cdots & \cdots & \cdots \\ \frac{\partial f}{\partial z_1}(a^{(n)}) & \cdots & \frac{\partial f}{\partial z_n}(a^{(n)}) \end{pmatrix} = n.$$

Let $g_i(z) := f(z + a^{(i)})$, $i = 1, \ldots, n$. Then $g_i \in K$ and $(g_1, \ldots, g_n)$ is a locally biholomorphic mapping on $\mathbb{C}^n$ except an analytic subset. Then we can take $b^{(0)} := 0, b^{(1)}, \ldots, b^{(r)} \in \mathbb{C}^n$ such that if we set

$$g_{ij}(z) := g_j(z + b^{(i)}), \quad i = 0, 1, \ldots, r; j = 1, \ldots, n,$$

then (g_{ij}) is locally biholomorphic at any point $a \in \mathbb{C}^n$. Of course $g_{ij} \in K$. Let $\{h_1, \ldots, h_N\} := \{g_{ij}\} \cup \{f_0, f_1, \ldots, f_n\}$. We have holomorphic functions $\varphi_0, \ldots, \varphi_N$ on $\mathbb{C}^n$ such that $\varphi_0, \ldots, \varphi_N$ have no common divisor, they give a holomorphic immersion

$$\Phi := [\varphi_0 : \cdots : \varphi_N] : \mathbb{C}^n \longrightarrow \mathbb{P}^N$$

and $h_i = \varphi_i / \varphi_0$, $i = 1, \ldots, N$.

Consider all the algebraic relations among $h_1, \ldots, h_N$. We denote by $\mathcal{P}$ the set of all corresponding homogeneous polynomials. Then $\mathcal{P}$ gives an algebraic subvariety Y of $\mathbb{P}^N$. It follows from the definition of Y that $\Omega := \Phi(\mathbb{C}^n) \subset Y$ and Y is the Zariski closure of Ω.

For $p, q \in \Omega$ we define

$$p \cdot q := \Phi(z + w), \quad z \in \Phi^{-1}(p), w \in \Phi^{-1}(q).$$

Then Ω is a connected complex abelian Lie group by this operation. And $\Phi : \mathbb{C}^n \longrightarrow \Omega$ is an epimorphism. Let $\Gamma := \mathrm{Ker}\Phi$. We note that Γ is a discrete subgroup of $\mathbb{C}^n$. From the above epimorphism, we obtain an isomorphism

$$\overline{\Phi} : G := \mathbb{C}^n / \Gamma \longrightarrow \Omega,$$

and K is considered as a subfield of $\mathfrak{M}(G)$.

We summarize the above results in the following theorem which is basic to our argument.

Theorem 3.1 ([2, Theorem 2.6]) *There exist holomorphic functions* $\varphi_0, \dots, \varphi_N$ *on* $\mathbb{C}^n$, *a discrete subgroup* Γ *of* $\mathbb{C}^n$, *an algebraic subvariety* Y *of the* N*-dimensional complex projective space* $\mathbb{P}^N$ *and a connected complex abelian Lie group* Ω *in* Y *such that*

(a) $\varphi_0, \dots, \varphi_N$ *give a Lie group isomorphism*

$$\overline{\Phi} = [\varphi_0 : \cdots : \varphi_N] : G := \mathbb{C}^n/\Gamma \longrightarrow \Omega,$$

(b) $\varphi_1/\varphi_0, \dots, \varphi_N/\varphi_0$ *generate* K *and* K *is considered as a subfield of* $\mathfrak{M}(G)$, *where* $\mathfrak{M}(G)$ *is the field of meromorphic functions on* G,

(c) Y *is the Zariski closure of* Ω *and*

$$\overline{\Phi}^* : \mathbb{C}(Y) \longrightarrow K, \ R \longmapsto R \circ \overline{\Phi}$$

is an isomorphism, then $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} \Omega = n$, *where* $\mathbb{C}(Y)$ *is the rational function field of* Y.

We call Y a Picard variety of K. Let K' be a subfield of $\mathfrak{M}(\mathbb{C}^n)$ satisfying the same assumptions, and let Y' be a Picard variety of K'. Then, K and K' are isomorphic if and only if Y and Y' are birationally equivalent.

Applying the above theorem, I gave a short proof of the statement of Weierstrass when $n = 1$ (the proof of Theorem 2.7 in [2]).

Theorem 3.2 (Weierstrass) *A function* $f \in \mathfrak{M}(\mathbb{C})$ *admits the (AAT*) if and only if it is an elliptic function or a rational function or a rational function of* e^{az}.

4 Continuation of closed subgroups

We assume the situation in Theorem 3.1. Let Ω be the connected complex abelian Lie group embedded in $\mathbb{P}^N$. It has the Zariski closure Y with $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} \Omega = n$. Let $(x_0, \ldots, x_N)$ be homogeneous coordinates of $\mathbb{P}^N$. We write $x(p) = (x_0(p), \ldots, x_N(p))$ for any $p \in \Omega$. For any $p, q \in \Omega$, we have

$$x_i(p \cdot q) = R_i(x(p), x(q)), \quad i = 0, \ldots, N, \tag{3}$$

where R_i is a rational function. Let $\mathfrak{g}$ be the Lie algebra of Ω. The following lemma is an immediate consequence of (3).

Lemma 4.1 *For any $X \in \mathfrak{g}$ and any $p \in \Omega$, there exists a neighborhood U of p in $\mathbb{P}^N$ such that*

$$X = \sum_{j=1}^{N} \tilde{R}_j(t) \frac{\partial}{\partial t_j} \quad \text{on } U \cap \Omega,$$

where $t = (t_1, \ldots, t_N)$ are affine coordinates on U and $\tilde{R}_j(t)$ is a rational function of t.

We now consider a connected closed complex Lie subgroup H of Ω. Let $\mathfrak{h}$ be the Lie algebra of H. We can take a basis $\{X_1, \ldots, X_m, Y_1, \ldots, Y_r\}$ of $\mathfrak{g}$ such that $\{Y_1, \ldots, Y_r\}$ is a basis of $\mathfrak{h}$, where $r = \dim_{\mathbb{C}} H$. Let $\{\omega_1, \ldots, \omega_m, \eta_1, \ldots, \eta_r\}$ be a set of holomorphic 1-forms on Ω which forms the dual system of $\{X_1, \ldots, X_m, Y_1, \ldots, Y_r\}$. For any $p \in \Omega$ we assign

$$\mathcal{D}_p := \{v \in T_p(\Omega); (\omega_1)_p(v) = \cdots = (\omega_m)_p(v) = 0\}.$$

Then $\mathcal{D} : \Omega \ni p \longmapsto \mathcal{D}_p \subset T_p(\Omega)$ is an r-dimensional complex differential system. Since Ω is abelian, $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Then, all holomorphic 1-forms ω_i $(i = 1, \ldots, m)$ and η_j $(j = 1, \ldots, r)$ are d-closed.

We may assume by the resolution of singularities that Y is non-singular. By Lemma 4.1 each $X_i, Y_j, \omega_i, \eta_j$ is meromorphically extendable to Y. We use the same notations $X_i, Y_j, \omega_i, \eta_j$ for their extensions to Y, without confusion. It holds that $d\omega_i = 0$ on Y for $i = 1, \dots, m$.

Let $\mathcal{D}_i$ be the $(n-1)$-dimensional complex differential system on Ω defined by the local equation $\omega_i = 0$, for $i = 1, \dots, m$. Since $\mathcal{D}_i$ is completely integrable on Ω, there exists an $(n-1)$-dimensional integral manifold Z_i of $\mathcal{D}_i$ such that

$$H = \bigcap_{i=1}^{m} Z_i.$$

We obtain the following proposition by careful consideration of the singular points of ω_i.

Proposition 4.2 *For any $i = 1, \dots, m$, there exists an irreducible analytic subset $\widetilde{Z}_i$ of Y of pure codimension 1 such that*

$$Z_i = \widetilde{Z}_i \cap \Omega.$$

We set

$$Z := \bigcap_{i=1}^{m} \widetilde{Z}_i.$$

Then Z is an r-dimensional irreducible analytic subset of Y with $Z \cap \Omega = H$. Therefore, it is the Zariski closure of H. We summarize the above results in the following theorem for the later use.

Theorem 4.3 *Let H be a connected closed complex Lie subgroup of Ω. Then the Zariski closure Z of H has the same dimension as H.*

5 Restriction to a closed subgroup

Let H be a connected closed complex Lie subgroup of $G = \mathbb{C}^n/\Gamma$. We consider the restriction of K to H. For any $f \in \mathfrak{M}(G)$ we denote by $P(f)$ the polar set of f. We define the restriction f_H of f to H by

$$f_H := \begin{cases} 0, & \text{if } H \subset P(f) \\ f|_H, & \text{otherwise.} \end{cases}$$

Let $K_H := \{f_H; f \in K\}$ be the restriction of K to H. It is obvious that K_H is non-degenerate and admits the (AAT).

Proposition 5.1 *It holds that* $\operatorname{Trans} K_H = \dim_{\mathbb{C}} H$.

Proof. Let $\overline{\Phi} : G \longrightarrow \Omega$ be the isomorphism in Theorem 3.1. Then $\widetilde{H} := \overline{\Phi}(H)$ is a connected closed complex Lie subgroup of Ω. Let Z be the Zariski closure of $\widetilde{H}$. By Theorem 4.3 we have $\dim_{\mathbb{C}} Z = \dim_{\mathbb{C}} \widetilde{H} = \dim_{\mathbb{C}} H$. Since

$$K_H \cong \{f_{\widetilde{H}}; f \in \mathbb{C}(Y)\} \cong \mathbb{C}(Z)$$

and $\operatorname{Trans} \mathbb{C}(Z) = \dim_{\mathbb{C}} Z$, we obtain the conclusion. □

6 Separately extendable meromorphic functions

In this section, we discuss the extendability of separately extendable meromorphic functions improving the arguments in [7].

Let D and E be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. We consider meromorphic functions $F_1(z, w), \ldots, F_N(z, w)$ on $D \times E$, which are not all identically zero. Let P_i be the polar set of F_i. We set

$$P := \bigcup_{i=1}^{N} P_i.$$

Then P is an analytic subset of $D \times E$ with $\mathrm{codim}_{\mathbb{C}} P = 1$. There exist subdomains $D_0 \subset D$ and $E_0 \subset E$ such that

$$D_0 \times E_0 \subset (D \times E) \setminus P.$$

We obtain the following lemma along an argument in [7] (Chapter IX, Section 5, Lemma 6).

Lemma 6.1 *Assume that there exist functions $c_1(w), \ldots, c_N(w)$ on E_0 such that*

$$c_1(w)F_1(z,w) + \cdots + c_N(w)F_N(z,w) \equiv 0 \quad \text{on } D_0 \times E_0,$$

where $c_1(w), \ldots, c_N(w)$ are not all zero for any $w \in E_0$. Then, there exist meromorphic functions $C_1(w), \ldots, C_N(w)$ on E, which are not all identically zero, such that

$$C_1(w)F_1(z,w) + \cdots + C_N(w)F_N(z,w) \equiv 0 \quad \text{on } D \times E.$$

Remark 6.2 *If $F_i(z,w)$ $(i = 1, \ldots, N)$ is a rational function of w for any fixed $z \in D$ in addition to the assumption in Lemma 6.1, then we can take $C_i(w)$ $(i = 1, \ldots, N)$ as a rational function.*

Using this fact, we can prove the following proposition.

Proposition 6.3 ([7, Chapter IX, Section 5, Theorem 5]) *Let $D \times E \subset \mathbb{C}^n \times \mathbb{C}^m$ be a domain, and let $f(z,w)$ be a holomorphic function on $D \times E$. If $f(z,w)$ is rational in w for any $z \in D$ and rational in z for any $w \in E$, then it is a rational function of (z,w).*

Let $f(z)$ be a meromorphic function on $D := \mathbb{C}^p \times (\mathbb{C}^*)^q$, $p + q = n$, and let P be its polar set. We define

$$P_i := \{(z', z'') = (z_1, \ldots, z_{i-1}; z_{i+1}, \ldots, z_n); \{z'\} \times \mathbb{C} \times \{z''\} \subset P\}$$

for $i = 1, \dots, p$ and

$$P_i := \{(z', z'') = (z_1, \dots, z_{i-1}; z_{i+1}, \dots, z_n); \{z'\} \times \mathbb{C}^* \times \{z''\} \subset P\}$$

for $i = p+1, \dots, n$, where $z = (z_1, \dots, z_n)$ are coordinates of D. We set $D_i = \{(z', z'') = (z_1, \dots, z_{i-1}; z_{i+1}, \dots, z_n)\}$ for $i = 1, \dots, n$. The following proposition is immediate from Proposition 6.3.

Proposition 6.4 *If for any $i = 1, \dots, n$ and any $(z', z'') \in D_i \setminus P_i$, $f(z', z_i, z'')$ is rational in z_i, then $f(z)$ is rational in z.*

Let D be a domain in $\mathbb{C}^k$, and let $E := \mathbb{C}^p \times (\mathbb{C}^*)^q$, $p + q = \ell$, $k + \ell = n$. We consider a meromorphic function $f(z, w)$ on $D \times E$. Let P be the polar set of f. If we set

$$P_z := \{z \in D; \{z\} \times E \subset P\},$$

then P_z is an analytic subset of D. Let $\{p_\nu(w)\}_{\nu=1}^\infty$ be the sequence of all monomials $\{w_1^{\alpha_1} \cdots w_\ell^{\alpha_\ell}\}$. Following the argument in the proof of [7, Chapter IX, Section 5, Theorem 5], we obtain the following theorem.

Theorem 6.5 *Assume that $f(z_0, w)$ is rational in w for any $z_0 \in D \setminus P_z$. Then, there exist meromorphic functions $a_1(z), \dots, a_M(z)$, $b_1(z), \dots, b_N(z)$ on D such that*

$$f(z, w) = \frac{Q(z, w)}{P(z, w)} \quad \text{on } D \times E,$$

where $P(z, w) = \sum_{\mu=1}^M a_\mu(z) p_\mu(w)$ and $Q(z, w) = \sum_{\nu=1}^N b_\nu(z) p_\nu(w)$. Therefore, $f(z, w)$ meromorphically extends to $D \times (\mathbb{P}^1)^\ell$.

7 Extendable line bundles on toroidal groups

A connected complex Lie group G_0 is called a toroidal group if $H^0(G_0, \mathcal{O}) = \mathbb{C}$. Every toroidal group is abelian ([11]). Then we can write $G_0 = \mathbb{C}^r/\Gamma^*$, where Γ^* is a discrete subgroup of $\mathbb{C}^r$ with rank $\Gamma^* = r + m$ $(1 \leqq m \leqq r)$. We denote by $\Gamma^*_{\mathbb{R}}$ the real linear subspace of $\mathbb{C}^r$ spanned by Γ^*. Let $\Gamma^*_{\mathbb{C}} := \Gamma^*_{\mathbb{R}} \cap \sqrt{-1}\Gamma^*_{\mathbb{R}}$ be the maximal complex linear subspace contained in $\Gamma^*_{\mathbb{R}}$. It is easy to see that $\dim_{\mathbb{C}} \Gamma^*_{\mathbb{C}} = m$.

Definition 7.1 *A toroidal group $G_0 = \mathbb{C}^r/\Gamma^*$ is said to be a quasi-abelian variety if there exists a Hermitian form $\mathcal{H}$ on $\mathbb{C}^r$ such that*

(a) $\mathcal{H}$ is positive definite on $\Gamma^*_{\mathbb{C}}$,

(b) the imaginary part $\mathcal{A} := \mathrm{Im}\ \mathcal{H}$ of $\mathcal{H}$ is $\mathbb{Z}$-valued on $\Gamma^* \times \Gamma^*$.

We call such a Hermitian form $\mathcal{H}$ an ample Riemann form for Γ^* or G_0.

From the projection $\mathbb{C}^r \longrightarrow \Gamma^*_{\mathbb{C}}$, we obtain a fiber bundle structure $\sigma : G_0 \longrightarrow \mathbb{T}$ on an m-dimensional complex torus $\mathbb{T}$ with fibers $(\mathbb{C}^*)^\ell$, $\ell = r - m$ ([21]). Replacing fibers $(\mathbb{C}^*)^\ell$ with $(\mathbb{P}^1)^\ell$, we obtain the associated $(\mathbb{P}^1)^\ell$-bundle $\overline{\sigma} : \overline{G}_0 \longrightarrow \mathbb{T}$.

Proposition 7.2 ([20, Satz 3.2.8]) *Let $L \longrightarrow G_0$ be a holomorphic line bundle which is holomorphically extendable to $\overline{G}_0$. Then there exists a theta bundle $L_\theta \longrightarrow \mathbb{T}$ such that*

$$L \cong \sigma^* L_\theta.$$

8 Extension to a compactification of G

We return to our situation. By the theorem of Remmert-Morimoto ([9] and [12]), we have

$$G \cong \mathbb{C}^p \times (\mathbb{C}^*)^q \times X,$$

where $X = \mathbb{C}^r/\Gamma^*$ is a toroidal group of rank $\Gamma^* = r+m$ $(1 \leqq m \leqq r)$ and $p+q+r = n$. Since there exists a non-degenerate meromorphic function on X, X is a quasi-abelian variety ([1] and [8]). We have a $(\mathbb{C}^*)^s$-bundle $\sigma : X \longrightarrow \mathbb{T}$ on an m-dimensional complex torus $\mathbb{T}$, where $s = r - m$. Let $\overline{\sigma} : \overline{X} \longrightarrow \mathbb{T}$ be the associated $(\mathbb{P}^1)^s$-bundle. These bundles give fiber bundles $\tau : G \longrightarrow \mathbb{T}$ with fibers $\mathbb{C}^p \times (\mathbb{C}^*)^q \times (\mathbb{C}^*)^s$ and $\overline{\tau} : \overline{G} \longrightarrow \mathbb{T}$ with fibers $(\mathbb{P}^1)^\ell$, $\ell = p + q + s$, where $\overline{G} = (\mathbb{P}^1)^{p+q} \times \overline{X}$. For any $a \in \mathbb{T}$ we set

$$F_a := \tau^{-1}(a) \cong \mathbb{C}^p \times (\mathbb{C}^*)^q \times (\mathbb{C}^*)^s,$$

$$\overline{F}_a := \overline{\tau}^{-1}(a) \cong (\mathbb{P}^1)^\ell.$$

Theorem 8.1 *Every $f \in K$ meromorphically extends to $\overline{G}$.*

Proof. Let $e \in \mathbb{T}$ be the unit element of $\mathbb{T}$. We take coordinates $(z_1, \dots, z_\ell)$ on F_e. For any $i = 1, \dots, \ell$ we define

$$L_i := \{(0, z_i, 0) \in F_e\}.$$

Then L_i is a connected closed complex Lie subgroup of G with $\dim_{\mathbb{C}} L_i = 1$. It follows from Proposition 5.1 that Trans $K_{L_i} = 1$. K_{L_i} is non-degenerate, admits the (AAT) and is not periodic. Then, any $g \in K_{L_i}$ is a rational function of z_i by Theorem 3.2. Therefore, f_{F_e} is rational for any $f \in K$ by Proposition 6.4.

Let f be any function in K. Take any point $a \in \mathbb{T}$. We define $g(x) := f(x + \tilde{a})$ for some $\tilde{a} \in G$ with $\tau(\tilde{a}) = a$. Using the (AAT), we can verify that $g \in K$. From the above observation, we know that g_{F_e} is rational. Since $F_a = F_e + \tilde{a}$ and $f_{F_a} = g_{F_e}$, f_{F_a} is rational. Furthermore, there exists an open set $U \subset \mathbb{T}$ such that

$$\tau^{-1}(U) \cong U \times (\mathbb{C}^p \times (\mathbb{C}^*)^q \times (\mathbb{C}^*)^s) .$$

As we have seen in the above, $f_{\tau^{-1}(U)}$ satisfies the assumption in Theorem 6.5. Then $f_{\tau^{-1}(U)}$ meromorphically extends to $\overline{\tau}^{-1}(U) \cong U \times (\mathbb{P}^1)^\ell$. This completes the proof.
$\square$

9 Proof of Theorem 1.1

We state the situation again, in order to confirm the problem.

Let $K \subset \mathfrak{M}(\mathbb{C}^n)$ be a non-degenerate algebraic function field of n variables over $\mathbb{C}$. We assume that K admits the (AAT). It is considered as a subfield of $\mathfrak{M}(G)$, where $G = \mathbb{C}^n/\Gamma$. We have the decomposition

$$G \cong \mathbb{C}^p \times (\mathbb{C}^*)^q \times X,$$

where $X = \mathbb{C}^r/\Gamma^*$ is a quasi-abelian variety.

Proposition 9.1 *The quasi-abelian variety X is an abelian variety.*

Proof. There exists a function $f \in K$ such that $g := f_X$ is non-degenerate. Let $\tilde{L}$ be the holomorphic line bundle on G given by the zero-divisor of f. Since f is meromorphically extendable to $\overline{G}$ (Theorem 8.1), $\tilde{L}$ has the holomorphic extension to $\overline{G}$. Then, $L := \tilde{L}|_X$ extends to $\overline{X}$.

Let rank $\Gamma^* = r + s$. Suppose that $1 \leqq s < r$. Then there exists a $(\mathbb{C}^*)^{r-s}$-bundle $\sigma : X \longrightarrow \mathbb{T}$ on a complex torus $\mathbb{T}$ with $\dim_{\mathbb{C}} \mathbb{T} = s < r$. By Proposition 7.2 we can take a theta bundle $L_\theta \longrightarrow \mathbb{T}$ such that $L \cong \sigma^* L_\theta$. Let $\overline{\sigma} : \overline{X} \longrightarrow \mathbb{T}$ be the associated $(\mathbb{P}^1)^{r-s}$-bundle. We denote by $\overline{g}$ and $\overline{L}$ the extensions of g and L, respectively. Then there exist $\overline{\varphi}, \overline{\psi} \in H^0(\overline{X}, \mathcal{O}(\overline{L}))$ such that $\overline{g} = \overline{\psi}/\overline{\varphi}$. Since

$$H^0(\overline{X}, \mathcal{O}(\overline{L})) = \overline{\sigma}^* H^0(\mathbb{T}, \mathcal{O}(L_\theta)),$$

$\overline{g}$ is constant on the fibers. This contradicts the assumption that g is non-degenerate.

□

Proof of Theorem 1.1. It follows from Proposition 9.1 that

$$G \cong \mathbb{C}^p \times (\mathbb{C}^*)^q \times A,$$

where $A = \mathbb{C}^r/\Gamma^*$ is an abelian variety. By Theorem 8.1, any $f \in K$ meromorphically extends to $\overline{G} \cong (\mathbb{P}^1)^{p+q} \times A$. Then we obtain the conclusion. □

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ON HYPERBOLIC POLYNOMIAL DIFFEOMORPHISMS OF $\mathbb{C}^2$

YUTAKA ISHII

Contents

1. Introduction and Main Result

Hyperbolic polynomial diffeomorphisms of $\mathbb{C}^2$ have been extensively studied, e.g., from the viewpoint of Axiom A theory by [BS1] and the combinatorial point of view *à la* Douady–Hubbard by [BS7]. Here, a polynomial diffeomorphism f of $\mathbb{C}^2$ is said to be *hyperbolic* if its Julia set J is a hyperbolic set for f (see Sections 2 and 3). In [HO2, FS] it has been shown that a sufficiently small perturbation of any expanding polynomial $p(x)$ of one variable in the *generalized Hénon family*:

$$f_{p,b} : (x, y) \longmapsto (p(x) - by, x)$$

is hyperbolic. However, this is so far the only known example of a polynomial diffeomorphism of $\mathbb{C}^2$ which is rigorously shown to be hyperbolic. Moreover, the dynamics of such $f_{p,b}$ can be modeled by the projective limit of the one–dimensional map p on its Julia set. Thus, it is still not known if there exists a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$ which can not be obtained in this way.

The purpose of this talk is to present a framework for verifying hyperbolicity of holomorphic dynamical systems in $\mathbb{C}^2$. This framework in particular enables us to construct the first example of a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$ whose dynamics is essentially two–dimensional. Consider a cubic complex Hénon map:

$$f_{a,b} : (x, y) \longmapsto (-x^3 + a - by, x)$$

with $(a, b) = (-1.35, 0.2)$.

Main Theorem. *The cubic complex Hénon map above is hyperbolic but is not topologically conjugate on J to a small perturbation of any expanding polynomial in one variable.*

In the rest of this article, we will outline the proof of Main Theorem which relies on the combination of some analytic tools from complex analysis (see Section 4), a combinatorial idea called the *fusion* (see Section 5), and rigorous numerics technique by using interval arithmetic (see Section 6).

2. Hyperbolicity: A Background

Let $f : M \to M$ be a diffeomorphism from a Riemannian manifold M to itself. We say that a point $p \in M$ belongs to the *non–wandering set* Ω_f if for any neighborhood U of p there exists n so that $U \cap f^n(U) \neq \emptyset$. Apparently, periodic points of f belong to Ω_f.

Definition 2.1. *A compact invariant subset $\Lambda \subset M$ is said to be hyperbolic if there exist constants $C > 0$ and $0 < \lambda < 1$, and a (Df)–invariant splitting $T_pM = E^u_p \oplus E^s_p$ for $p \in \Omega_f$ so that $\|Df_p^{+/-n}(v)\| \leq C\lambda^n \|v\|$ for all $n > 0$, $v \in E_p^{s/u}$ and $p \in \Omega_f$.*

A fundamental concept in the dynamical system theory since 1960's is

Definition 2.2. *We say that a diffeomorphism $f : M \to M$ satisfies Axiom A if Ω_f is a compact hyperbolic set and periodic points are dense in Ω_f.*

It is often the case that the hyperbolicity of Ω_f implies the density of the periodic points in Ω_f (and this is true for polynomial diffeomorphisms of $\mathbb{C}^2$ which we will discuss in this article), thus the most crucial point is to prove the hyperbolicity of Ω_f.

Axiom A or hyperbolic diffeomorphisms (with some additional conditions) have several nice properties such as (i) they are structurally stable (see, e.g., [Sh]), (ii) their statistical properties are described by some invariant measures which are constructed through symbolic dynamics (see, e.g., [B]), and (iii) in the context of one–dimensional complex dynamics, one can analyze topology and combinatorics of the hyperbolic Julia sets (see, e.g. [D, T]).

Since the celebrated paper [Sm], it was widely believed that the maps satisfying Axiom A are dense in the space of all systems. Although this belief was turned out to be false in some cases, it has been always a driving force for research of dynamical systems.

For polynomial diffeomorphisms of $\mathbb{C}^2$, the only known examples of hyperbolic maps are small perturbation of expanding (=hyperbolic) polynomials in one variable [HO2, FS]. Moreover, the dynamics of such map can be modeled by the projective limit of the one–dimensional map on its Julia set, so it does not present essentially two–dimensional dynamics. In view of the belief above, it is thus natural to ask the following

Question. *Does there exist a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$ which can not be obtained as a small perturbation of any expanding polynomial in one variable?*

The answer to this question was not known for the last 15 years, and our Main Theorem gives the affirmative answer to it.

3. Some Preliminary Results

Let f be a polynomial diffeomorphism of $\mathbb{C}^2$. It is known by a result of Friedland and Milnor [FM] that f is conjugate to either (i) an affine map, (ii) an elementary map, or (iii) the composition of finitely many generalized complex Hénon maps. Since the affine maps and the elementary maps do not present dynamically interesting behavior, we will hereafter focus only on a map in the class (iii), i.e. a map of the form $f = f_{p_1,b_1} \circ \cdots \circ f_{p_k,b_k}$ throughout this article. The product $d \equiv \deg p_1 \cdots \deg p_k$ is called the *(algebraic) degree* of f. Note also that we have $b \equiv \det(Df) = \det(Df_{p_1,b_1}) \cdots \det(Df_{p_k,b_k}) = b_1 \cdots b_k$.

For a polynomial diffeomorphism f, let us define

$$K^\pm = K^\pm_f \equiv \left\{(x, y) \in \mathbb{C}^2 : \{f^{\pm n}(x, y)\}_{n>0} \text{ is bounded in } \mathbb{C}^2\right\},$$

i.e. K^+ (resp. K^-) is the set of points whose forward (resp. backward) orbits are bounded in $\mathbb{C}^2$. We also put $K \equiv K^+ \cap K^-$ and $J^\pm \equiv \partial K^\pm$. The *Julia set* of f is defined as $J = J_f \equiv J^+ \cap J^-$ [HO1]. Obviously these sets are invariant by f.

Hereafter, we will often consider two different spaces $\mathcal{A}^* \subset \mathbb{C}^2$ where $* = \mathfrak{D}$ or $\mathfrak{R}$, and consider a polynomial diffeomorphism $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ (notice that this does not necessarily mean $f(\mathcal{A}^{\mathfrak{D}}) \subset \mathcal{A}^{\mathfrak{R}}$). Here, $\mathfrak{D}$ signifies the domain and $\mathfrak{R}$ signifies the range of f.

A subset of $T_p\mathbb{C}^2$ is called a *cone* if it can be expressed as the union of complex lines through the origin of $T_p\mathbb{C}^2$. Let $\{C^*_p\}_{p\in\mathcal{A}^*}$ ($* = \mathfrak{D}, \mathfrak{R}$) be two cone fields in $T_p\mathbb{C}^2$ over $\mathcal{A}^*$ and $\|\cdot\|_*$ be metrics in C^*_p.

Definition 3.1 (**Pair of Expanding Cone Fields).** *We say that* $(\{C^{\mathfrak{D}}_p\}_{p\in\mathcal{A}^{\mathfrak{D}}}, \|\cdot\|_{\mathfrak{D}})$ *and* $(\{C^{\mathfrak{R}}_p\}_{p\in\mathcal{A}^{\mathfrak{R}}}, \|\cdot\|_{\mathfrak{R}})$ *form a pair of weakly expanding cone fields for f (or, f weakly expands the pair of cone fields) if there exists a constant $\lambda \geq 1$ so that*

$$Df(C^{\mathfrak{D}}_p) \subset C^{\mathfrak{R}}_{f(p)} \quad \text{and} \quad \lambda\|v\|_{\mathfrak{D}} \leq \|Df(v)\|_{\mathfrak{R}}$$

hold for all $p \in \mathcal{A}^{\mathfrak{D}} \cap f^{-1}(\mathcal{A}^{\mathfrak{R}})$ and all $v \in C^{\mathfrak{D}}_p$. When we can take $\lambda > 1$ uniformly with respect to p and v, we call the cone fields a pair of expanding cone fields for f (or, f expands the pair of cone fields). Similarly, a pair of (weakly) contracting cone fields for f is defined as a pair of (weakly) expanding cone fields for f^{-1}.

In particular, if $\mathcal{A} \equiv \mathcal{A}^{\mathfrak{D}} = \mathcal{A}^{\mathfrak{R}}$, $\|\cdot\| \equiv \|\cdot\|_{\mathfrak{D}} = \|\cdot\|_{\mathfrak{R}}$ and $C^u_p \equiv C^{\mathfrak{D}}_p = C^{\mathfrak{R}}_p$ for all $p \in \mathcal{A} \cap f^{-1}(\mathcal{A})$ and the above condition holds, then we say $(\{C^u_p\}_{p\in\mathcal{A}}, \|\cdot\|)$ forms an *(weakly) expanding cone field* (or, *f (weakly) expands the cone field*). Similarly, the notion of *(weakly) contracting cone field* (or, *f (weakly) contracts the cone field*) can be defined.

The next claim tells that, to prove hyperbolicity, it is sufficient to construct some expanding/contracting cone fields.

Lemma 3.2. *If $f : \mathcal{A} \to \mathcal{A}$ has both nonempty expanding/contracting cone fields $\{C^{u/s}_p\}_{p\in\mathcal{A}}$, then f is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$.*

On the hyperbolicity of the polynomial diffeomorphisms of $\mathbb{C}^2$, the following fact is known (see [BS1], Lemma 5.5 and Theorem 5.6).

Lemma 3.3. *f is hyperbolic on its Julia set J iff so is on its nonwandering set Ω_f.*

Thanks to this fact, one may simply say that a polynomial diffeomorphism f is *hyperbolic* when one of the two sets in the above lemma is a hyperbolic set. In what follows, we thus prove hyperbolicity of some f on its Julia set J.

4. A Criterion for Hyperbolicity

Let A_x and A_y be bounded regions in $\mathbb{C}$. Let us put $\mathcal{A} = A_x \times A_y$, and let $\pi_x : \mathcal{A} \to A_x$ and $\pi_y : \mathcal{A} \to A_y$ be two projections. Below, we will define two types of cone fields. The first one (to which we do not assign a metric) looks more general than the other.

Definition 4.1 (**Horizontal/Vertical Cone Fields).** *A cone field on $\mathcal{A}$ is called a horizontal cone field if each cone contains the horizontal direction but not the vertical direction. A vertical cone field can be defined similarly.*

Next, a very specific cone field is defined in terms of Poincaré metrics. Let $|\cdot|_D$ be the Poincaré metric in a bounded domain $D \subset \mathbb{C}$. Define a cone field in terms of the "slope" with respect to the Poincaré metrics in A_x and A_y as follows:

$$C^h_p \equiv \big\{v = (v_x, v_y) \in T_p\mathcal{A} : |v_x|_{A_x} \geq |v_y|_{A_y}\big\}.$$

A metric in this cone is given by $\|v\|_h \equiv |D\pi_x(v)|_{A_x}$.

Definition 4.2 (Poincaré Cone Fields). *We call* $(\{C^h_p\}_{p\in\mathcal{A}}, \|\cdot\|_h)$ *the horizontal Poincaré cone field. The vertical Poincaré cone field* $(\{C^v_p\}_{p\in\mathcal{A}}, \|\cdot\|_v)$ *can be defined similarly.*

A product set $\mathcal{A} = A_x \times A_y$ equipped with the horizontal/vertical Poincaré cone fields is called a *Poincaré box.* A Poincaré box will be a building block for verifying hyperbolicity of polynomial diffeomorphisms throughout this work.

Let $\mathcal{A}^* = A^*_x \times A^*_y$ $(* = \mathfrak{D}, \mathfrak{R})$ be two Poincaré boxes, $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ be a holomorphic injection and $\iota : \mathcal{A}^{\mathfrak{D}} \cap f^{-1}(\mathcal{A}^{\mathfrak{R}}) \to \mathcal{A}^{\mathfrak{D}}$ be the inclusion map. The following two conditions will be used to state our criterion for hyperbolicity.

Definition 4.3 (Crossed Mapping Condition). *We say that* $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ *satisfies the crossed mapping condition (CMC) of degree* d *if*

$$\rho_f \equiv (\pi^{\mathfrak{R}}_x \circ f, \pi^{\mathfrak{D}}_y \circ \iota) : \iota^{-1}(\mathcal{A}^{\mathfrak{D}}) \cap f^{-1}(\mathcal{A}^{\mathfrak{R}}) \longrightarrow A^{\mathfrak{R}}_x \times A^{\mathfrak{D}}_y$$

is proprer of degree d.

Let $\mathcal{F}^{\mathfrak{D}}_h = \{A^{\mathfrak{D}}_x(y)\}_{y\in A^{\mathfrak{D}}_y}$ be the horizontal foliation of $\mathcal{A}^{\mathfrak{D}}$ with leaves $A^{\mathfrak{D}}_x(y) = A^{\mathfrak{D}}_x \times \{y\}$ and $\mathcal{F}^{\mathfrak{R}}_v = \{A^{\mathfrak{R}}_y(x)\}_{x\in A^{\mathfrak{R}}_x}$ be the vertical foliation of $\mathcal{A}^{\mathfrak{R}}$ with leaves $A^{\mathfrak{R}}_y(x) = \{x\} \times A^{\mathfrak{R}}_y$.

Definition 4.4 (No–Tangency Condition). *We say that* $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ *satisfies the no–tangency condition (NTC) if* $f(\mathcal{F}^{\mathfrak{D}}_h)$ *and* $\mathcal{F}^{\mathfrak{R}}_v$ *have no tangencies. Similarly we say that* $f^{-1} : \mathcal{A}^{\mathfrak{R}} \to \mathcal{A}^{\mathfrak{D}}$ *satisfies the (NTC) if* $\mathcal{F}^{\mathfrak{D}}_h$ *and* $f^{-1}(\mathcal{F}^{\mathfrak{R}}_v)$ *have no tangencies.*

Notice that we do not exchange h and v of the foliations in the definition of the non–tangency condition for f^{-1}. Hence, f satisfies the (NTC) iff so does f^{-1}.

Example. Given a polynomial diffeomorphism f, choose a sufficiently large $R > 0$. Put $\mathcal{D}_R = \Delta_x(0; R) \times \Delta_y(0; R)$, $V^+ = V^+_R \equiv \{(x, y) \in \mathbb{C}^2 : |x| \geq R,\ |x| \geq |y|\}$ and $V^- = V^-_R \equiv \{(x, y) \in \mathbb{C}^2 : |y| \geq R,\ |y| \geq |x|\}$. Then, f induces a homomorphism:

$$f_* : H_2(\mathcal{D}_R \cup V^+, V^+) \longrightarrow H_2(\mathcal{D}_R \cup V^+, V^+).$$

Since $H_2(\mathcal{D}_R \cup V^+, V^+) = \mathbb{Z}$, one can define the *(topological) degree* of f to be $f_*(1)$. It is easy to see that the topological degree of f is equal to the algebraic degree d of f.

Consider $f : \mathcal{D}_R \to \mathcal{D}_R$ and $\rho_f : \mathcal{D}_R \cap f^{-1}(\mathcal{D}_R) \to \mathcal{D}_R$. Given $(x, y) \in \mathcal{D}_R$, $f(\rho^{-1}(x, y))$ is equal to $f(D_x(y)) \cap D_y(x)$, where we write $D_x(y) = \Delta_x(0; R) \times \{y\}$ and $D_y(x) = \{x\} \times \Delta_y(0; R)$. Since $f(V^+) \subset V^+$ and $f^{-1}(V^-) \subset V^-$ hold, the number card $(f(D_x(y)) \cap D_y(x))$ can be counted by the number of times $\pi_x \circ f(\partial D_x(y))$ rounds around $\Delta_x(0; R)$ by the Argument Principle. This is equal to the degree of f, so it follows that card $(f(D_x(y)) \cap D_y(x)) = d$ counted with multiplicity for all $(x, y) \in \mathcal{D}_R$. Thus, $f : \mathcal{D}_R \to \mathcal{D}_R$ satisfies the (CMC). Notice that $f : \mathcal{D}_R \to \mathcal{D}_R$ satisfies the (NTC) iff card $(f(D_x(y)) \cap D_y(x)) = d$ counted without multiplicity for all $(x, y) \in \mathcal{D}_R$. (End of Example.)

Now, the central claim for verifying hyperbolicity is stated as

Theorem 4.5 (Hyperbolicity Criterion). *Assume that* $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ *satisfies the crossed mapping condition (CMC) of degree* d. *Then, the following are equivalent:*

(i) f *preserves some pair of horizontal cone fields,*
(ii) f^{-1} *preserves some pair of vertical cone fields,*
(iii) f *weakly expands the pair of the horizontal Poincaré cone fields,*
(iv) f^{-1} *weakly expands the pair of the vertical Poincaré cone fields,*
(v) f *satisfies the no–tangency condition (NTC),*
(vi) f^{-1} *satisfies the no–tangency condition (NTC).*

Moreover, when $\mathcal{A}^{\mathfrak{D}} = \mathcal{A}^{\mathfrak{R}} = \mathcal{B} = B_x \times B_y$*, where* B_x *and* B_y *are bounded open topological disks in* $\mathbb{C}$*, then any of the six conditions above is equivalent to the following:*

(vii) $\mathcal{B} \cap f^{-1}(\mathcal{B})$ *has* d *connected components.*

In fact, the (CMC) and the (NTC) can be rewritten as more checkable conditions so that we can verify the hyperbolicity of some specific polynomial diffeomorphisms of $\mathbb{C}^2$. For example, as a by–product of this criterion, we can give explicit bounds on parameter regions of hyperbolic maps in the (quadratic) *Hénon family*:

$$f_{c,b} : (x, y) \longmapsto (x^2 + c - by, x),$$

where $b \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ are complex parameters.

Corollary 4.6. *If* (c, b) *satisfies either*

(i) $|c| > 2(1 + |b|)^2$ *(a hyperbolic horseshoe case),*

(ii) $c = 0$ *and* $|b| < (\sqrt{2} - 1)/2$ *(an attractive fixed point case) or*

(iii) $c = -1$ *and* $|b| < 0.02$ *(an attractive cycle of period two case),*

then the complex Hénon map $f_{c,b}$ *is hyperbolic on* J*.*

Notice that [HO2, FS] did not give any specific bounds on the possible perturbation width which keeps the hyperbolicity of $f_{c,b}$.

We can extend the hyperbolicity criterion above to the case where some Poincaré boxes have overlaps in the following way. Let $\{\mathcal{A}_i\}_{i=0}^N$ be a family of Poincaré boxes in $\mathbb{C}^2$ each of which is biholomorphic to a product set of the form $A_x^i \times A_y^i$ with its horizontal Poincaré cone field $\{C_p^{\mathcal{A}_i}\}_{p \in \mathcal{A}_i}$ in $\mathcal{A}_i$. Let us put $\mathcal{A} = \bigcup_{i=0}^N \mathcal{A}_i$ and $\Omega_{\mathcal{A}} \equiv \bigcap_{n \in \mathbb{Z}} f^n(\mathcal{A})$.

Definition 4.7 (Gluing of Poincaré Boxes). *For each* $p \in \mathcal{A}$*, let us write* $I(p) \equiv \{i : p \in \mathcal{A}_i\}$*. We shall define a cone field* $\{C_p^{\cap}\}_{p \in \mathcal{A}}$ *by*

$$C_p^{\cap} \equiv \bigcap_{i \in I(p)} C_p^{\mathcal{A}_i}$$

for $p \in \mathcal{A}$ *and a metric* $\|\cdot\|_\cap$ *in it by*

$$\|v\|_\cap \equiv \min\{\|v\|_{\mathcal{A}_i} : i \in I(p)\}$$

for $v \in C_p^{\cap}$*.*

Remark 4.8. *A priori we do not know if* $C_p^{\cap}$ *is non–empty for* p *with* $\mathrm{card}(I(p)) \geq 2$*.*

Given a subset $I \subset \{0, 1, \cdots, N\}$, let us write

$$\langle I \rangle \equiv \left(\bigcap_{i \in I} \mathcal{A}_i \right) \setminus \left(\bigcup_{j \in I^c} \mathcal{A}_j \right) = \{p \in \mathcal{A} : I(p) = I\}.$$

In what follows, we only consider the case $\mathrm{card}(I(p)) \leq 2$. One then sees, for example, $\langle i \rangle = \mathcal{A}_i \setminus \bigcup_{j \neq i} \mathcal{A}_j$ and $\langle i, j \rangle = \mathcal{A}_i \cap \mathcal{A}_j$.

A crucial step in the proof of Main Theorem is to combine the hyperbolicity criterion with the following:

Lemma 4.9 (Gluing Lemma). *Let* $p \in \mathcal{A} \cap f^{-1}(\mathcal{A})$*. If for any* $i \in I(f(p))$ *there exists* $j = j(i) \in I(p)$ *such that* $f : \mathcal{A}_j \to \mathcal{A}_i$ *satisfies the (CMC) and the (NTC), then* $Df(C_p^{\cap}) \subset C_{f(p)}^{\cap}$ *and* $\|Df(v)\|_\cap \geq \lambda \|v\|_\cap$ *for some* $\lambda \geq 1$*.*

5. Fusion of Two Polynomials

In this section we present a model study of fusion.

Think of two cubics $p_1(x)$ and $p_2(x)$ so that $p_2(x) = p_1(x) + \delta$ for some $\delta > 0$, both have negative leading coefficients and have two real critical points $c_1 > c_2$. Let $\Delta_x(0; R) = \{|x| < R\}$ and $\Delta_y(0; R) = \{|y| < R\}$. Take $R > 0$ sufficiently large so that $\partial\Delta_x(0; R) \times \Delta_y(0; R) \subset \mathrm{int}V^+$ and $\Delta_x(0; R) \times \partial\Delta_y(0; R) \subset \mathrm{int}V^-$ hold. Assume that p_i satisfies $p_1(c_2) < -R$, $p_2(c_2) < -R$ and $p_2(c_1) > R$ so that the orbits $|p_1^k(c_2)|$, $|p_2^k(c_1)|$ and $|p_2^k(c_2)|$ go to infinity as $k \to \infty$. Assume also that c_1 is a super–attractive fixed point for p_1. Define $B_{y,1}$ to be the connected component of $p_1^{-1}(\Delta_y(0; R))$ containing c_1 and $B_{y,2}$ to be the other component. Let H be a closed neighborhood of c_1 which is contained in the attractive basin of c_1. Put $\mathcal{A}_1 = (\Delta_x(0; R) \setminus H) \times B_{y,1}$ and $\mathcal{A}_2 = \Delta_x(0; R) \times B_{y,2}$. Now, we assume that there exists a generalized Hénon map f with

$$f|_{\mathcal{A}_i}(x, y) \approx (p_i(x), x) \tag{1}$$

for $i = 1, 2$.

(a) Consider $f : \mathcal{A}_1 \to \mathcal{A}_1 \cup \mathcal{A}_2$. Then, the (CMC) would hold since

$$\overline{f(H \times B_{y,1})} \approx \overline{p_1(H) \times H} \subset \mathrm{int}(H \times B_{y,1})$$

by the approximation (1) above and $R > 0$ is large. Also the (NTC) would hold since

$$\overline{f(\{c_1\} \times B_{y,1})} \approx \overline{\{p_1(c_1)\} \times \{c_1\}} \subset \mathrm{int}(H \times B_{y,1})$$

and

$$\overline{f(\{c_2\} \times B_{y,1})} \approx \overline{\{p_1(c_2)\} \times \{c_2\}} \subset \mathrm{int}V^+$$

again by (1). Thus we may conclude that $f : \mathcal{A}_1 \to \mathcal{A}_1 \cup \mathcal{A}_2$ satisfies the (NTC) and the (CMC) if the argument above is verified rigorously.

(b) Consider $f : \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$. Since $\mathcal{A}_2$ does not have any holes like H and $R > 0$ is large, the (CMC) would hold for f on $\mathcal{A}_2$. Also the (NTC) would hold since

$$\overline{f(\{c_1\} \times B_{y,2})} \approx \overline{\{p_2(c_1)\} \times \{c_1\}} \subset \mathrm{int}V^+$$

and

$$\overline{f(\{c_2\} \times B_{y,2})} \approx \overline{\{p_2(c_2)\} \times \{c_2\}} \subset \mathrm{int}V^+.$$

Thus we may conclude that $f : \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ satisfies the (NTC) and the (CMC) if the argument above is verified.

Combining these two considerations, we may expect that $f : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A}_1 \cup \mathcal{A}_2)$ by the hyperbolicity criterion. In this way, the generalized Hénon map $f_{p,b}$ restricted to $\mathcal{A}_1 \cup \mathcal{A}_2$ can be viewed as a *fusion* of two polynomials $p_1(x)$ and $p_2(x)$ in one variable. This method enables us to construct a topological model of the dynamics of a generalized Hénon map which have essentially two–dimensional dynamics.

6. Rigorous Numerics Technique

Computer do not understand all real numbers. Let $\mathbb{F}^*$ be the set of real numbers which can be represented by binary floating point numbers no longer than a certain length of digits and put $\mathbb{F} \equiv \mathbb{F}^* \cup \{\infty\}$. Denote by $\mathfrak{I}$ the set of all closed intervals with their end points in $\mathbb{F}$. Given $x \in \mathbb{R}$, let $\downarrow x \downarrow$ be the largest number in $\mathbb{F}$ which is less than x and let $\uparrow x \uparrow$ be the smallest number in $\mathbb{F}$ which is greater than x (when such number does not exist in $\mathbb{F}^*$, we assign ∞). It then follows that

$$x \in [\downarrow x \downarrow, \uparrow x \uparrow] \in \mathfrak{I}.$$

Interval arithmetic is a set of operations to output an interval in $\mathfrak{I}$ from given two intervals in $\mathfrak{I}$. It contains at least four basic operations: addition, differentiation, multiplication and division. Specifically, the addition of given two intervals $I_1 = [a, b]$, $I_2 = [c, d] \in \mathfrak{I}$ is defined by

$$I_1 + I_2 \equiv [\downarrow a + c\downarrow, \uparrow b + d\uparrow].$$

It then rigorously follows that $\{x+y : x \in I_1,\ y \in I_2\} \subset I_1 + I_2$. The other three operations can be defined similarly. A point $x \in \mathbb{R}$ is represented as the small interval $[\downarrow x\downarrow, \uparrow x\uparrow] \in \mathfrak{I}$. We also write $[a, b] < [c, d]$ when $b < c$.

In this article interval arithmetic will be employed to prove rigorously the (CMC) and the (NTC) for a given polynomial diffeomorphism of $\mathbb{C}^2$. It should be easy to imagine how this technique is used for checking the (CMC); we simply cover the vertical boundary of $\mathcal{A}^{\mathfrak{D}}$ by small real four–dimensional cubes (i.e. product sets of four small intervals) in $\mathbb{C}^2$ and see how they are mapped by $\pi_x \circ f$. Thus, below we explain how interval arithmetic will be applied to check the (NTC).

The problem of checking the (NTC) for a given generalized Hénon map $f_{p,b}$ reduces to finding the zeros of the derivative $\frac{d}{dx}(p(x) - by_0)$ for each fixed $y_0 \in A_y^{\mathfrak{R}}$. Essentially, this means that one has to find the zeros for a family of polynomials $q_y(x)$ in x parameterized by $y \in A \subset \mathbb{C}$. To do this, we first apply Newton's method to know approximate locations of its zeros. However, this method can not tell how many zeros we found in the region since it does not detect the multiplicity of zeros.

In order to count the multiplicity we employ the idea of winding number. That is, we first fix $y \in A$ and write a small circle in the x–plane centered at the approximate location of a zero (which we had already found by Newton's method). We map the circle by q_y and count how it rounds around the image of the approximate zero, which gives both the existence and the number of zeros inside the small circle. Our method to count the winding number on computer is the following. We may assume that the image of the approximate zero is the origin of the complex plane. Cover the small circle by many tiny squares and map them by q_y. We then verify the following two points (i) check that the images of the squares have certain distance from the origin which is much larger than the size of the image squares, and (ii) count the number of changes of the signs in the real and the imaginary parts of the sequence of image squares. These data tell how the image squares move one quadrant to another (note that the transition between the first and the third quadrants and between the second and the fourth are prohibited by (i)), and if the signs change properly, we are able to know the winding number of the image of the small circle.

An advantage of this method is that, since the winding number is integer–valued, its mathematical rigorous justification becomes easier (there is almost no room for round–off errors to be involved). Another advantage of this winding number method is its stability; once we check that the image of the circle by q_y rounds a point desired number of times for a fixed parameter y, then this is often true for any nearby parameters. So, by dividing the parameter set A into small squares and verifying the above points for each squares, we can rigorously trace the zeros of q_y for all $y \in A$.

7. Proof of Main Theorem

Let $f = f_{a,b}$ be the cubic complex Hénon map under consideration as in the Introduction. We first define four specific Poincaré boxes $\{\mathcal{A}_i\}_{i=0}^3$ with associated Poincaré cone fields $\{C_p^{\mathcal{A}_i}\}_{p\in\mathcal{A}_i}$ for $0 \le i \le 3$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are biholomorphic to a bidisk and $\pi_1(\mathcal{A}_i) = \mathbb{Z}$ for $i = 0, 3$. Let us write $\mathcal{A} = \bigcup_{i=0}^3 \mathcal{A}_i$ and $\Omega_{\mathcal{A}} = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$. As was seen in Definition 4.7, we can define the new cone field $(\{C_p^{\cap}\}_{p\in\mathcal{A}}, \|\cdot\|_{\cap})$ by using $\{C_p^{\mathcal{A}_i}\}_{p\in\mathcal{A}_i}$. See Figure 1 below, where we described how the boxes are sitting in $\mathbb{C}^2$, how they are overlapped and how they are mapped by f. The shaded regions are the holes of $\mathcal{A}_0$ and $\mathcal{A}_3$ and their images. Note that the two Poincaré boxes $\mathcal{A}_i$ $(i = 1, 2)$ are figured out in the same place in Figure 1.

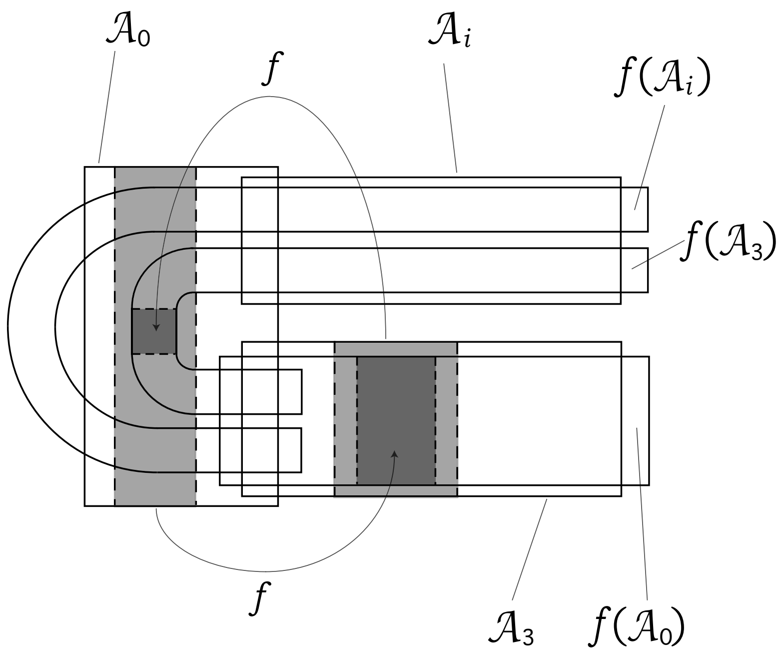

Figure 1. Four Poincaré boxes for the cubic Hénon map $f_{a,b}$.

With a help of rigorous numerics technique described in the previous section we are able to get the

Proposition 7.1. *We have six C++ programs which rigorously verify the following assertions using interval arithmetic:*

(i) $J_f \subset \mathcal{A}$.
(ii) *The cone* $C_p^{\cap}$ *is nonempty for all* $p \in \Omega_{\mathcal{A}}$.
(iii) *The following transitions:* $\mathcal{A}_0 \to \mathcal{A}_3$, $\mathcal{A}_1 \to \mathcal{A}_0$, $\mathcal{A}_1 \to \mathcal{A}_1$, $\mathcal{A}_1 \to \mathcal{A}_2$, $\mathcal{A}_2 \to \mathcal{A}_0$, $\mathcal{A}_2 \to \mathcal{A}_1$, $\mathcal{A}_2 \to \mathcal{A}_2$, $\mathcal{A}_3 \to \mathcal{A}_0$, $\mathcal{A}_3 \to \mathcal{A}_1$ *and* $\mathcal{A}_3 \to \mathcal{A}_2$ *by* f *satisfy the (CMC) and the (NTC).*
(iv) *There exists an open set* $\mathcal{B} \supset \mathcal{A}_0 \cap \mathcal{A}_3$ *which is biholomorphic to a bidisk so that* $f : \mathcal{B} \to \mathcal{B}$ *satisfies the (CMC) of degree one.*

Combining this proposition with the Hyperbolicity Criterion and the Gluing Lemma, we conclude that the cubic Hénon map $f_{a,b}$ under consideration is hyperbolic on its Julia set. A more argument shows that $f_{a,b}$ is not topologically conjugate to a small perturbation of any hyperbolic polynomial in one variable, which finishes the proof of Main Theorem. Q.E.D.

For more details of the proof, consult [I].

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Department of Mathematics, Kyushu University, Ropponmatsu, Fukuoka, Japan

Currents of higher bidegree and dynamics

Tien-Cuong Dinh and Nessim Sibony

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0 Introduction

The present note is a summary of a talk given by the first author at the Hayama Symposium 2004. Our aim is to explain some ideas on positive closed currents of higher bidegree which allow us to define the cup product of such currents and to study the dynamics of a large family of holomorphic maps : *the horizontal-like maps*. The reader will find all details in the article "*Dynamics of horizontal-like maps in higher dimension*" available at arXiv:math.DS/0409272. Here we avoid all technicalities.

It seems that the use of potentials is not adapted for the study of positive closed currents of higher bidegree in holomorphic dynamics. Our point of view is to consider the set of such currents as a space of infinite dimension which is a union of *structural discs* and admits a *rich family of plurisubharmonic functions*. We use the complex structure of the discs.

We also attract attention on the construction of the equilibrium measure and on the proof of mixing. We use the map $(z, w) \mapsto (f(z), f^{-1}(z))$ in order to reduce these non-linear problems to a linear situation. The method seems to be more efficient than the classical one.

We first introduce the horizontal-like maps and give a list of examples. The main theorem will be given in Section 2, the ideas for the proof are in Section 3 and the intersection of currents in Section 4. (the Green measure associated to a horizontal-like map is equal to the intersection of its Green currents).

1 Horizontal-like maps

Let $M \Subset \mathbb{C}^p$ and $N \Subset \mathbb{C}^{k-p}$ be two open convex sets. Define $D := M \times N$. An object (set, form, current) on D is called *vertical* (resp. *horizontal*) if it is supported in a domain $M' \times N$ (resp. $M \times N'$) with $M' \Subset M$ (resp. $N' \Subset N$).

Let f be a proper holomorphic map defined on a vertical open set D_1 of D with values in a horizontal open set D_2 of D. We assume that f sends the vertical part of ∂D_1 onto the vertical part of ∂D_2 and the horizontal part of ∂D_1 onto the horizontal part of ∂D_2. We describe now some particular horizontal-like maps.

Example 1. When $p = k$ we have $N = \{0\}$, $D \simeq M$, $D_1 \Subset D$ and $D_2 = D$. The map f is called *polynomial-like*. The case of dimension 1 was considered by Douady-Hubbard [14]. The restriction of every polynomial of degree ≥ 2 of one variable to a big disc is polynomial-like.

Polynomial-like maps in higher dimension were studied by Dinh-Sibony in [9, 10, 7]. If f is an endomorphism of $\mathbb{P}^{k-1}$ of degree ≥ 2, one can lift f to $\mathbb{C}^k$. The restriction of the lifted map to a big ball is polynomial-like. This case was considered by Fornæss-Sibony [17, 20], Ueda [23], Briend-Duval [4, 5], ...

Example 2. When $p = 1$, $k = 2$ and f is invertible, f is called *Hénon-like*. These maps were studied by Dujardin [15] and Dinh-Dujardin-Sibony [8]. The restriction of a Hénon map $(x, y) \mapsto (y + P(x), x)$ to a good bidisc is Hénon-like. Hénon maps were considered in Bedford-Lyubich-Smillie [1, 2] and Fornæss-Sibony [16].

Example 3. The restriction of some regular (generic) automorphisms of $\mathbb{C}^k$ to a good polydisc is horizontal-like. These maps were introduced and studied by Sibony [20]. See also [13].

Example 4. If f is a horizontal-like map, one can pertub f to obtain large families of examples.

Dynamical degree. There exists an integer d, called *the dynamical degree* of f, satisfying the following properties.

Let L be a vertical analytic set of the right dimension and of degree 1, i.e. the projection of L onto N is bijective. Then $f^{-1}(L)$ is a vertical set of degree d, i.e. the projection of $f^{-1}(L)$ onto N defines a (ramified) covering of degree d. If L is a vertical analytic set of degree m then $f^{-1}(L)$ is a vertical analytic set of degree dm.

More generally, if R is a vertical positive closed current of slice mass 1 then $f^*(R)$ is a vertical positive closed current of slice mass d. The slice mass of R is the mass of the slice R_b of R by the subspace $M \times \{b\}$, $b \in N$ generic. The slice R_b is a positive measure whose mass is independent of b.

If L' is a horizontal analytic set of the right dimension and of degree m then $f(L')$ is a horizontal analytic set of degree dm. If S is a horizontal positive closed current of slice mass 1 then $f_*(S)$ is a horizontal positive closed current of slice mass d.

For Hénon maps $(x, y) \mapsto (y + P(x), x)$, we have $d = \deg P$.

Julia sets. Define the *filled Julia sets* of f as

$$K^+ := \bigcap_{n\geq 0} f^{-n}(D),\ K^- := \bigcap_{n\geq 0} f^n(D) \text{ and } K := K^+ \cap K^-.$$

A point z belongs to $K^{\pm}$ (resp. K) iff $f^{\pm n}(z)$ is defined for every $n \geq 0$ (resp. for every n). We can study the dynamics of f on K and $K^{\pm}$.

Problem. Construct and study the Green currents $T^{\pm}$ and the equilibrium measure μ of f, i.e., in some sense, the dynamically interesting invariant currents and invariant measure associated to f. This is a central question in dynamics.

2 Main Theorem

Assume that f is invertible (some results below hold for non-invertible maps).

Main Theorem. Let R (resp. S) be a smooth vertical (resp. horizontal) positive closed current of slice mass 1 on D. Then

a) $d^{-n}(f^n)^*R$ converges to a vertical positive closed current T^+ of slice mass 1 which is independent of R and is supported in ∂K^+.

b) $d^{-n}(f^n)_*S$ converges to a horizontal positive closed current T^- of slice mass 1 which is independent of S and is supported in ∂K^-.

c) $d^{-2n}(f^n)^*R \wedge (f^n)_*S$ converges to a probability measure μ which is independent of R, S and is supported in ∂K.

d) $f^*T^+ = dT_+$, $f_*T^- = dT^-$ and $f^*\mu = f_*\mu = \mu$.

e) μ has maximal entropy $\log d$ and is mixing. We also have $\mu = T^+ \wedge T^-$.

Remarks. The assertions a) and b) are still valid for forms R, S with weaker regularity (continuous, bounded, with bounded potentials ...). The positivity of R and S is not necessary. Indeed, if R is smooth, we can write $R = R_1 - R_2$ with R_i positive closed. When R and S are continuous forms, the hypothesis that R, S are closed are not necessary, but the limit currents $T^\pm$ are always closed. We have some uniform convergence on R, S and analogous results for random iterations.

Maximal entropy. In some sense, the property that μ has maximal entropy means that the main part of the dynamics is on the support of μ. The proof of this property uses classical arguments of Gromov [19], Yomdin [24], Bedford-Smillie [2]. See also [9, 11, 12].

Mixing. The measure μ is mixing iff $\mu(f^{-n}B \cap A) \to \mu(A)\mu(B)$ as $n \to \infty$ for every measurable sets A, B. That is, the properties "$x \in A$" and "$f^n(x) \in B$" are asymptotically independent.

The mixing is a non-linear property. Our idea is to use the dynamics of the maps $F(z, w) := (f(z), f^{-1}(w))$ and $(u, v) \mapsto (F(u), F^{-1}(v))$ in order to reduce the problem to a linear situation.

Product of currents. The measure $T^+ \wedge T^-$ has to be defined.

3 Ideas of the proof

d) It is clear. For example, we have

$$f^*T^+ = \lim f^*\big(d^{-n}(f^n)^*R\big) = d \lim d^{-n-1}(f^{n+1})^*R = dT^+.$$

c) Let φ be a test function. Let Δ be the diagonal of $D \times D$. Since $\Delta \simeq D$, one can lift all integrals on the first factor of $D \times D$ to an integral on Δ.

Define $\widehat{\varphi}(z,w) := \varphi(z)$. When R and S are smooth we have

$$\langle R \wedge S, \varphi \rangle_D = \langle R_z \otimes S_w, \widehat{\varphi}[\Delta] \rangle_{D\times D}.$$

Consider the horizontal-like map on $D \times D$ of degree d^2 defined by

$$F(z,w) := (f(z), f^{-1}(w)).$$

We then have

$$\langle (f^n)^* R \wedge (f^n)_* S, \varphi \rangle = \langle (F^n)^*(R_z \otimes S_w), \widehat{\varphi}[\Delta] \rangle =: \langle (F^n)^* \widehat{R}, \widehat{\Phi} \rangle.$$

So we can deduce the convergence of $d^{-2n}(f^n)^* R \wedge (f^n)_* S$ (a non-linear problem) from the convergence of $d^{-2n}(F^n)^* \widehat{R}$ (a linear problem). We can apply the assertion a) for F (even if the "test form" $\widehat{\Phi}$ is singular).

a) Let

$$\mathcal{C} := \{\text{vertical positive closed currents of slice mass } 1\}.$$

We use this space of infinite dimension in order to prove the convergence in a). The operator $\frac{1}{d} f^*$ acts on $\mathcal{C}$ and defines a dynamical system on $\mathcal{C}$; the Green current T^+ will be a fixed point. We consider $\mathcal{C}$ as a set with a privileged structure. We construct *structural discs* in $\mathcal{C}$ and *plurisubharmonic functions* on $\mathcal{C}$. These points of view correspond to the ones of Kobayashi (for structural discs) and of Oka-Lelong (for the psh functions) in the case of manifolds of finite dimension.

Structural discs. Let $\Delta \subset \mathbb{C}$ be a holomorphic disc. We define "structural maps" τ from Δ into $\mathcal{C}$ as follows. Let $\mathcal{R}$ be a positive closed current of the right bidegree in $\Delta \times D$. Let T_θ be the slice of $\mathcal{R}$ by $\{\theta\} \times D$ for θ generic. We assume that these currents are vertical and of slice mass 1. Hence we have a map

$$\tau : \Delta \to \mathcal{C}, \qquad \theta \mapsto T_\theta$$

which is defined almost everywhere. This is *a structural map* in our sense. They play the role of *holomorphic maps.*

Plurisubharmonic functions. Let Φ be a horizontal real test form of the right bidegree. Define

$$\Lambda_\Phi : \mathcal{C} \to \mathbb{R}, \qquad T \mapsto \langle T, \Phi \rangle.$$

When $dd^c\Phi \geq 0$, Λ_Φ is *a psh function* in our sense : it is subharmonic on structural discs, i.e. subharmonic in the usual sense after composition by τ. The psh functions Λ_Φ separate points of $\mathcal{C}$. Indeed, every smooth Φ' can be written as a difference $\Phi_1 - \Phi_2$ of forms with $dd^c\Phi_i$ positive.

We summarise our construction by the following diagram :

$$\begin{tikzcd} \Delta \arrow[r, "\tau"] & \mathcal{C} \arrow[r, "\Lambda_\Phi"] \arrow[loop below, "\frac{1}{d}f^*"] & \mathbb{R} \end{tikzcd}$$

Of course, the function $\Lambda_\Phi \circ \tau$ is subharmonic on Δ.

Main Proposition. Let T be a current in $\mathcal{C}$. Let $0 < \theta_0 < 1$, close enough to 1 and let Δ be a small simply connected neighbourhood of $[0, \theta_0]$ in $\mathbb{C}$. Then there exists a structural disc $(T_\theta)_{\theta \in \Delta}$ of currents such that

a) $T_{\theta_0} = T$.

b) T_0 is independent of T.

c) T_θ is smooth for $\theta \neq \theta_0$. Moreover, the family (T_θ) is locally equicontinuous on $\Delta \setminus \{\theta_0\}$.

d) If T is a continuous form, (T_θ) is locally equicontinuous on Δ.

The proof of the proposition uses some "convolution on currents" and Skoda's extension theorem of currents [22].

Remarks. When T varies, one obtains a family of structural discs in $\mathcal{C}$ passing through the same point T_0. The equicontinuity in c) and d) is uniform on T. We can prove that $\mathcal{C}$ is "Brody hyperbolic". This property might be useful to study the dynamics on $\mathcal{C}$.

End of the proof of a). We want to prove the convergence of $d^{-n}(f^n)^*R$ for R smooth. Fix a horizontal real test form Φ with $dd^c\Phi \geq 0$. We need to prove that $\langle d^{-n}(f^n)^*R, \Phi \rangle$ converges.

Let (m_n) be an increasing sequence and $R_{m_n} \subset \mathcal{C}$ such that

$$\langle d^{-m_n}(f^{m_n})^* R_{m_n}, \Phi\rangle \to c_\Phi.$$

We choose m_n and R_{m_n} so that c_Φ is the maximal value that we can obtain in this way. Replacing R_{m_n} by $R_n := d^{-m_n+n}(f^{m_n-n})^* R_{m_n}$, we may assume that $m_n = n$.

We have

$$\langle d^{-n}(f^n)^* R_n, \Phi\rangle = \langle R_n, d^{-n}(f^n)_*\Phi\rangle.$$

We construct the structural disc $(R_{n,\theta})$ as in the Main Proposition with $R_{n,\theta_0} = R_n$ and $T := R_{n,0}$ independent of R_n. Define the subharmonic functions φ_n on Δ by

$$\varphi_n(\theta) := \langle R_{n,\theta}, d^{-n}(f^n)_*\Phi\rangle.$$

The sequence $\varphi_n(\theta_0)$ converge to the maximal value c_Φ. Hence, by the maximum principle and the properties of subharmonic functions,

$$\varphi_n \to c_\Phi \text{ in } L^1_{\text{loc}}.$$

The equicontinuity of $(R_{n,\theta})$ at 0 implies that

$$\varphi_n(0) \to c_\Phi.$$

Hence since $R_{n,0} = T$ is independent of n we get

$$\langle T, d^{-n}(f^n)_*\Phi\rangle \to c_\Phi.$$

Now consider R smooth as in the Main Theorem and construct (R_θ) as in the Main Proposition with $R_{\theta_0} = R$ and $R_0 = T$. Define subharmonic functions

$$\psi_n(\theta) := \langle R_\theta, d^{-n}(f^n)_*\Phi\rangle.$$

We have

$$\psi_n(0) = \langle T, d^{-n}(f^n)_*\Phi\rangle \to c_\Phi \text{ the maximal value.}$$

Then

$$\psi_n \to c_\Phi \text{ in } L^1_{\text{loc}}.$$

Since R is smooth, (R_θ) is equicontinuous at θ_0. We deduce that

$$\psi_n(\theta_0) \to c_\Phi.$$

Finally, since $R_{\theta_0} = R$, we get

$$\langle d^{-n}(f^n)^* R, \Phi\rangle = \langle R, d^{-n}(f^n)_*\Phi\rangle \to c_\Phi.$$

4 Intersection of currents

Let R be a vertical positive closed current and let S be a horizontal positive closed current on D of the right bidegrees. Let φ be a psh function on D. Define

$$\langle R \wedge S, \varphi \rangle := \limsup \langle R' \wedge S', \varphi \rangle$$

where R' is a smooth vertical positive closed current with $R' \to R$, $\mathrm{supp}(R') \to \mathrm{supp}(R)$ and S' is a smooth horizontal positive closed current with $S' \to S$, $\mathrm{supp}(S') \to \mathrm{supp}(S)$.

If φ' is a smooth function, we write $\varphi' = \varphi_1 - \varphi_2$ with φ_i psh and define

$$\langle R \wedge S, \varphi' \rangle := \langle R \wedge S, \varphi_1 \rangle - \langle R \wedge S, \varphi_2 \rangle.$$

The following proposition shows that $R \wedge S$ is well defined.

Proposition. The previous 'limsup' depends linearly on R, S and φ.

The proof uses structural discs of currents.

Some properties. $R \wedge S$ is a positive measure. The definition does not depend on coordinates. We can construct explicitely R_n and S_n smooth such that $R_n \wedge S \to R \wedge S$, $R \wedge S_n \to R \wedge S$ and $R_n \wedge S_n \to R \wedge S$. When $\mathrm{supp}(R) \cap \mathrm{supp}(S)$ is small enough, all R_n and S_n satisfy this property. The intersection of smooth forms coincides with the usual product. When R or S has bidegree $(1, 1)$, the wedge product was defined for various classes of currents, see Bedford-Taylor [3], Sibony [21], Demailly [6] and Fornæss-Sibony [18].

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Tien-Cuong Dinh and Nessim Sibony,
Mathématique - Bât. 425, UMR 8628, Université Paris-Sud, 91405 Orsay, France.
E-mails: TienCuong.Dinh@math.u-psud.fr and Nessim.Sibony@math.u-psud.fr.

Effective Bounds on Holomorphic Maps to Complex Hyperbolic Manifolds

Jun-Muk Hwang

KIAS, Seoul, Korea

1 Introduction

Let X be a compact Riemann surface of genus $g \geq 0$ and let $M = \mathbf{B}^n/\Gamma$ be a compact complex hyperbolic manifold. Denote by $\mathrm{Hol}(X, M)$ the set of all non-constant holomorphic maps from X to M. In [Su], Sunada proved that $\mathrm{Hol}(X, M)$ is finite.

This result of Sunada was generalized to the non-compact situation by Noguchi [No] in the following form. Let $\bar{X}$ be a compact Riemann surface of genus $g \geq 0$ and let

$$X = \bar{X} - \{P_1, \ldots, P_m\}$$

be an m-punctured Riemann surface. Let $M = \mathbf{B}^n/\Gamma$ be a complex hyperbolic manifold of finite volume. Here the volume of M is measured with respect to the Bergmann metric of Ricci curvature -1. Then the set $\mathrm{Hol}(X, M)$ of non-constant holomorphic maps from X to M is finite.

The results of [No] and [Su] did not give any information about the cardinality of $\mathrm{Hol}(X, M)$. A priori, the number may depends on the moduli of X or properties of M other than the volume. In a joint-work with Wing-Keung To [HT], we have given an effective version of this result of Noguchi as follows.

Theorem 1. *Let X be an m-punctured Riemann surface of genus g. Set $\omega := 2g - 2 + m$. Let M be a complex hyperbolic manifold of dimension n with finite volume v. Then the number of non-constant holomorphic maps from X to M is bounded by*

$$q^{\omega} \sum_{d=1}^{A\omega} \sum_{j=1}^{2^{9\omega} d} \begin{pmatrix} (2n+3)\max(j, B) \\ 2n+3 \end{pmatrix}^{(2n+4)(j^2+j)}$$

where

$$q := \left(\frac{v(5n+4)^n}{2^{2n-2}\pi^n} \right)^9$$

$$A = 4 + \frac{1}{4}(n^2 + 3n + 4)(n+1)$$

$$B = 32\pi\omega v(n^2 + 3n + 4)^n n! \left(\frac{2^{9\omega}(n+1)}{8\pi} \right)^{n+1}.$$

Note that if $\omega \leq 0$, $\mathrm{Hol}(X, M)$ is empty. Thus we may assume that $\omega > 0$. In this case, X is a hyperbolic Riemann surface and it is well-known that ω is exactly the hyperbolic volume, up to a normalizing constant. Thus Theorem 1 has the following consequence.

Corollary *$\#\mathrm{Hol}(X, M)$ is bounded by a number depending only on the dimensions and the hyperbolic volumes of X and M.*

The strategy of proof of Theorem 1 is to bound the number of components of $\mathrm{Hol}(X, M)$ by using effective results in algebraic geometry, related to Fujita conjecture and effective Nullstellensatz. Then by Noguchi's result, this gives a bound on the cardinality of $\mathrm{Hol}(X, M)$. We will give a sketch of the proof in the following sections. Using the same method, it is not difficult to generalize Theorem 1 to higher dimension in the following form.

Theorem 2 *Let*

$$X = \mathbf{B}^{n_1}/\Gamma_1 \quad \textit{and} \quad M = \mathbf{B}^{n_2}/\Gamma_2$$

be two complex hyperbolic manifolds of finite volume. Then the number of non-constant holomorphic maps from X to M is bounded by a number depending only on the dimensions and the hyperbolic volumes of X and M.

2 Proof of Compact Case

To illustrate the method of proof of Theorem 1, let us first discuss the case when X and M are compact. In this case, the proof is rather straight forward and consists of the following three ingredients.

- Bound on degrees $(K_M)^n$ and $f(X) \cdot K_M$.
- Effective projective embedding of M.
- Bound on Chow variety of $X \times M$.

Let us examine them one by one.

To start with, we have the obvious relation between the volume and the canonical degree of M:

$$v = \frac{(4\pi)^n}{n!(n+1)^n}(K_M)^n.$$

Also for $f : X \to M$, Schwarz lemma gives

$$f(X) \cdot K_M \leq \frac{n+1}{2}(2g-2).$$

These two relations can be viewed as bounds on $(K_M)^n$ and $f(X) \cdot K_M$ in terms of volumes of X and M.

Next, recall the following result of [AS].

Theorem 3 *Let Y be an n-dimensional projective manifold with a positive line bundle L. Then sections of*

$$K_Y + \frac{n^2+3n+2}{2}L$$

separate any two points of Y.

Applying this to $Y = M$ and $L = K_M$, we see that sections of $\frac{n^2+3n+4}{2}K_M$ inject M into a projective space. The image is a projective subvariety of degree $\frac{(n^2+3n+4)^n}{2^n}(K_M)^n$.

Finally let $Y \subset \mathbf{P}_N$ be a projective subvariety of degree δ. Denote by $\mathrm{Chow}_d(Y)$ the Chow variety of curves of degree d on Y. In the 1990's there appeared several results based on effective

Nullstellensatz, which give an effective bound on the number of components of $\mathrm{Chow}_d(Y)$. For example, the result of [Gu] says that it is bounded by

$$\binom{(2\dim Y+2)\max\{d,\delta\}}{2\dim Y+1}^{(2\dim Y+2)(d^2+1)}.$$

Using these three ingredients, we can bound $\#\mathrm{Hol}(X,M)$ as follows. Note that a component of $\mathrm{Hol}(X,Y)$ corresponds to a component of the Chow variety of $X\times M$. In fact, if the graph of $f:X\to M$ is in a component of the Chow variety of $X\times M$, then a general member of the component is also the graph of a map $f:X\to M$. Thus to bound $\#\mathrm{Hol}(X,M)$, we may use a bound on the number of the components of $\mathrm{Chow}_d(X\times M)$ where d is a bound on the degree of the graphs of $\mathrm{Hol}(X,M)$ under a specific imbedding of $X\times M$. We can effectively inject $X\times M$ into a projective space with a bound in terms of g and v. The graphs of $\mathrm{Hol}(X,M)$ are curves of degree bounded by the bound on the degree of the image. Thus the number of components of $\mathrm{Hol}(X,M)$ can be bounded by the bound on Chow variety.

3 Proof of non-compact case

To generalize the proof of the compact case to the non-compact case, we use natural compactifications of the complex hyperbolic manifold M of finite volume. First, by [BB] and [SY], there exists a compactification M, to be denoted by $\bar{M}$, such that

$$\bar{M}\setminus M=\text{ finitely many cusps.}$$

Moreover, there exists a positive line bundle $K_{\bar{M}}$ on $\bar{M}$. By [Mu] and [To], there exists a resolution of singularity $\varphi:\hat{M}\to\bar{M}$ such that

$$\hat{M}\setminus M=\text{ smooth hypersurfaces}$$

and the hyperbolic metric on M has good behavior near the boundary of M in $\hat{M}$.

We can extend $f:X\to M$ to $\bar{f}:\bar{X}\to\bar{M}$. We may assume $\bar{f}$ sends punctures to cusps. The existence of the resolution $\varphi:\hat{M}\to M$ implies that the volume formula

$$v=\frac{(4\pi)^n}{n!(n+1)^n}(K_{\bar{M}})^n$$

holds as in the compact case and for $\bar{f}:\bar{X}\to\bar{M}$,

$$\bar{f}(\bar{X})\cdot K_{\bar{M}}\le\frac{n+1}{2}\omega$$

holds which generalizes the inequality on $f(X)\cdot K_M$ in the compact case.

Next, Theorem 3 was generalized in [Ko] to non-compact case as follows.

Theorem 4 *Let Y be an n-dimensional projective manifold, $Y^o\subset Y$ be an open subset and L be a nef and big line bundle which is positive on curves intersecting Y^o. Then sections of*

$$K_Y+\frac{n^2+3n+2}{2}L$$

separate any two points of Y^o.

Applying this to $Y = \hat{M}, Y^o = M$ and $L = \varphi^* K_{\bar{M}}$ under the resolution $\varphi : \hat{M} \to \bar{M}$, we see that sections of

$$\frac{n^2 + 3n + 4}{2} K_{\bar{M}}$$

inject M into a projective space. However, this is not enough for our purpose. We need to inject $\bar{M}$. For this, we need to separate cusps. An essential new ingredient is the following simple observation.

Key Lemma *Sections of $2K_{\bar{M}}$ separate any two cusp points in $\bar{M} \setminus M$.*

The idea of the proof of Key Lemma can be explained as follows. To separate points, we need to construct a multiplier ideal supported at isolated points. Since cusps are isolated singular points which are sufficiently bad singular points, we automatically get a multiplier ideal supported at cusps.

As a consequence of Key Lemma and Theorem 4, we see that sections of $\frac{n^2+3n+4}{2} K_{\bar{M}}$ inject $\bar{M}$ into a projective space. The image is a projective subvariety of degree $\frac{(n^2+3n+4)^n}{2^n}(K_{\bar{M}})^n$.

Now there is one difference from the compact case when we apply the bound on Chow varieties: a component of $\mathrm{Hol}(X, M)$ is not necessarily a component of $\mathrm{Hol}(\bar{X}, \bar{M})$. Thus it may not correspond to a component of the Chow variety of $\bar{X} \times \bar{M}$. However, a component of $\mathrm{Hol}(X, M)$ is a component of the space

$$\mathrm{Hol}((\bar{X}, \text{punctures}), (\bar{M}, \text{cusps}))$$

of holomorphic maps sending punctures of X to cusps of $\bar{M}$. Thus we need to consider the Chow variety of curves passing through these marked points on $\bar{X} \times \bar{M}$. An effective bound on such Chow variety can be derived from the bound on the usual Chow variety. Thus as in the compact case, we can get a bound on the number of components of $\mathrm{Hol}(X, M)$. However, because of the marking, the final bound depends on the number of cusps of $\bar{M}$, not just v.

To finish the proof we need to bound the number of cusps of $\bar{M}$ in terms of v. A result of this type was already proved by geometric topologists. For example, [Pa] showed

$$\frac{n(6\pi)^{2n^2-3n+1}}{2^{n-1}} v \geq \sharp(\text{cusps}).$$

Actually, using Key Lemma, [Hw] gave a better bound:

$$\frac{(5n+4)^n}{2^{2n-1}\pi^n} v \geq \sharp(\text{cusps}).$$

The proof uses Key Lemma and the formula of Mumford [Mu] for the dimensions of the space of cups forms.

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AUTOMORPHISMS OF HYPERKÄHLER MANIFOLDS

KEIJI OGUISO

1. In [Mc], McMullen has found a very interesting K3 automorphism, namely, an automorphism having a Siegel disk. One of remarkable properties of such a K3 automorphism is that it is of positive entropy but has no dense orbit in the Euclidean topology. My surprise is that the target K3 surface is necessarily of algebraic dimension 0 (though it admits an automorphism of infinite order) and, contrary to the projective case, the character of some automorphism on the space of the two forms is not a root of unity. One can also make a simply-connected 4-dimensional counterexample of Kodaira's problem about algebraic approximation of compact Kähler manifolds from his K3 surface [Og2], as a supplement of a work of Voisin [Vo].

2. At the conference, I have presented my work about the bimeromorphic automorphism group of a non-projective hyperkähler manifold (Theorems (0.1), (0.2) below) with outline of the proof. This work is motivated by a natural question: *how complicated is the full automorphism group of a McMullen's K3 surface?*

3. *A hyperkähler manifold* (*a HK mfd*, for short) is a compact complex simply-connected Kähler manifold M admitting an everywhere non-degenerate global holomorphic 2-form ω_M such that $H^0(M, \Omega^2_M) = \mathbf{C}\omega_M$. Contrary to the Calabi-Yau cases, projective HK mfds form a countable union of hypersurfaces in the Kuranishi space (of a given HK mfd). We denote by $\mathrm{Bir}\,(M)$ the group of bimeromorphic automorphisms of M. Recall that $H^2(M, \mathbf{Z})$ admits a natural $\mathbf{Z}$-valued symmetric bilinear form called *BF-form* or *Bogomolov-Beauville-Fujiki's form*, which sometimes allows one to study HK mfds as if they were K3 surfaces (= 2-dimensional HK mfds). (For BF-form, see [Be], and also an excellent survey [Hu, Section 1]).

4. The BF-form is of signature $(3, 0, b_2(M) - 3)$ (where the three-dimensional positive part corresponds to the Kähler class and the real and imaginary parts of the holomorphic 2-form) and the restriction of BF-from on the Néron-Severi group $NS(M)$ is of signature $(1, 0, \rho(M) - 1)$, $(0, 1, \rho(M) - 1)$ or $(0, 0, \rho(M))$. Here $\rho(M) := \mathrm{rank}\, NS(M)$ is the Picard number of M. We call $NS(M)$ *hyperbolic*, *parabolic*, *elliptic* according to these three cases. By a very deep result of Huybrechts [Hu], M is projective iff $NS(M)$ is hyperbolic (just for K3 surfaces). See also below (**9.**) for a relevant conjecture. We also note that the Hodge decomposition of $H^2(M, \mathbf{Z})$ and the BF-form are stable under $\mathrm{Bir}\,(M)$, and that the kernel of the natural representation $\mathrm{Bir}\,(M) \longrightarrow \mathrm{O}\,(H^2(M, \mathbf{Z}))$ is finite ([Hu, Sections 1 and 10]).

5. The next theorem has been proved in [Og2, 3] (Precisely speaking, these papers treated only biholomorphic automorphisms, but will be soon replaced by new versions including bimeromorphic automrphisms, for which the main part of the proof is essentially the same.):

Theorem 0.1. *Let M be a HK mfd. Let $\rho(M)$ be the Picard number of M. Then:*

(1) If M is not projective, then $\mathrm{Bir}\,(M)$ *is almost abelian of finite rank. More precisely, if the Néron-Severi group $NS(M)$ is elliptic (resp. parabolic), then*

$\mathrm{Bir}\,(M)$ is almost abelian of rank at most one (resp. at most $\rho(M)-1$). Moreover, in the first case, it is of rank one iff $\mathrm{Bir}\,(M)$ *has an element of positive entropy at H^2-level. In the second case,* $\mathrm{Bir}\,(M)$ *is always of null-entropy at H^2-level. In particular, the automorphism group of a McMullen's K3 surface (whose $NS(M)$ is necessarily elliptic) is isomorphic to* $\mathbf{Z}$ *up to finite group.*

(2) Let $G < \mathrm{Bir}\,(M)$. Assume that M is projective, and that every element of G is of null-entropy at H^2-level. Then G is almost abelian of rank at most $\rho(M)-2$.

(3) The estimates in (1) and (2) are optimal for K3 surfaces.

Here, a group G is called *almost abelian of finite rank r* if there are a normal subgroup $G^{(0)}$ of G of finite index, a finite group K and a non-negative integer r which fit in the exact sequence $1 \longrightarrow K \longrightarrow G^{(0)} \longrightarrow \mathbf{Z}^r \longrightarrow 0$. The rank r is well-defined.

We also say an element $g \in \mathrm{Bir}\,(M)$ is *of positive* (resp. *null*) *entropy at H^2-level* if the natural logarithm of the spectral radius of $g^*|H^2(M, \mathbf{C})$ is greater than (resp. equal to) 0.

6. Due to the works of Yomdin, Gromov and Friedland ([Yo], [Gr], [Fr]), the *topological entropy* $e(g)$ of a *biholomorphic* automorphism $g \in \mathrm{Aut}\,(M)$ of a compact Kähler manifold M can be defined by $e(g) := \log \delta(g)$. Here $\delta(g)$ is the spectral radius, i.e. the maximum of the absolute values of eigenvalues, of $g^*|H^*(M)$. One has $e(g) \geq 0$, and $e(g) = 0$ iff the eigenvalues of g^* are on the unit circle S^1. A subgroup G of $\mathrm{Aut}\,(M)$ is said to be of *null-entropy* (resp. of *positive-entropy*) if $e(g) = 0$ for $\forall g \in G$ (resp. $e(g) > 0$ for $\exists g \in G$). By [DS], $g \in \mathrm{Aut}\,(M)$ is of positive (resp. null) entropy iff so is at H^2-level.

7. For a K3 surface X, we have $\mathrm{Aut}\,(X) = \mathrm{Bir}\,(X)$ and the topological entropy of an automorphism g coincides with the entropy of H^2-level. By using Theorem (0.1), we have the following algebro-geometric characterization of the topological entropy of a K3 automorphism ([Og2, 3]):

Theorem 0.2. *Let X be a K3 surface, $G < \mathrm{Aut}\,(X)$, and $g \in \mathrm{Aut}\,(X)$. Then:*

(1) G is of null-entropy iff either G is finite or G makes an elliptic fibration on X, say $\varphi : X \longrightarrow \mathbf{P}^1$, stable.

(2) g is of positive entropy iff g has a Zariski dense orbit.

This result is also inspired by the following question of McMullen [Mc]:

Question 0.3. Does a K3 automorphism g have a dense orbit (in the Euclidean topology) when a K3 surface is projective and g is of positive entropy?

Note that McMullen's automorphism has a Zariski dense orbit but no dense orbit in the Euclidean topology.

8. During the conference, Professor Yutaka Ishii asked me an example of a rational surface with an automorphism of positive entropy. Here is one answer using Theorem (0.2):

Proposition 0.4. *Let E and F be elliptic curves which are not isogenous and $X := \mathrm{Km}\,(E \times F)$ be the associated Kummer surface. Let ι be the automorphism of X induced by the automorphism $(z, w) \mapsto (-z, w)$ of $E \times F$. Then $Y := X/\langle\iota\rangle$ is a smooth rational surface (of $\rho(Y) = 18$) and $\mathrm{Aut}\,(Y)$ is of positive entropy.*

Sketch of Proof. Our involution ι satisfies that $\iota^* = -1$ on $H^0(\Omega_X^2)$ and $X^\iota \neq \emptyset$. Thus, the surface Y is rational and non-singular. It is known that $\iota^*|NS(X) = id$

and ι is a center of Aut (X) (see for instance [Og1]). So, one has a natural embedding $\mathrm{Aut}(X)/\langle\iota\rangle \subset \mathrm{Aut}(Y)$ being compatible with natural identification $H^2(Y, \mathbf{Q}) = NS(Y)_{\mathbf{Q}} = NS(X)_{\mathbf{Q}}$. This together with the fact that $\mathrm{Aut}(X)$ is of positive entropy at $NS(X)$-level implies the result. Here the last fact follows from Theorem (0.2) and the fact that X is projective and admits at least two different elliptic fibrations of Mordell-Weil rank > 0 ([Og1]). Q.E.D.

9. Let us return back to Theorem (0.1). In the statement (1), the first (resp. second) case exactly corresponds to the case $a(M) = 0$, $a(M) = 1 (= \dim M/2)$ when M is a K3 surface. Here $a(*)$ is the algebraic dimension of $*$. From this and a work of Matsushita [Ma] about fiber space structures on a projective HK mfd (which says, among other things, that the dimension of the base space, when it is projective, is either 0, $\dim M/2$ or $\dim M$), it may be natural to pose the following:

Conjecture 0.5. *$a(M) \in \{0, \dim M/2, \dim M\}$ for a HK mfd M.*

At the moment (February 7 2005), what I can say towards this conjecture is that $a(M) \le \dim M/2$ if $a(M) \neq \dim M$.

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Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mail address: oguiso@ms.u-tokyo.ac.jp

SPECTRAL THEORY FOR THE LAPLACE OPERATOR ON HIRZEBRUCH SURFACES

CHRISTOPHE MOUROUGANE

1. Introduction

Let (X, ω) be a compact Kähler manifold and $(E, h) \to X$ a Hermitian holomorphic vector bundle on X. Denote by $\langle\!\langle \, , \, \rangle\!\rangle_{L^2}$ the hermitian inner product on the spaces $A^{p,q}(X, E)$ of forms on X with values in E built thanks to ω and h. The Dolbeault operator $\overline{\partial}_q \; : \; A^{0,q}(X, E) \to A^{0,q+1}(X, E)$ then has an adjoint operator $\overline{\partial}_q^{\star} \; : \; A^{0,q+1}(X, E) \to A^{0,q}(X, E)$. The associated Laplace operator is defined to be $\Delta''_q = \overline{\partial}_q^{\star}\overline{\partial}_q + \overline{\partial}_{q-1}\overline{\partial}_{q-1}^{\star}$. It has a discrete spectrum

$$0 = 0 = \cdots = 0 < \lambda_{1,q} \leq \lambda_{2,q} \leq \cdots \leq \lambda_{N,q} \leq \cdots$$

each eigenvalue is of finite multiplicity and they tend to $+\infty$.

Note that for positive λ the Dirac operator $\overline{\partial} + \overline{\partial}^{\star} \; : \; E^{+}_{\lambda}(\Delta'') \to E^{-}_{\lambda}(\Delta'')$ is an isomorphism but not an isometry : it is of norm λ. When trying to construct a direct image operation in the category of hermitian vector bundles it is therefore more natural to consider the following hermitian inner product

$$\langle\!\langle \, , \, \rangle\!\rangle_{Quillen} = \langle\!\langle \, , \, \rangle\!\rangle_{L^2} e^{-\tau}$$

where $e^{-\tau} = e^{-\tau(X,\omega,E,h)}$ has the properties of " $\prod_{\lambda \in spec \Delta''_q} \lambda$".

The correct definition of τ requires the so-called ζ-regularization process. Define the spectral function $\zeta_q(s) := \sum_{N=1}^{+\infty} \lambda_{N,q}^{-s}$. From the study of the short time asymptotic expansion of the heat kernel of the Laplace operator, one can infer that the spectral function extends to a meromorphic function on $\mathbb{C}$, holomorphic at 0. As expected from a formal differentiation of the spectral function, set

$$det' \Delta''_q := \exp\left(-\frac{d\zeta_q}{ds}_{\, s=0} \right).$$

Ray and Singer defined the analytic torsion $\tau(X, \omega, E, h)$ of (X, ω, E, h) by

$$exp(-\tau) := \prod_{q \geq 0} (det' \Delta''_q)^{(-1)^q q}.$$

There are three main strategies to compute τ.

- Compute the spectrum of Δ''_q. This can be done for example when the manifold X is homogeneous.

- Use Bismut formula for the curvature of the Quillen metric (Index formula for families). Given a hermitian holomorphic fiber bundle on the total space of fibration

$$\begin{array}{ccc} (E,h) & \to & (\mathcal{X}, g_\pi) \\ & & \downarrow^{\pi} \\ & & Z \end{array}$$

the first Chern form of (determinant of) the direct image of E endowed with the Quillen metric is

$$Ch(R\pi_\star E, Quillen)_{[2]} \quad = \quad \pi_\star(Ch(E,h)Td(T\pi, g_\pi))_{[2]}.$$

Quillen metric hence realizes the degree two Grothendieck-Riemann-Roch formula at the level of forms. This may be used to compute the variation of the analytic torsion when the data (X, ω, E, h) sweeps some moduli.
- Use the arithmetic Riemann Roch formula. For this, an integral model of the pair (X, E) is required.

2. THE ARITHMETIC RIEMANN ROCH FORMULA

Let χ be a scheme over $Spec\mathbb{Z}$. An arithmetic scheme $\overline{\mathcal{X}}$ is χ together with a Hermitian metric on $X := \chi_{\mathbb{C}}(\mathbb{C})$ playing the role of the integral structure in non-archimedean places. Accordingly, $\overline{\mathcal{E}}$ is a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_\chi$-modules together with a Hermitian metric h on $E := \mathcal{E}_{\mathbb{C}}(\mathbb{C})$.

There is an arithmetic intersection theory which

- which turns the height theory into a degree theory.
- which overlaps the algebraic theory and the Chern-Weil theory. Just note that there is a map

$$\begin{array}{rcccl} \omega & : & \widehat{CH}^p(\mathcal{X}) & \to & A^{p,p}(X) \\ & & [(\mathcal{Z}, g_Z)] & \mapsto & \delta_Z + dd^c g_Z \end{array}$$

$$\widehat{c_1}(\overline{\mathcal{L}}) = [(s=0), -\log\|s\|^2] \mapsto c_1(L,h)$$

which provides us with the Chern forms out of the arithmetic Chern classes. Here g_Z is a Green current for the analytic cycle $Z := \mathcal{Z}_{\mathbb{C}}(\mathbb{C})$ and δ_Z the current of integration along Z.
- which accounts for lifting to original integral and hermitian structures, and which therfore has to take care of secondary objects. Green currents, analytic torsion are secondary objects. We describe the third kind of secondary objects.

  For any short exact sequence $(\Sigma) = (0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0)$, and any choice of metrics $h = (h_E, h_S, h_Q)$, the Bott-Chern machinery is a functorial way to choose a form $\widetilde{c}(\Sigma, h) \in \widetilde{A^{d,d}}(X) := \frac{A^{d,d}(X,\mathbb{C})}{Imd' + Imd''}$ which fulfills

$$\widehat{c}(\overline{\mathcal{S}} \oplus \overline{\mathcal{Q}}, h_S \perp h_Q) - \widehat{c}(\overline{\mathcal{E}}, h_E) = a(\widetilde{c}(\Sigma, h))$$

  where $a \ : \ \widetilde{A^{d,d}}(X) \to \widehat{CH}^p(\mathcal{X})$ maps u to $(0, u)$.
- where there is a Riemann Roch theorem

$$\begin{array}{rcl} \widehat{Ch}(R\pi_\star\overline{\mathcal{E}}, Quillen)_{[2]} & = & \widehat{Ch}(R\pi_\star\overline{\mathcal{E}}, L^2)_{[2]} + a(\tau(X_z, g_\pi, L, h)) \\ & = & \pi_\star(\widehat{Ch}(\overline{\mathcal{E}}, h)\widehat{Td}^R(\overline{T\pi}, g_\pi))_{[2]} \end{array}$$

3. On Hirzebruch surfaces

The scheme of rank one quotients of the vector bundle $\mathcal{E}_n := \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n)$ on $\mathbb{P}^1_{\mathbb{Z}}$

$$\mathcal{S}_n = \mathbb{P}(\mathcal{E}_n) \xrightarrow{\pi} \mathbb{P}^1_{\mathbb{Z}} \xrightarrow{f} Spec\,\mathbb{Z}$$

is an integral model of the Hirzebruch surface S_n. From the choice of a Fubini-Study metric $\omega_{\mathbb{P}^1_{\mathbb{C}}}$ on $\mathbb{P}^1_{\mathbb{C}}$, we get a natural metric on E_n and on $\mathcal{O}_{E_n}(1)$ of semi-positive curvature α_n. We choose $\alpha_n + \pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}$ as a metric on S_n.

Main theorem The analytic torsion of the Hirzebruch surface S_n is

$$\tau(S_n, \mathcal{O}_{\mathcal{S}_n}) = \log Vol(S_n) + \frac{n\log(n+1)}{24} - \frac{n}{6} + 2\tau(\mathbb{P}^1).$$

4. The sketch of proof

We compute the arithmetic first Chern class of the determinant line bundle

$$detR^0F_\star\overline{\Omega^p_{\mathcal{S}_n}} \otimes (detR^1F_\star\overline{\Omega^p_{\mathcal{S}_n}})^{-1} \otimes detR^2F_\star\overline{\Omega^p_{\mathcal{S}_n}}$$

of the direct image $RF_\star\overline{\Omega^p_{\mathcal{S}_n}}$.

- with the definition

$$\begin{aligned} &\widehat{c_1}(RF_\star\overline{\Omega^p_{\mathcal{S}_n}}, Quillen) \\ = \;& a\Big[\sum_{q=0}^{2}(-1)^{q+1}\log\left(Vol_{L^2}\frac{H^q(S_n,\Omega^p_{\mathcal{S}_n})}{H^q(S_n,\Omega^p_{\mathcal{S}_n})_{\mathbb{Z}}}\right)^2 + \tau(S_n,\Omega^p_{\mathcal{S}_n})\Big]. \end{aligned}$$

- using the arithmetic Riemann Roch theorem

$$\begin{aligned} \widehat{c_1}(RF_\star\overline{\Omega^p_{\mathcal{S}_n}}, Quillen) &= \widehat{F}_\star\left(\widehat{Td}^R(\overline{T\mathcal{S}_n})\widehat{ch}(\overline{\Omega^p_{\mathcal{S}_n}})\right) \\ &= \widehat{F}_\star\left(\widehat{Td}(\overline{T\mathcal{S}_n})\widehat{ch}(\overline{\Omega^p_{\mathcal{S}_n}})\right) \\ &\quad -a\left(F_\star\left(Td(TS_n)R(TS_n)ch(\mathcal{O}_{S_n})\right)\right). \end{aligned}$$

4.1. **With the definition.** We use the Hodge metric on $H^q(S_n, \Omega^p_{\mathcal{S}_n})$. We therefore have to find representatives of cohomology classes harmonic with respect to $\alpha_n + \pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}$.

$$\begin{aligned} H^\bullet(S_n, \mathcal{O}_{S_n})_{\mathbb{Z}} &= H^0(S_n,\mathcal{O}_{S_n})_{\mathbb{Z}} = \mathbb{Z}\{1\} \\ H^1(S_n,\Omega^1_{S_n})_{\mathbb{Z}} &= \mathbb{Z}\{\pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}\} + \mathbb{Z}\{\alpha_n\} \\ H^2(S_n,\Omega^2_{S_n})_{\mathbb{Z}} &= \mathbb{Z}\{\pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}\} \cup \{\alpha_n\} = \mathbb{Z}\frac{\{\alpha_n\}^2}{n}. \end{aligned}$$

The only difficult case is for the class $\{\pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}\}$: the form

$$\omega_H := \pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}} - \frac{1}{n+2}dd^c\log\frac{\langle\Theta(E^\star,h)a^\star,a^\star\rangle_h}{\pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}\langle a^\star,a^\star\rangle_h}$$

is harmonic.

4.2. **Using the arithmetic Riemann Roch theorem.** The sequence given by the differential of the fibration map π

$$0 \to \overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}} \xrightarrow{\iota} \overline{T\mathcal{S}_n} \xrightarrow{d\pi} \overline{\pi^\star T\mathbb{P}^1_{\mathbb{Z}}} \to 0$$

and the relative Euler sequence

$$0 \to \overline{\mathcal{O}_{\mathbb{P}(\mathcal{E}_n)}} \to \pi^\star \overline{\mathcal{E}_n^\star} \otimes \overline{\mathcal{O}_{\mathcal{E}_n}(1)} \xrightarrow{q} \overline{T_{\mathcal{S}_n/\mathbb{P}^1_{\mathbb{Z}}}} \to 0.$$

enables to compute $\widehat{c}(\overline{T\mathcal{S}_n})$ after having computed the corresponding Bott-Chern classes.

We roughly explain how a careful look at the Chern-Weil theory enables to compute the Bott-Chern classes.

For $0 \to S \xrightarrow{\iota} E \xrightarrow{p} Q \to O$ endowed with metrics constructed from a metric h on E. Denote by ∇ the Chern connection of (E, h). Consider the family of connections on E

$$\nabla_u := \nabla + (u-1) P_Q \nabla P_S$$

where $P_S = \iota\iota^\star$ (resp. $P_Q = p^\star p$) denotes the orthogonal projection of E onto $\iota(S)$ (resp. $\iota(S)^\perp$). The idea is to transfer the space differentiations d' and d'' into $\frac{d}{du}$-differentiations in order to find $d'd''$-potentials. In a frame adapted to the $\mathcal{C}^\infty$ splitting $E \simeq S \oplus Q$ given by $\iota^\star \oplus p$, the curvature of the connection ∇_u is

$$\Theta(\nabla_u) = \left| \begin{array}{cc} (1-u)\Theta_S + u\iota^\star\Theta_E\iota & \iota^\star\Theta_E p^\star \\ \\ up\Theta_E\iota & (1-u)\Theta_Q + up\Theta_E p^\star \end{array} \right|$$

Consider the polarization Det_k of $\det_k$, that is the symmetric k-linear form on $M_r(\mathbb{C})$ whose restriction on the small diagonal is $\det_k$. Denote by $Det_k(A; B) = kDet_k(A, A, \cdots, A, B)$. The differential version of the $Gl(r, \mathbb{C})$-invariance of $\det_k$ shows that

$$-\frac{i}{2\pi} d'd'' Det_k(\Theta(\nabla_u); P_S) = u\frac{d}{du} det_k(\Theta(\nabla_u)).$$

where $Det_k(\Theta(\nabla_u); P_S) = \mathrm{coeff}_\lambda det_k(\Theta(\nabla_u) + \lambda P_S)$. This is the key fact for writing the wanted double transgression.

5. Varying the Kähler metric

We in fact can rule the computations with different metrics sweeping the whole Kähler cone of the Hirzebruch surfaces. For positive T, choose $\alpha_n + T^2\pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}$ as a metric on S_n. Then,

Theorem

$$\begin{aligned} &\tau(S_n, \alpha_n + T^2\pi^\star\omega_{\mathbb{P}^1_{\mathbb{C}}}) \\ = \ & \tau(\mathbb{P}^1_{\mathbb{C}}, T^2\omega_{\mathbb{P}^1_{\mathbb{C}}}) + \tau(\mathbb{P}^1_{\mathbb{C}}, \omega_{\mathbb{P}^1_{\mathbb{C}}}) - \frac{n}{6} + \log\frac{n+2T^2}{2T^2} + \frac{n}{24}\log\frac{n+T^2}{T^2} \\ & \frac{T^2-1}{3}\left(\frac{T^2}{n}\log\frac{n+T^2}{T^2} - 1 + \frac{13}{8}\log\frac{n+T^2}{T^2}\right). \end{aligned}$$

5.1. **The singular limit.** For $T = 0$, α_n comes from

$$S_n \to \mathbb{P}(H^0(S_n, \mathcal{O}_{E_n}(1)))$$

which contract the $-n$-curve. The image is a cone over a rational normal curve. The expansion of $\tau(S_n, \alpha_n + \epsilon^2 \pi^\star \omega_{\mathbb{P}^1_{\mathbb{C}}})$ is

$$\tau = -\frac{n+3}{12} \log \epsilon + O(1).$$

For the singular cone with its metric having conical singularities, Cheeger showed that the function $\zeta_q(s)$ has a simple pole at 0. Its residue can be computed from the coefficient of $(\log \epsilon)^2$ in the expansion of $\frac{d\zeta}{ds}_{s=0}(s, \epsilon)$. For there is no $(\log \epsilon)^2$-terms, the analytic torsion of the singular cone can be defined by the usual formula. This reproves a result of Yoshikawa.

5.2. **The adiabatic limit.** As T tends to $+\infty$ the fibration behaves like a trivial fibration with small fibers. Accordingly, the asymptotic for $T \to +\infty$ is

$$\tau(S_n, \alpha_n^T) \sim \tau(\mathbb{P}^1_{\mathbb{C}}, T\omega_{\mathbb{P}^1_{\mathbb{C}}}).$$

For $T \to +\infty$, the harmonic representative

$$\omega_H := \pi^\star \omega_{\mathbb{P}^1_{\mathbb{C}}} - \frac{1}{n + 2T^2} dd^c \log \frac{\langle \Theta(E_n^\star(T^2), h) a^\star, a^\star \rangle}{\pi^\star \omega_{\mathbb{P}^1_{\mathbb{C}}} \langle a^\star, a^\star \rangle}$$

tends to $\pi^\star \omega_{\mathbb{P}^1_{\mathbb{C}}}$. This confirms Mazzeo-Melrose theory. Set $x = T^{-1}$. The spaces of harmonic forms fits into the Hodge-Leray cohomology fiber bundle

$$\begin{array}{ccc} \mathcal{H}_{Hodge-Leray} & \leftarrow & \mathcal{H}arm(S_n, \alpha_n + x^{-2}\pi^\star \omega_{\mathbb{P}^1_{\mathbb{C}}}) \\ \downarrow & & \downarrow \\ [0, +\infty[& \ni & x \end{array}$$

and $\mathcal{H}_{Hodge-Leray}$ has a basis which extends to be smooth on $[0, +\infty[$.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU / PLATEAU 7D / 175, RUE DU CHEVALERET / 75013 PARIS
E-mail address: christophe.mourougane@math.jussieu.fr

Monge-Ampère Currents and Masses

Dan Popovici

Abstract. This is the text of a talk given at the Hayama Symposium on Several Complex Variables in December 2004. A compact complex manifold is Moishezon if and only if there exists a big line bundle over it. A new characterization of big line bundles, and implicitly of Moishezon manifolds, is conjectured in terms of the existence of a possibly singular Hermitian metric satisfying a relatively weak positivity condition, which would generalize previous characterizations of Siu and Demailly sprung from the solution to the Grauert-Riemenschneider conjecture. The key issue is to obtain a regularization of currents with an effective control of the Monge-Ampère masses of the regularizing currents. Our main goal here is to present a generalization of the Ohsawa-Takegoshi L^2 extension theorem for jets of holomorphic sections of a Hermitian line bundle, which has an interest of its own, as a first step in this direction. Multiplier ideal sheaves are also discussed. We have forgone the proofs of the results presented here, contenting ourselves with indicating the references where they can be found.

0.1 Introduction

The object of our study in this note is a class of compact complex manifolds which can be seen as a birational version of projective manifolds, the so-called Moishezon manifolds.

The Context

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$, and fix ω a Hermitian metric on X. For a holomorphic line bundle $L \to X$, we remind the following definitions.

Definition 0.1.1 *The line bundle L is said to be big if there exists a constant $C > 0$ such that $\dim H^0(X,\, L^k) \geq C\, k^n$, for $k >> 0$.*

This amounts to the Kodaira-Iitaka dimension of L being maximal (i. e. equal to the complex dimension n of X), which means that the space of global sections $H^0(X,\, L^k)$ defines a bimeromorphic embedding of X into some projective space $\mathbb{P}^{N_k}$, for $k >> 0$.

Definition 0.1.2 *The manifold X is Moishezon if there exist n algebraically independent meromorphic functions on X. This is equivalent to the algebraic dimension $a(X)$ of X being maximal (i. e. $a(X) = n$).*

The following statement parallels Kodaira's embedding theorem in this bimeromorphic setting.

Theorem 0.1.3 *(Moishezon) Let X be a compact complex manifold. Then, X is Moishezon if and only if there exists a big line bundle $L \to X$.*

Differential point of view

A singular Hermitian metric h on L is defined in any local trivialization $L_{|U} \overset{\theta}{\simeq} U \times \mathbb{C}$ as

$$L_x \ni \xi \mapsto ||\xi||_h := |\theta(\xi)|\, e^{-\varphi(x)},$$

for a local weight φ which is assumed to be only locally integrable.

The associated curvature current of $(L,\, h)$ is a $(1,\,1)$-current on X which is locally defined as :

$$i\theta_h(L) = i\partial\bar{\partial}\varphi \text{ on } U, \qquad \text{in the sense of currents}$$

for any trivializing open set U.

Algebraic vs. Analytic.

It is important to understand the algebraically defined concept of bigness in terms of the existence of possibly singular Hermitian metrics on the line bundle under consideration. The first result in this direction was obtained by Siu ([Siu84]) who gave a sufficient criterion for bigness.

Theorem 0.1.4 *(Siu, [Siu84]) L is big if there exists a C^∞ metric h on L such that the associated curvature form satisfies the following positivity conditions :*

$$i\theta_h(L) \geq 0 \qquad \textit{on } X,$$

and

$$\int_X (i\theta_h(L))^n > 0.$$

A necessary and sufficient criterion for bigness was later obtained by Ji and Shiffman ([JS93]), and also independently by Bonavero ([Bon98]). It can be seen as a complement to Siu's sufficient criterion insofar as it dispenses with the regularity condition on the metric but imposes in exchange a stronger positivity condition on the curvature.

Theorem 0.1.5 *(Ji-Shiffman [JS93], Bonavero [Bon98]) L is big if and only if there exists a (possibly singular) metric h on L such that*

$$i\theta_h(L) > 0 \qquad \text{on } X.$$

This strict positivity condition on the curvature current means that there exists a small $\varepsilon > 0$ such that $i\theta_h(L) \geq \varepsilon\omega$.

One of our main motivations has been to generalize the previous two criteria for bigness into a statement that would incorporate them both, with no regularity condition on the metric and only a comparatively weak positivity assumption on the curvature.

Conjecture 0.1.6 *L is big if and only if there exists a (possibly singular) metric h on L such that*

$$i\theta_h(L) \geq 0 \text{ on } X,$$

and

$$\int_X (i\theta_h(L)_{ac})^n > 0,$$

where $i\theta_h(L)_{ac}$ is the absolutely continuous part of the curvature current in the Lebesgue decomposition.

We could ask if one can do without the absolutely continuous part in the above conjectural statement. The nth power of the curvature current may not be well-defined in that case, but we could still integrate the nth power of the cohomology class of the curvature current, by integrating the nth power of an arbitrary smooth representative. The result is independent of the choice of a smooth representative, thanks to Stokes's theorem. However, the following example shows that the conjecture would fail in this case.

Example 0.1.7 *Let X be an arbitrary compact complex manifold, let $\tilde{X} \longrightarrow X \ni x$ be the blow-up of an arbitrary point $x \in X$, and let $E \geq 0$ be the exceptional divisor. It is well-known that the associated line bundle $L =: \mathcal{O}(E)$ on $\tilde{X}$ has a singular metric h such that its curvature current is $i\theta_h(L) = [E] \geq 0$. Then we have :*

$$\mathcal{O}(E)_{|E} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1),$$

and

$$\begin{aligned}
\int_{\tilde{X}} \{i\theta_h(L)\}^n &= \int_{\tilde{X}} \{[E]\}^n = \int_E \{[E]\}^{n-1} \\
&= \int_{\mathbb{P}^{n-1}} c_1(\mathcal{O}(-1))^{n-1} \\
&= (-1)^{n-1} > 0,
\end{aligned}$$

if n is odd.

We shall now list the main ingredients used in our approach of the above conjecture. We shall but briefly discuss the first two of them in the sequel to dwell on the third one, a new extension theorem for jets of sections of Hermitian line bundles.

(a) Demailly's regularization of currents.

(b) Multiplier ideal sheaves.

(c) Generalization of the Ohsawa-Takegoshi-Manivel L^2 extension Theorem.

0.2 Demailly's regularization of currents.

Let $T \geq 0$, be a d-closed (1, 1)-current on X. Locally we have $T = dd^c\varphi$, for some local plurisubharmonic potential φ.

The current T is said to have analytic singularities of coefficient $c > 0$, if

$$\varphi = \frac{c}{2} \log(|f_1|^2 + \cdots + |f_N|^2) + C^\infty,$$

for some holomorphic functions $f_1, \ldots, f_N$.

The Lelong number at a point x is given in this case by :

$$\nu(T, x) = \nu(\varphi, x) = c \min_{j=1,\ldots,N} \mathrm{ord}_x f_j,$$

to quantify the singularity of the current T at x.

A variant of Demailly's regularization-of-currents theorem ([Dem92]) states that T can be approximated in the weak topology of currents by currents with analytic singularities lying in the same cohomology class as T. The price to pay is a loss in positivity which becomes negligible as the approximation gets more and more accurate. Due to a scarcity of global objects on a compact manifold, we allow some leeway by requiring the current T to be only almost positive, namely with a possible negative part which must be bounded.

Theorem 0.2.1 *(Demailly [Dem92]) Let $T = dd^c\varphi$ be a closed $(1, 1)$-current on X, and assume that $T \geq \gamma$ for some continuous $(1, 1)$-form γ on X.*

Then, for all $m \in \mathbb{N}^\star$, there exists a current $T_m = dd^c\varphi_m$ with analytic singularities of coefficient $\frac{1}{m}$, such that :

(i) $T_m \to T$ *weakly ;*

(ii) $\nu(T, x) - \frac{n}{m} \leq \nu(T_m, x) \leq \nu(T, x)$, *for all m and all $x \in X$;*

(iii) $T_m \geq \gamma - \varepsilon_m \omega$, *for* $\varepsilon_m \searrow 0$.

Remark. The loss in positivity ε_m was not explicit in Demailly's original result. However, it is possible to control the rate of convergence to zero of ε_m by careful estimates of curvature terms in the approximation procedure. This is a recent result.

Theorem 0.2.2 *([Pop04a]) Under the above hypotheses, if the form γ on X is closed, we can achieve the same conclusion with $\varepsilon_m = \frac{C}{m}$, for a constant $C > 0$ independent of m.*

If the form γ is arbitrary, the conclusion of Demailly's theorem holds with $\varepsilon_m = \frac{C}{\sqrt[4]{m}}$, for a constant $C > 0$ indeendent of m.

However, conjecture the 0.1.6 deals with the nth power of the curvature current rather than with this current itself. Therefore, we would like to control the behaviour of the wedge powers (when they are well-defined) of the regularizing currents T_m as m tends to $+\infty$. These Monge-Ampère currents are unpredictable in general. The following conjectural statement aims at controlling their masses, a control which would be sufficient in the perspective of the above conjecture. The following integrals are considered in the complement of the singular set $\operatorname{Sing} T_m$ of T_m.

Conjecture 0.2.3 *(Control of the Monge-Ampère masses) There exists a regularization of $T \geq \gamma$ by currents T_m with analytic singularities of coefficient $\frac{1}{m}$ which satisfy, besides the properties in theorem 0.2.1, the following condition :*

$$\varepsilon_m \int\limits_{X \setminus \operatorname{Sing} T_m} (T_m - \gamma + \varepsilon_m \omega)^k \wedge \omega^{n-k} \qquad \text{converges to } 0 \text{ as } m \to +\infty,$$

for all $k = 1, \ldots, n$.

The regularizing currents T_m, as constructed in Demailly's theorem, may not satisfy this extra condition on the Monge-Ampère masses. A better construction of regularizing currents may be needed. Let us now briefly remind their original definition. We shall subsequently indicate a way of modifying it to suit our purposes.

Local approximation. (Demailly)

Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex open set, and let φ be a plurisubharmonic function on Ω. Define the following Hilbert space

$$\mathcal{H}_\Omega(m\varphi) = \{f \in \mathcal{O}(\Omega)\,;\ \int_\Omega |f|^2\, e^{-2m\varphi} < +\infty\},$$

which is a weighted Bergman space with a singular weight. Pick an arbitrary orthonormal basis $(\sigma_{m,\,j})_{j\in\mathbb{N}}$, and put

$$\varphi_m := \frac{1}{2m}\,\log \sum_{j=0}^{+\infty} |\sigma_{m,\,j}|^2,$$

a plurisubharmonic function with analytic singularities. Then, the Ohsawa-Takegoshi L^2 extension theorem can be applied at a point (cf.[Dem92]) to prove that $\varphi_m \to \varphi$ in L^1_{loc} topology, and consequently that

$T_m := dd^c\varphi_m \to T = dd^c\varphi$ as currents.

These local approximations are subsequently glued together into a global approximation of T (see [Dem92] for the details).

Modify the T_m's

However, these currents still have too many singularities for their Monge-Ampère masses to be kept under control. It is natural to reduce their singular set by taking extra functions in their definition. The idea is to consider all the derivatives $D^\alpha\sigma_{m,\,j}$ of the functions $(\sigma_{m,\,j})_{j\in\mathbb{N}}$ up to a given order p, for multiindices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Set therefore

$$\psi_m := \tfrac{1}{2m}\,\log \sum_{|\alpha|\leq p} |D^\alpha\sigma_{m,\,j}|^2.$$

It can be easily checked that the new currents $T_m := dd^c\varphi_m$ still have the property $T_m \to T$, as $m \to +\infty$, if $p \in \mathbb{N}^\star$ is chosen independent of m. An extension theorem of the Ohsawa-Takegoshi type, that would account for the derivatives of the extension, arises then as a natural tool to deal with these newly constructed currents. This extension theorem will be explained in section 0.4.

0.3 Multiplier ideal sheaves.

Let us now make a brief detour through multiplier ideal sheaves which enable us to better deal with singular metrics on holomorphic line bundles. With a plurisubharmonic function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ defined on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$, one associates the ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ defined as

$$\mathcal{I}(\varphi)_x = \{f \in \mathcal{O}_{\Omega,\, x}\,;\ \exists V \ni x,\ \int_V |f|^2\, e^{-2\varphi} < +\infty\},$$

at every point $x \in \Omega$. This is called the multiplier ideal sheaf associated with φ, and it reflects rather accurately the singularites of φ. Intuitively, the multiplier ideal sheaf gets smaller and smaller as the singularities of φ increase. Likewise, if h is a singular metric on some holomorphic line bundle of local weight φ, the multiplier ideal sheaf $\mathcal{I}(h)$ is defined as $\mathcal{I}(\varphi)$.

Basic fact. (Nadel [Nad90], see also [Dem93]) The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is coherent. Moreover, it is globally generated by an arbitrary orthonormal basis of the Hilbert space $\mathcal{H}_\Omega(\varphi) = \{f \in \mathcal{O}(\Omega)\,;\ \displaystyle\int_\Omega |f|^2\, e^{-2\varphi} < +\infty\}$.

Basic question. What is the variation of $\mathcal{I}(m\varphi)$ as $m \to +\infty$?

We have the following partial answers.

$\cdot$ $\mathcal{I}((m+1)\varphi) \subset \mathcal{I}(m\varphi)$, for every m

$\cdot$ Subadditivity (Demailly-Ein-Lazarsfeld [DEL00]) :

$$\mathcal{I}(m\varphi) \subset \mathcal{I}(\varphi)^m, \quad \text{for every } m.$$

The lack of actual additivity leads us to ask if a lower bound for $\mathcal{I}(m\varphi)$ can be found. The following result points in this direction.

Theorem 0.3.1 *([Pop04b], effective coherence) Let $d = diam(\Omega)$, $x \in \Omega$, and let $B(x, r') \subset\subset B(x, r) \subset\subset \Omega$ be two fixed balls. Pick an arbitrary orthonormal basis $(\sigma_{m,\, j})_{j \geq 0}$ of $\mathcal{H}_\Omega(m\varphi)$.*

Then, for every $f \in \mathcal{O}(B(x, r))$ such that $C_f := \displaystyle\int_{B(x,\, r)} |f|^2\, e^{-2m\varphi} < +\infty$,

there exist holomorphic functions $b_{m,\, j}$, $j = 1, \ldots, +\infty$, on $B(x, r')$, such that

$$f(z) = \sum_{j=0}^{+\infty} b_{m,\, j}(z)\, \sigma_{m,\, j}(z), \qquad \text{for all } z \in B(x, r'),$$

and

$$\sup_{B(x,r')} \sum_{j=0}^{+\infty} |b_{m,j}|^2 \leq \frac{1}{(1-\frac{r}{d})^2} \, \frac{C(n)}{(\frac{r}{d})^{2(n+2)}} \, C_f.$$

The proof of this result relies heavily on Skoda's L^2 division theorem ([Sko78]). It was inspired by a similar theorem of Siu's ([Siu02]) obtained for global holomorphic sections of high tensor powers of ample line bundles over projectve manifolds.

0.4 Generalization of the Ohsawa-Takegoshi-Manivel L^2 extension Theorem.

We now pick up where we left off at the end of the section 0.2. Our goal is to give an extension theorem for a section of a Hermitian line bundle, defined on a submanifold of a given manifold and satisfying an L^2 condition, to the whole of the ambient manifold, such that we prescribe the jets of the extension along the submanifold, and such that we control the L^2 norm of the extension in terms of the L^2 norm of the original section on the submanifold.

• Local version : L^2 extension of jets of functions

For the sake of perspicuity, we shall first deal with the case of functions with prescribed derivatives (or jets) up to an arbitrary order given beforehand at a point. This is what is actually needed in the section 0.2.

Theorem 0.4.1 *([Pop05]) Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex open set, $z_0 \in \Omega$, and $k \in \mathbb{N}^\star$. Let φ be a plurisubharmonic function on Ω.*

Then, for every $a_\alpha \in \mathbb{C}$, $|\alpha| \leq k$, there exists a holomorphic function f on Ω, such that

$$f(z_0) = a_0, \qquad \frac{\partial^\alpha f}{\partial z^\alpha}(z_0) = a_\alpha, \text{ and}$$

$$\int_\Omega \frac{|f|^2}{|z-z_0|^{2(n-\varepsilon)}} \, e^{-\varphi(z)} \, dV_\Omega(z) \leq \frac{C_n^{(k)}}{\varepsilon^2 \, (diam \, \Omega)^{2(n-\varepsilon)}} \left(\sum_{|\alpha| \leq k} |a_\alpha|^2 \right) e^{-\varphi(z_0)},$$

where $C_n^{(k)} > 0$ is a constant depending only on the modulus of continuity of φ.

• Global version : L^2 extension of jets of sections

We shall now present the general global version of the jet extension theorem in the more geometric context of a fairly general ambient manifold X

and sections of Hermitian holomorphic vector bundles. Unlike the previous L^2 extension results from submanifolds of X, we deal here with extensions from a possibly unreduced subscheme of X defined by a sheaf of jets. Intuitively, our subscheme consists of several layers of a same submanifold.

Initial data

Suppose the following are given :

$(X,\,\omega)$ a weakly pseudoconvex Kähler manifold, $\dim_{\mathbb{C}} X = n$,
(e.g. compact or Stein)

$(L,\,h) \to (X,\,\omega)$, a Hermitian holomorphic line bundle,

$E \to (X,\,\omega)$, a Hermitian holomorphic vector bundle bundle, $\mathrm{rk}\, E = r \geq 1$,

$s \in H^0(X, E)$ a section, generically transverse to the zero section,

$Y := \{x \in X\,;\, s(x) = 0\}$, $\mathrm{codim} Y = r$.

$\mathcal{J}_Y \subset \mathcal{O}_X$, the ideal sheaf of Y,

$k \in \mathbb{N}^\star$, an arbitrary positive integer,

$\mathcal{O}_X/\mathcal{J}_Y^{k+1}$, the sheaf of "vertical" or "transversal" k-jets.

$\Omega \subset\subset X$ an arbitrary relatively compact open subset. Associate the weight function :

$$\rho(y) = \frac{1}{||Ds_y^{-1}||\,\sup\limits_{\xi\in\Omega}(||D^2 s_\xi|| + ||Ds_\xi||)},$$

defined at every point $y \in Y$.

Goal : to extend vertical k-jets from Y to X with L^2 growth conditions, i.e. to extend sections

$$f \in H^0(X, \Lambda^n T^\star X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}).$$

Equivalently, extend sections from the unreduced subscheme $Y^{(k+1)}$ defined by the quotient sheaf $\mathcal{O}_X/\mathcal{J}_Y^{k+1}$, to the ambient manifold X.

Construction of relevant metrics on jets

As the sheaf of vertical k-jets $\mathcal{O}_X/\mathcal{J}_Y^{k+1}$ is not locally free, we have to construct a metric on it which is relevant to this particular case at hand. Let $f \in H^0(U, \Lambda^n T^\star X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$ be a transversal k-jet, let $y \in Y$ be a point, $U \ni y$ a Stein open neighbourhood, and $\tilde{f} \in H^0(U, L')$ a local lifting of f. Without specifying the details, here are the steps of our construction :

· Construct inductively $\nabla^j \tilde{f} \in C^\infty(U, L' \otimes S^j N^\star_{Y/X})$ for all nonnegative integers j, using the Chern connections ∇ of the Hermitian line bundles involved at every step of the induction.

· Define the pointwise ρ-weighted norm :

$$|f|^2_{s,\rho,(k)}(y) := |\tilde{f}|^2(y) + \frac{|\nabla^1 \tilde{f}|^2}{|\Lambda^r(ds)|^{2\frac{1}{r}}\, \rho^{2(r+1)}}(y) + \cdots + \frac{|\nabla^k \tilde{f}|^2}{|\Lambda^r(ds)|^{2\frac{k}{r}}\, \rho^{2(r+k)}}(y),$$

depending on the weight function ρ, on the section s defining Y, and on the order k of the jet.

· Define the $L^2_{(k)}$ weighted Sobolev-type norm :

$$||f||^2_{s,\,\rho,\,(k)} = \int_Y |f|^2_{s,\,\rho,\,(k)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\,\omega}.$$

Initial curvature assumptions

The curvature form $i\Theta_h(L)$ of the Hermitian line bundle $(L,\, h)$ is required to satisfy the following positivity assumptions for the jet extensions to be possible. These assumptions depend on the order k of the prescribed jets, and take into account the curvature of the submanifold Y through the use of s.

(a) $i\Theta_h(L) + (r+k)\, id'd'' \log |s|^2 \geq \alpha^{-1} \dfrac{\{i\Theta(E)s, s\}}{|s|^2}$,

(b) $|s| \leq e^{-\alpha}$, for a continuous function $\alpha \geq 1$ on X.

Notation

Let $J^k : H^0(X,\, \Lambda^n T^\star_X \otimes L) \to H^0(X,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$ be the map induced between the spaces of corresponding global sections by the projection of sheaves $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_Y^{k+1}$.

Conclusion

We can now state our result.

Theorem 0.4.2 *([Pop05]) In the above setting, every vertical k-jet* $f \in H^0(X, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$ *satisfying*

$$\int_Y |f|^2_{s,\,\rho,\,(k)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\,\omega} < +\infty,$$

can be extended to a global section $F_k \in H^0(X, \Lambda^n T^\star_X \otimes L)$ *such that* $J^k F_k = f$*, and*

$$\int_\Omega \frac{|F_k|^2}{|s|^{2r}\,(-\log|s|)^2}\, dV_{X,\,\omega} \leq C_r^{(k)} \int_Y |f|^2_{s,\,\rho,\,(k)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\,\omega},$$

where $C_r^{(k)} > 0$ *is a constant depending only on* r*,* k*,* E*, and* $\sup_\Omega ||i\Theta(L)||$*.*

Remarks on the proof

In the first part of the proof of the jet extension theorem, we adapt the techniques of the original proof of the Ohsawa-Takegoshi-Manivel extension theorem ([OT87], [Man93]) to our more general situation. We construct the global extension of our original jet by induction on the order of the jet. Particular attention is paid to the curvature conditions needed in terms of the order k of the jets involved and the curvature of the submanifold Y from which the extension is made. This approach produces an extension F_k of the kth order jet f to X. In the second part of the proof, we estimate the global L^2 norm of the extension in terms of the Sobolev-type norm of the original k-jet f on Y. This part of the proof is more delicate since we have to ensure uniformity in the final estimate. In particular, in order to get a constant $C_r^{(k)} > 0$ which is independent of the uncontrolable radii of the local holomorpic coordinate balls on the ambient manifold X, we make use of the exponential map to replace locally X by the tangent space at a point. We transfer the situation over to the tangent space, but we have to compare the Euclidian metric on the tangent space with the pull-back of our original Hermitian metric ω on X. To this effect, we infer and use the following slight modification of Rauch's comparison theorem.

Proposition 0.4.3 *If there exists a constant* $k > 0$ *such that the sectional curvature* $K(p, P)$ *of* X *satisfies the inequalities*

$$-k \leq K(p, P) \leq k,$$

for every point $p \in X$ *and every plane* $P \subset T_pX$*, then*

$$||T_x \exp_m - \mathrm{Id}|| \leq \frac{\sinh(\sqrt{k}||x||)}{\sqrt{k}||x||} - 1, \qquad \textit{for all } x \in T_mX.$$

The details can be found in [Pop03] or [Pop05].

A few comments. The idea of getting a jet extension theorem was originally motivated by the study of a possible regularization of currents with a control of the Monge-Ampère masses, as pointed out in the section 0.2. However, we hope that the jet extension theorem will find applications of independent interest. It has been intended as a tool of producing global holomorphic objects from simpler objects defined on a submanifold or merely at a point. We could imagine, for instance, possible applications to questions related to the Fujita conjecture.

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Dan Popovici,
Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

e-mail : popovici@maths.warwick.ac.uk

Cohomology and analysis of locally symmetric spaces — Many quantum integrable systems produced by Dirac operators

Takayuki Oda

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1 Cohomology of discrete subgroups

In this section, we recall basic facts on cohomology groups of discrete subgroups in real semisimple Lie groups. A good reference is a survey artilce by A. Borel [1]. The book [2] of Borel-Wallach also has been a very important reference, though this is a bit difficult to penetrate since it is not written as a textbook. The original paper about this theme is Matsushima's [23], [24]. In the case when G/K is Hermitian, there is a paper by the author which is more specialized to the Hodge theoretic aspect of the problem [34].

1.1 Shift to the relative Lie algebra cohomology groups

Given an arithmetic discrete subgroup Γ in a semisimple real Lie group G, we consider its Eilenberg-Maclane cohomology group $\mathrm{H}^i(\Gamma, \mathbf{C})$. Or more generally if a finite dimensional rational representation $r : G \to GL(V)$ of G is given, we may regard its as Γ-module by restriction, and we can form cohomology group $\mathrm{H}^i(\Gamma, V)$.

Fix a maximal compact subgroup K of G to get a Riemannnian symmetric space $X = G/K$. For simplicity assume that we have no elements of finite order in Γ, then Γ acts on X from the right side without fixed point, and the quotinet $\Gamma\backslash X$ becomes a manifold. In this case Γ is isomorphic to the fundamental group of this manifold (X is contractible to a point), and the Γ-modules V defines a local system $\tilde{V}$ on this quotient manifold. Then we have an isomorphism of cohomology groups: $\mathrm{H}^*(\Gamma, V) \cong \mathrm{H}^*(\Gamma\backslash X, \tilde{V})$.

Let $\sigma : X \to \Gamma\backslash X$ be the canonical map, by pulling back differential forms with respect to σ we have a monomorphism of de Rham complexes $\sigma^0 : \Omega^*_{\Gamma\backslash X} \to \Omega^*_X$. Moreover on the target complex, Γ acts naturally. Let $(\Omega^*_X)^\Gamma$ be the invariant subcomplex. Then it coincides with the image of σ^0, and by the de Rham theorem, we have an isomorphism of cohomology groups:

$$\mathrm{H}^*(\Gamma, V) = \mathrm{H}^*(\Gamma\backslash X, \tilde{V}) = \mathrm{H}^*(\Omega_X(V)^\Gamma).$$

The last complex in the above isomorphims is identified with the complex of differentail forms on G as follows.

Let π^0 be the pull-back homomorphism of differential forms with respect to the canonical map $\pi : \Gamma\backslash G \to \Gamma\backslash X$. Then a form $\omega \in \Omega_X^d(V)^\Gamma$ defines a differential form ω^0 on G by

$$x \in G \mapsto r(x)^{-1}\pi^0(\omega)(x),$$

and denote by $A_0^*(G,\Gamma,V)$ the image of this homomorphism. Then since σ^0 is a monomorphism, we have an isomorphism of complexes $\Omega_X^*(V)^\Gamma \cong A_0^*(G,\Gamma,V)$.

Here the last complex is identified with the right K-invariant subcomplex of the de Rham complex $\Omega^*_{\Gamma\backslash G}(V) = \Omega^*_{\Gamma\backslash G} \otimes V$, which is defined over $\Gamma\backslash G$ and takes values in V.

Since the tangent space at each point of G/K is identified with the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with resepect to the Killing form, the module of the i-th cochains becomes $\mathrm{Hom}_K(\wedge^i\mathfrak{p}, C^\infty(\Gamma\backslash G) \otimes V)$. Therefore, the cochain comlex defined in this manner gives the relative Lie algebra cohomology groups. When G is connected we have an isomorphism:

$$\mathrm{H}^*(\Gamma, V) \cong \mathrm{H}^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G), V).$$

1.2 Matsushima isomorphism

Let G be a connected semisimple real Lie group with finite center. Assume that the discrete subgroup Γ is cocompact, i.e. the quotinet $\Gamma\backslash G$ is compact.

Let $L^2(\Gamma\backslash G)$ be the space of L^2-functions on G with respect the Haar measure on G, on which G acts unitarily by the right action. By assumption the space $C^\infty(\Gamma\backslash G)$ is a subspace of this space.

Proposition (1.1) (Gelfand-Graev,Piateskii-Shapiro) If Γ is cocompact, we have the direct sum decomposition of the unitary representation $L^2(\Gamma\backslash G)$ into irreducible components:

$$L^2(\Gamma\backslash G) = \tilde{\oplus}_{\pi\in\hat{G}} m(\pi,\Gamma)M_\pi,$$

with finite multipicities $m(\pi,\Gamma)$. Here $\hat{G}$ is the unitary dual of G, i.e.the unitary equivalence classes of irreducible unitary representations of G, and M_π denotes the representation sapce of π.

By the above proposition, one has a decomposition

$$C^\infty(\Gamma\backslash G) = \tilde{\oplus}_{\pi\in\hat{G}} m(\pi,\Gamma)M_\pi^\infty$$

of topological linear spaces. Here M_π^∞ is the subspace consisting of C^∞-vectors in the representation space M_π.

Theorem (1.2) For a finite dimensional rational G-module V over the complex number field, we have an isomorphism:

$$\mathrm{H}^*(\Gamma, V) = \oplus_{\pi\in\hat{G}} m(\pi,\Gamma)\mathrm{H}^*(\mathfrak{g}, K, M_\pi^\infty \otimes V)$$

$$= \oplus_{\pi\in\hat{G}}\{\mathrm{Hom}_G(M_\pi, L^2(\Gamma\backslash G) \otimes_{\mathbb{C}} \mathrm{H}^*(\mathfrak{g}, K, M_\pi^\infty \otimes V)\}$$

The key point of the proof here is the complete direct sum $\tilde{\oplus}$ is replaced by a simple algebraic direct sum by passing to the cohomology (*cf.* Borel [1], or §1 of Vogan-Zuckermann [42]).

We can equipp a K-invaraint inner product on V. By using this, we may regard $H^m(\mathfrak{g}, K, M_\pi^\infty \otimes V)$ as a space of a kind of harmonic forms, i.e. the totality of cochains vanishing by the Laplace operator. Thus we have the following.

Proposition (1.3) Let (r, V) be an irreducible G-module of finite dimension. For any $\pi \in \hat{G}$ we have the follwing.

(i) If $\chi_r(C) = \chi_\pi(C)$, there is an isomorphism

$$\mathrm{Hom}_K(\wedge^m, M_\pi \otimes^\infty \otimes V) = C^m(\mathfrak{g}, K, M_\pi^\infty \otimes V) = \mathrm{H}^*(\mathfrak{g}, K, M_\pi^\infty \otimes V)$$

(ii) If $\chi_r(C) \neq \chi_\pi(C)$, there is an isomorphism

$$\mathrm{H}^m(\mathfrak{g}, K, M_\pi^\infty \otimes V) = \{0\}.$$

Here C denotes the Casimir operator.

In particular, when V is the trivial G-module $\mathbb{C}$, the above theorem is no other than the original formula of Betti numbers of $\Gamma\backslash X$ by Matsushima ([23], [24]).

Also there is a variant of this type vanishing theorem shown by D. Wigner. But it is omitted here.

We recall here that the relative Lie algebra cohomology group $\mathrm{H}^m(\mathfrak{g}, K, M_\pi^\infty \otimes V)$ is isomorphic to the continuous cohomology group $\mathrm{H}^m_{ct}(G, M_\pi^\infty \otimes V)$ if G is connected. This is shown by using differential cohomology and van Est spectral sequence.

1.3 Non-cocompact case

When $\Gamma\backslash G$ is not compact, $L^2(\Gamma\backslash)$ has continuous spectrum. This makes the problem techinically quite complicated. A satisfactory general solution is given by Franke, around mid-90's after a series of efforts of many people including Armand Borel's major contribution.

2 After the Matsushima isomorphism

There are two ingredients in the Matsushima isomorphism: one is the relative Lie algebra cohomology group

$$\mathrm{H}^m(\mathfrak{g}, K, M_\pi^\infty \otimes V)$$

which is independent of Γ (hence, a *local* problem at the real place from arithmetic view-point), the other the intertwining space

$$\mathrm{Hom}_G(M_\pi, L^2(\Gamma\backslash G))$$

which depends on Γ, hence a *global* problem from arithmetic view point. The tensor product

$$\mathrm{Hom}_G(M_\pi, L^2(\Gamma\backslash G) \otimes_{\mathbb{C}} \mathrm{H}^*(\mathfrak{g}, K, M_\pi^\infty \otimes V) =: H^*(\Gamma, V)[\pi]$$

gives the $[\pi]$-part of the cohomology group $H^*(\Gamma, V)$.

People working on the representation theory of real Lie groups had been taking care of the local problem, because it is related to the decomposition of the (generalized) principal series representations of semisimple Lie groups.

The global problem is believed to be fixed by Selberg trace formula. To explain the state of arts about this theme. One can say that in the case of Γ with point-cusps, this is essentially solved module evaluation of the special values of zeta functions of certain prehomogeneous vector spaces. But it is quite difficult to have effective computable results even for this case. For Γ in an algebraic group of $\mathbf{Q}$-rank ≥ 2, little is known, except for the case G/K hermirtian with $\mathrm{rank}_{\mathbf{R}}(G) = 2$ and automorphic forms are holomorphic.

Our view-point is to go back the state before taking cohomology classes, but want to have "harmonic forms" representiong cohomology classes.

2.1 Local problems (A): vanishing theorems

A number of vanishing theorems were found in '60's: Calabi-Vesentini, Weil etc. In their proof, the same type of computation of "curvature forms" is done, which is similar to a proof of Kodaira vanishing theorem.

Firstly, Matsushima's vanishing theorem of the 1-st Betti number of $\Gamma\backslash X$ was also proved by such method ([23], [24]).

This type of vanishing theorem is vastly improved by representation theoretic method. Probably the best resut of this category is the following result by Zuckermann ([45]) (see also [2], Chapter V, §2 − §3 (p.150–155)).

Theorem(1.6) Let G be a simple real algebraic group, (π, H) a nontrivial irreducible unitary representation of G, and (r, V) a finite dimensional representation of G. Then for $k < \mathrm{rank}_{\mathbb{R}} G$,

$$\mathrm{H}^k_{ct}(G, H^\infty \otimes V) = \{0\}.$$

Corollary(1.7) Given a cocompact discrete subgroup Γ of G, for $k < \mathrm{rank}_{\mathbb{R}} G$, the restriction homomorphism

$$\mathrm{H}^k_{ct}(G, V) \to \mathrm{H}^k(\Gamma, V)$$

is an isomorphism.

2.2 Local problems (B): Enumeration and construction of unitary cohomological representations

We shortly review the state of arts on the cohomological representations defined below.

Definition(1.8) An irreducible (unitary) representation $(\pi, H_\pi) \in \hat{G}$ is called *cohomological*, if there is a finite dimensional G-module F such that

$$\mathrm{H}^i(\mathfrak{g}, K; M_\pi^\infty \otimes_{\mathbb{C}} F) \neq \{0\}$$

for some $i \in \mathbb{N}$. The set of equivalent classes of cohomological representations is denoted by $\hat{G}_{coh}$.

Enumeration of such cohomological representations was done for the case trivial $F = \mathbf{C}$ by Kumaresan [**?**], and for general case by Vogan-Zuckermann

[42]. This was originally described by susing the cohomological induction functor $\mathcal{A}_\lambda(\mathfrak{g})$ first. And later a global realiztion of this Zuckermann module was obtained by H.-W. Wong [43].

Recently Tosiyuki Kobayashi is developping a theory of branching rule for such cohomological representations when they are restricted to a large reductive subgroup H of G ([15], [16], [17], [18]).

3 Cohomological automorphic forms and Dirac operators

3.1 Automorphic forms

Let G, K be a semisimple Lie group and a fixed maximal compact subgroup K. Let Γ be a discrete subgroup of G, ususally an arithmetic subgroup, which is of finite covolume, i.e., $vol(\Gamma\backslash G) < \infty$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g} = Lie(G)$, and let $Z(\mathfrak{g})$ be its center. Normally the following is consider the most general definition of automorphic forms.

Definition A C^∞-function $f : G \to \mathbf{C}$ is an automorphic form with respect to Γ if

(i) f is left Γ-invariant, i.e., $f(\gamma g) = f(g)$ for for any $\gamma \in \Gamma,\ g \in G$.

(ii) f is right K-finite, i.e., the linear span generated by the right K-translations $f(gk)$ of f is of finite dimension.

(iii) f is of moderate growth at infinity.

Let $\mathcal{A}(\Gamma\backslash G)$ be the space of automorphic forms.

Talking about the last condition, there are a few equivalent but apparently different formulations. One naive way is to embedd G into some $GL(N, \mathbf{C})$, and utilizing the matrix realization $g = (g_{ij})_{1\le i,j\le N}$ we define a norm $\|g\| = trace({}^t\bar{g} \cdot g)$. Then $f(g)$ is said to be of moderatre growth if there are constants $C,\ M$ such that $|f(g)| \le C\|g\|^M$ for all $g \in G$.

Given an irreducible admissible G-module $(\pi,\ M_\pi)$ or a $(\mathfrak{g}, K)$-module $(\pi,\ M_\pi)$, we can consider the evaluation map

$$ev_\pi : \mathrm{H}om_G(\pi, \mathcal{A}(\Gamma\backslash G)) \otimes M_\pi \to \mathcal{A}(\Gamma\backslash G) \text{ , or}$$
$$ev_\pi : \mathrm{H}om_{(\mathfrak{g},K)}(\pi, \mathcal{A}(\Gamma\backslash G)) \otimes M_\pi \to \mathcal{A}(\Gamma\backslash G).$$

We may denote the image of this evaluation map by $\mathcal{A}(\Gamma\backslash G)[\pi]$. The intertwining spaces $\mathrm{H}om_G(\pi, \mathcal{A}(\Gamma\backslash G))$, $\mathrm{H}om_{(\mathfrak{g},K)}(\pi, \mathcal{A}(\Gamma\backslash G))$ are of finite dimension, if Γ is an arithmetic subgroup of G. When $\Gamma\backslash G$ is non-compact, this is one of the fundamental results obtained as an application of "reduction theory of algebraic groups" (*cf.* Harish-Chandra's Springer Lecture Notes. For "reduction theory", there is a good monograph by Armand Borel published from Hermann).

Choose an irredicible K-module (τ, W_τ) and also choose a fixed K-homomorphism $i : \tau \hookrightarrow M_\pi$. Then the composition $ev_\pi \cdot i$ has an image $\mathcal{A}(\Gamma\backslash G)[\pi, i]$ of finite dimension. Moreover the Casimir operator C in $Z(\mathfrak{g})$ acts as a scalar $\chi_\pi(C)$ on M_π where $\chi_\pi : Z(\mathfrak{g}) \to \mathbf{C}$ is the infinitesimal character. We can decompose

$C = C_{\mathfrak{p}} + C_{\mathfrak{k}}$ according as the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ ($\mathfrak{k} = Lie(K)$). By definition $C_{\mathfrak{k}}$ acts on $\mathcal{A}(\Gamma\backslash G)[\pi, i]$ as a scalar multiple, hence the elliptic operator $C_{\mathfrak{p}}$ acts as a scalar multiple. Therefore all the elements in $\mathcal{A}(\Gamma\backslash G)[\pi, i]$ are real analytic functions on G/K or on G.

Let

$$ev_\tau : \mathrm{Hom}_K(\tau,\ M_\pi) \otimes W_\tau \to M_\pi$$

be the evaluation map. Let $M_\pi[\tau]$ be the iamge of this evaluation map. The admissibility of the irreducible $(\mathfrak{g}, K)$-module implies that $\mathrm{Hom}_K(\tau,\ M_\pi)$ is of finite dimension. Hence $M_\pi[\tau]$ is also of finite dimension.

Now we can consider the linear span $\mathcal{A}(\Gamma\backslash G$ of $\mathcal{A}(\Gamma\backslash G)[\pi, i]$, when i moves in $\mathrm{Hom}_K(\tau,\ M_\pi)$. This is also of finite dimension, and the general space of automorphic forms with specified K-types and specified $Z(\mathfrak{g})$-types has a composition series consisting of such spaces.

Remark. The above definition of automprphic forms (due to Harish-Chandra) is quite general. For example if $\pi \in \hat{G}$ is a principal series representation, we may call the associated space of automorphic forms in $\mathcal{A}(\Gamma\backslash G)$ *wave forms.* At present we have no way to have effectively computable results for such automorphic forms. Proably we might not be able to do so in the future. For me they look like "the dark matter" in our universe of automorphic forms.

3.2 Cohomological automorphic forms and Dirac operators

Let (π, M_π) be a cohomological representation of G. A deep result for such representations is that we know very precise information on their K-types.

For a cohomological representation π, there is a distinguished K-type (τ_0, W_{τ_0}) which occurs with multiplicity one in M_π, such that some irreducible factors τ_i in the tensor product $\mathfrak{p} \otimes \tau$ does not occurs in M_π (Remark: there might be some exception. But this is the case in many cases). This means in the natural K-homomorphism :

$$\mathfrak{p} \otimes \tau \to \mathfrak{p}\tau \hookrightarrow M_\pi$$

these factors $\tau_i \hookrightarrow \mathfrak{p} \otimes \tau$ vanish. Passing to the world of automorphic forms, this means that the composition of the gradient operator:

$$\nabla : \mathcal{A}(\Gamma\backslash G)[\pi, \tau_0] \to \mathcal{A}(\Gamma\backslash G)[\pi, \tau_0] \otimes_K \mathfrak{p}^*$$

with the projector:

$$pr_i : \mathcal{A}(\Gamma\backslash G)[\pi, \tau_0] \otimes_K \mathfrak{p}^* \cong \mathcal{A}(\Gamma\backslash G)[\pi, \tau \otimes \mathfrak{p}] \to \mathcal{A}(\Gamma\backslash G)[\pi, \tau_i]$$

vanishes. This defines a first order differential (-difference) operators. These have an interpretation as Dirac operators in some context.

Note that for the Casimir operator C, the element $C - \chi_\pi(C)$ in $Z(\mathfrak{g})$ defines an elliptic differential operator for functions in $\mathcal{A}(\Gamma\backslash G)[\pi, \tau_0]$. This is compatible with the Dirac operator.

Now ask the following question.

Question. How to investigate each element in $\mathcal{A}(\Gamma\backslash G)[\pi, \tau_0]$?

When Γ is cocompact, it is difficult to find some "grip". But when Γ has cusps, or equivalently when we have parabolic subgroups in G, which are rational with respect to *Gamma*, it is possible to consider Fourier exapnsions along them.

We have some experience when G/K is Hermitian and (π, M_π) is a highest weight module, or slightly more restricted, a discrete series representation. But our knowledge on Fourier expansions are quite limited, in spite of that this a fundamental tool in the investigation of automorphic forms.

4 Generalzied spherical functions

The investigation of Fouirer expansion of cohomological automorphic forms is a global problem. But before that we need to solve some basic problem to have reasonable Fourier expansion. This is a local problem at the real or complex place in arithmetic terminology. A general formulation of the problem is the following.

Given a parabolic subgroup P along which we consider the Fourier expansion of an automorphic form f. Let N be the unipotent radical of P. Then we have to find a closed subgroup R between N and P, i.e., $N \subset R \subset P$.

We choose a unitary irreducible representation(η, V_η) of R and forms C^∞ induction $C^\infty\text{-Ind}_R^G(\eta)$. Then we are interested in the intertwining space

$$I(\pi; (R, \eta)) := \mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C^\infty\text{-Ind}_R^G(\eta)).$$

We require that the subgroup R is large enough so that we have a double coset decomposition $G = RAK$ with the split component A of a maximally split Cartan subgroup in G. We hope that the interetwinig space $I(\pi; (R, \eta))$ is of finite dimension.

Let $\tau_0 \hookrightarrow M_\pi$ be the distinguished K-type in the cohomological representation π. Then we have restriction map

$$I(\pi; (R, \eta)) \to \mathrm{Hom}_K(\tau_0, C^\infty\text{-Ind}_R^G(\eta)) = \{C^\infty\text{-Ind}_R^G(\eta) \otimes \tau_0^*\}^K.$$

We are intereted in to determine the holonomic system to characterize the A-radial part of the finite dimensional image of this restriction map, that is derived from the Dirac operator mentioned in the previous section. Then we have quantum integrable system.

These are the special functions which appear in the Fourier expansion. But here we have to replace the space

$$C^\infty\text{-Ind}_R^G(\eta)$$

by a subspace stable under $(\mathfrak{g}, K)$-action :

$$C^\infty\text{-Ind}_R^G(\eta)^{mod}$$

consisting of functions of moderate growth.

4.1 State of arts, and examples

Unfortunately we have no general results yet for this problem. There are rather general results by Kenji Taniguchi and by Maso Tsuzuki for G of real rank 1 case.

We have some results for the split group $Sp(2, \mathbf{R})$ of rank 2 and the quasi-split group $SU(2, 2)$ of rank 2. Also there are some results for real semisimple groups of rank 1.

The computations of various spherical functions on small Lie groups were quite important problems in the very dawn of the real harmonic analysis on Lie groups. But now the fashion is changed, the trend of the representation theory of real semisimple groups is "algebraization", though there are still people who believe firmly that every realization is important, like Professor Gindikin.

The special computations look like endless, but a well-formulated and exhaustive result makes some case study come to end.

In the case of $Sp(2, \mathbf{R})$, the special functions, which should appear in the Fourier expansion of automorphic harmonic forms representation some cohomological classes (i.e, those automorphic forms which generate discrete series represenations of $Sp(2, \mathbf{R})$)s, are discussed in [35], Miyazaki [25], Hirano [9]. But these are vector-valued and we have to prepare some terminology of the represenations of $K \cong U(2)$.

To avoid this complication, and to cheat the readers a bit but not so much, we consider the case of P_J prinicipal series representations of $Sp(2, \mathbf{R})$. The point is the "shape" of K-types of these representations are the same as those of the large discrete series representations, and the obtained spherical functions in both cases are quite resembled.

4.2 The P_J principal series

The split group $G = Sp(2, \mathbf{R})$ of C_2 type has 3 kinds of standard parabolic subgroups: the minimal P_0, the maximal P_S assoacaited with the short root which has abelian unipotent radical that is called often *Siegel parabolic subgroup*, and the other maximal P_J associated with the long root, whose uniptent radical is the Heisenberg group of real dimension 3. In this note, we refer the last group P_J as the Jacobi parabolic subgroup.

A P_J principal series representation π_J is a parabolic induction form a discrete series representation $\sigma_J := \varepsilon \boxtimes D_k^{\pm}$ of $M_J = \{\pm 1\} \times SL(2, \mathbf{R})$ with $\mathrm{diag}(-1, 1, -1, 1)$ the generator of $\{\pm 1\}$ part. Here $D_k^{\pm}$ are the represenations of discrete series with Blattner parameter $\pm k$ (i.e., the minimal $SO(2)$-type). Here $P_J = M_J N_J A_J$ is the Langlands decomposition of P_J. Following the standard format, we fix a linear form $\nu_J \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$ and also the half sum ρ_J of roots in $\mathfrak{n}_J$. Here $\mathfrak{a}_J, \mathfrak{n}_J$ are the Lie algebras associated with A_J, N_J. We form the quasi-character $e^{\nu_J + \rho_J} : A_J \to \mathbf{C}^*$ and via the identification $P_J / N_J \cong M_J A_J$ it defines a representation of P_J. Finally the induced represenation $\pi_J := \mathrm{Ind}_{P_J}^{G}(\sigma_J \otimes e^{\nu_J + \rho_J})$ gives a P_J principal series representation.

In the even case $\varepsilon(\mathrm{diag}(-1, 1, -1, 1)) = (-1)^k$, in the represenattion space M_π of π_J there is a special K-type $\tau_{(\pm k, \pm k)}$ of $K \cong U(2)$ which occurs with multiplicity one, that lies 'at the corner' of the picture of the dominant weights of K-types with positive multiplicity in M_π. We can find a nice annihilator D with degree 2 of this K-type $\tau_{(\pm k, \pm k)}$ in the universal envelopping algebra $U(\mathfrak{g})$.

Together with $C - \chi_{\pi_J}(C)$ we have two important annihilators.

Next we have to consider the realization of these two operators as differential operators in appropriate space of functions. For this, as we already explained, we choose a spherical subgroup R and a unitary representation η of R, and consider the intertwining space $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J, \mathrm{Ind}_R^G(\eta))$. Let I be a non-zero intertwining operator and let v be a non-zero vector in M_π with K-type $\tau_{(\pm k, \pm k)}$. Then the image $I(v)$ which is a function on G with values in V_η the representation space of η, with intertwining prpperty

$$I(v)(rxk) = \eta(r) \otimes \tau_{(\pm k, \pm k)}(k) I(v)(x) \quad (r \in R,\ x \in G,\ k \in K).$$

The function $I(v)$ is determined by its restriction to a subgroup A, if we have a double coset decomposition $G = RAK$, and in many good case we can take A as the split Cartan subgroup or its subgroup.

By a standard way, we can compute the A-radial part of the annihilators $(I \cdot \pi_J)(D)$ and $(I \cdot \pi_J)(C) - \chi_{\pi_J}(C)$. These two operators make up a holonomic system (of rank 4) for various choice of the pair (R, η).

4.3 Explicit formulae

We give here two cases, one case (A) is when $R = N_0$ the unipotent radical of the minimal parabolic subgroup P_0 of G, and η a non-degenerate character of N_0; the other case (B) is when R is a subgroup of P_S containing the unipotent radical N_S, which is a semidirect product of N_S and the connenced component $SO(\xi)$ of the stabilizer of a 'definite' character ξ of N_S, and η is a twised tensor product of ξ and a unitary character χ of $SO(\eta)$.

Let v be an element belonging to $\tau_{(k,k)} \hookrightarrow M_\pi$, and let I be a non-zero intertwining operator in $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J, \mathrm{Ind}_R^G(\eta))$. The restriction of the image $I(v)$ to A, i.e., the A-radial part, is denoted by ϕ_W in the case (A), and by ϕ_{SW} in the case (B). An element of A denoted by $\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})$.

(A): Whittaker functions of P_J principal series

The holonomic system for the A-radial part

Let $\pi = \mathrm{Ind}_{P_J}^G(\varepsilon \boxtimes D_k^+, \nu)$ be an irreducible even P_J-principal series representation of G with $\varepsilon(\mathrm{diag}(-1, 1, -1, 1)) = (-1)^k$, and $\tau^* = \tau_{(k,k)}$ is the corner K-type of π. We first prepare some basic facts on the Whittaker functions for (π, η_1, τ). Throughout this section we use a coordinate $x = (x_1, x_2)$ on A defined by

$$x_1 = \left(\pi c_0 \frac{a_1}{a_2}\right)^2, \quad x_2 = 4\pi c_3 a_2^2.$$

We have ([27, Proposition 7.1, Theorem 8.1]):

Proposition 4.1. *Let π and τ be as above. Then we have the following:*

(i) We have $\dim \mathcal{I}_{\eta_1,\pi} = \dim \mathrm{Wh}(\pi, \eta_1, \tau) = 4$, and a function

$$\phi_W(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi c_3 a_2^2) h_W(a)$$

on A is in the space $\mathrm{Wh}(\pi, \eta_1, \tau)|_A$ *if and only if* $h_W(a) = h_W(x)$ *is a smooth solution of the following holonomic system of rank* 4:

$$\Big\{\partial_{x_1}\Big(-\partial_{x_1} + \partial_{x_2} + \frac{1}{2}\Big) + x_1\Big\}h_W(x) = 0, \tag{4.1}$$

$$\Big\{\Big(\partial_{x_2} + \frac{k+\nu}{2}\Big)\Big(\partial_{x_2} + \frac{k-\nu}{2}\Big) - x_2\Big(-\partial_{x_1} + \partial_{x_2} + \frac{1}{2}\Big)\Big\}h_W(x) = 0, \tag{4.2}$$

where $\partial_{x_i} = x_i(\partial/\partial x_i)$ $(i = 1, 2)$ *is the Euler operator with respect to* x_i.
(ii) $\dim \mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}} \leq 1$. *Moreover this inequality is an equality if and only if* $c_3 > 0$.

Remark 1. Since [27] treated the case $\sigma = \varepsilon \boxtimes D_k^-$, we need a minor change by using the explicit formulas of 'shift operators' ([26, Proposition 8.3]).

Explicit formulas of good Whittaker functions

When $c_3 < 0$, Proposition 2.1 tells us that there is no non-zero moderate growth Whittaker function. Therefore let us assume $c_3 > 0$ in the following discussion. The integral expression for the Whittaker functions of moderate growth was obtained by Miyazaki and Oda.

Proposition 4.2. *([27, Theorem 8.1]) Let π and τ be as before. Define*

$$g_W(a) = g_W(x) := x_2^{-1/2}\int_0^\infty t^{-k+1/2}W_{0,\nu}(t)\exp\Big(-\frac{t^2}{16x_2} - \frac{16x_1x_2}{t^2}\Big)\frac{dt}{t},$$

with $W_{\kappa,\mu}$ the classical Whittaker function. Then the function

$$\phi_W(a) = a_1^{k+1}a_2^{k+1}\exp(-2\pi c_3 a_2^2)g_W(a)$$

gives a non-zero element in $\mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}}|_A$ *which is unique up to constant multiple.*

Siegel-Whittaker functions belonging to P_J principal series

The holonomic system for the A-radial part

Miyazaki ([25]) studied the Siegel-Whittaker functions for P_J-principal series and obtained the multiplicity one property and the explicit integral representation for rapidly decreasing function. As in the previous section, we introduce the coordinate $y = (y_1, y_2)$ on A by

$$y_1 = \frac{h_1a_1^2}{h_2a_2^2}, \quad y_2 = 4\pi h_2a_2^2.$$

We remark on a compatibility condition. For a non-zero element ϕ of $C^\infty_{\eta_i,\tau_{(-k,-k)}}(R_i\backslash G/K)$, we have

$$\phi(a) = \phi(mam^{-1}) = (\chi_{m_0} \boxtimes \xi)(m)\tau_{(-k,-k)}(m)\phi(a),$$

where $a \in A$ and $m \in SO(\xi) \cap Z_K(A) = \{\pm 1_4\}$. If we take $m = -1_4$, $(\chi_{m_0} \boxtimes \xi)(m) = \chi_{m_0}(m) = \exp(\pi\sqrt{-1}m_0)$ and $\tau_{(-k,-k)}(m) = 1$ imply that m_0 is an even integer.

Proposition 4.3. *([25, Proposition 7.2]) Let π and τ be as in §2.1. Then we have the following:*
(i) We have $\dim \mathcal{I}_{\eta_2,\pi} = \dim \mathrm{SW}(\pi,\eta_2,\tau) \leq 4$ *and a function*

$$\phi_{SW}(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi(h_1 a_1^2 + h_2 a_2^2)) h_{SW}(a)$$

is in the space $\mathrm{SW}(\pi,\eta_2,\tau)|_A$ *if and only if* $h_{SW}(a) = h_{SW}(y)$ *is a smooth solution of following system*:

$$\Big\{\partial_{y_1}\Big(-\partial_{y_1} + \partial_{y_2} + \frac{1}{2}\Big) + \frac{y_1}{y_1 - 1}\Big(-\partial_{y_1} + \frac{1}{2}\partial_{y_2}\Big) + \frac{m_0^2}{4}\frac{y_1}{(y_1-1)^2}\Big\} h_{SW}(y) = 0, \tag{4.3}$$

$$\Big\{\Big(\partial_{y_2} + \frac{k+\nu}{2}\Big)\Big(\partial_{y_2} + \frac{k-\nu}{2}\Big) - y_1 y_2\Big(\partial_{y_1} + \frac{1}{2}\Big) - y_2\Big(-\partial_{y_1} + \partial_{y_2} + \frac{1}{2}\Big)\Big\} h_{SW}(y) = 0, \tag{4.4}$$

with $\partial_{y_i} = y_i(\partial/\partial y_i)$.
(ii) $\dim \mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}} \leq 1$.

Remark 2. The above system has singularities along the three divisors $y_1 = 0$, $y_1 = 1$ and $y_2 = 0$, and they are regular singularities.

Explicit formulas of good Siegel-Whittaker functions

The integral representation of the unique element in $\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}}|_A$ is given by Miyazaki ([25, Theorem 7.5]). For our purpose, however, we need another integral expression for this function. we obtain the following Euler type integral. (See also Iida ([12]) and Gon ([3])).

Proposition 4.4. *Define*

$$g_{SW}(a) = g_{SW}(y) := (1-y_1)^{|m_0|/2} y_2^{|m_0|/2} \cdot \int_0^1 t^{(|m_0|-1)/2}(1-t)^{(|m_0|-1)/2} F\Big(\frac{y_2}{2}\{1 - t(1-y_1)\}\Big) dt,$$

with

$$F(z) = e^z (2z)^{(-k-|m_0|-1)/2} W_{(k-|m_0|-1)/2,\nu/2}(2z).$$

Then the function

$$\phi_{SW}(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi(h_1 a_1^2 + h_2 a_2^2)) g_{SW}(a)$$

gives a non-zero element in $\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}}|_A$ *which is unique up to constant multiple.*

Proof. See [**?**, 8.4]. □

5 Postscript: How these are related to the complex function theory of many variables?

Needless to say, $Sp(2,\mathbf{R})/K$ is the Siegel upper half space of dimension 3, and the quotient $\Gamma\backslash Sp(2,\mathbf{R})/K$ is a complex analytic variety of dimension 3. But let me exlain my *hope* more generally.

5.1 Expansions around singularities

Cheeger's L^2-theory computations at the conical singularity resemble to the Fourier expansion of automorphic forms on real semisimple groups of rank 1 at cusps (*cf.* [46], Chapter 8 Spectral Theory, §8 The Laplace operator on cones). Probably this is not by chance. Fourier expansions of automorphic forms are conisdered as expansions around cusp singularities.

5.2 Residues of many variables

This problem was suggested by Miki Hirano of Ehime University in the course of writing a joint paper [47].

In the classical literature on special functions or various hypergeomtric functions of one varaible, the Barnes integrals, i.e., a kind of inverse Mellin transformation, sometimes play key roles.

In the invetigation of special functions of many variables, one can expect that similarly an analogy of Barnes's integral expression in many variable plays an important role. But there is little references about this theme.

Sometimes here, the convergence of integral, and the order of residue compuations causes sometimes non-trivial problems. The loci of poles are higher dimension, and the chain of integration is also higher dimension. One cannot do the computation just drawing pictures on the planes, as in the case of one variable.

Similar problem seems to appear in Feynmann integrals.

There are much more references than necessary. This is because I transcribed this from other paper(s). May be it causes little harm.

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Graduate School of Mathematical Sciences
The University of Tokyo
Komaba 3-8-1, Meguro Ward, Tokyo 153-8914
JAPAN

ON THE SINGULARITY OF QUILLEN METRICS

Ken-Ichi Yoshikawa (University of Tokyo)

§0. Introduction.

Notation Let

(X, g_X) : a compact Kähler manifold of dimension $n+1$,
S : a compact Riemann surface,
$\pi \colon X \to S$: a proper surjective holomorphic map with fiber $X_t := \pi^{-1}(t)$,
$\Sigma_\pi := \{x \in X;\, d\pi(x) = 0\}$: the critical locus of π,
$\Delta := \pi(\Sigma_\pi)$: the discriminant locus of $\pi \colon X \to S$.
We set

$$S^o := S \setminus \Delta, \qquad X^o := X|_{S^o}, \qquad \pi^o := \pi|_{X^o}.$$

Then $\pi^o \colon X^o \to S^o$ is a family of compact Kähler manifolds.
Set
$TX^o/S^o := \ker \pi^o_*$: the relative tangent bundle of $\pi^o \colon X^o \to S^o$,
$g_{X^o/S^o} := g_X|_{TX^o/S^o}$: the Hermitian metric on TX^o/S^o induced from g_X,
(ξ, h_ξ) : a Hermitian vector bundle on X.

Let
$\lambda(\xi) := \det R\pi_*\xi$: the determinant of the cohomologies of ξ whose fiber is $\lambda(\xi)_t = \otimes_{q \geq 0} (\det H^q(X_t, \xi_t))^{(-1)^q}$ for $t \in S$,
$\| \cdot \|^2_{\lambda(\xi),Q}$: the Quillen metric on $\lambda(\xi)|_{S^o}$ associated with g_{X^o/S^o}, h_ξ.

We consider the following:

Problem. *Let σ be a holomorphic section of $\lambda(\xi)$ which does not vanish at $0 \in \Delta$. Determine the singularity of the function $\log \|\sigma\|^2_{\lambda(\xi),Q}$ near 0.* $\square$

Our result is summarized as follows (see Section 3):
• the existence of an asymptotic expansion of Barlet type for $\log \|\sigma\|^2_{\lambda(\xi),Q}$ near 0 with at most a logarithmic singularity;
• the determination of the logaruthmic singularity and the constant term of the asymptotic expansion.

Plan

§1. Determinants of cohomologies and Quillen metrics: a quick review.

We recall the definition of Quillen metrics. We refer to [BGS] for more details.

Notation Let

$\overline{V} := (V, \gamma)$: A compact Kähler manifold with Kähler form,

$\overline{F} := (F, h)$: a holomorphic Hermitian vector bundle,

$A_V^{p,q}(F)$: the vector space of C^∞ F-valued (p,q)-forms on V,

$S_V(F) := \bigoplus_{q \geq 0} A_V^{0,q}(F)$: the space of spinors,

$\langle \cdot, \cdot \rangle_x$: the inner product on $\bigoplus_{q\geq 0} \bigwedge^q T^{*(0,1)} \otimes F$ induced from γ, h.

Then one can define the L^2-inner product $(\cdot,\cdot)_{L^2}$ on $S_V(F)$ by

$$(s, s')_{L^2} := \frac{1}{(2\pi)^{\dim V}} \int_V \langle s(x), s'(x) \rangle_x \frac{\gamma^{\dim V}}{(\dim V)!}, \qquad s, s' \in S_V(F).$$

1.1. Analytic torsion.

Define operators on $S_V(F)$ by

$\square_{(\overline{V},\overline{F})} := (\bar\partial + \bar\partial^*)^2$: the Laplacian;

$N(\varphi) := q\,\varphi$ $(\varphi \in A_V^{0,q}(F))$: the number operator;

$\epsilon(\varphi) := (-1)^q \varphi$ $(\varphi \in A_V^{0,q}(F))$: the parity operator.

Let $E_{(\overline{V},\overline{F})}(\lambda)$ be the eigenspace of $\square_{(\overline{V},\overline{F})}$ with eigenvalue λ

$$E_{(\overline{V},\overline{F})}(\lambda) := \{\varphi \in S_V(F);\, \square_{(\overline{V},\overline{F})}\varphi = \lambda\,\varphi\}, \qquad \lambda \in \sigma(\square_{(\overline{V},\overline{F})}).$$

Definition. Define zeta function by

$$\zeta_{(\overline{V},\overline{F})}(s) := \sum_{\lambda \in \sigma(\square_{(\overline{V},\overline{F})})\setminus\{0\}} \lambda^{-s}\, \mathrm{Tr}\left[\epsilon \cdot N|_{E_{(\overline{V},\overline{F})}(\lambda)}\right], \qquad \mathrm{Re}(s) \gg 0.$$

Then $\zeta_{(\overline{V},\overline{F})}(s)$ extends to a meromorphic function on $\mathbb{C}$, and holomorphic at $s=0$.

Definition. The analytic torsion of $(\overline{V},\overline{F})$ is defined by

$$\tau(\overline{V},\overline{F}) := \exp\left(-\zeta'_{(\overline{V},\overline{F})}(0)\right).$$

1.2. Determinants of cohomologies and Quillen metrics.

Definition. The determinant of the cohomologies of F is the complex line

$$\lambda(F) := \bigotimes_{q\geq 0} \det H^q(V,F)^{(-1)^q}.$$

Let $K(\overline{V},\overline{F}) \subset S_V(F)$ be the space of harmonic forms

$$K(\overline{V},\overline{F}) := \ker \square_{(\overline{V},\overline{F})} \cap S_V(F).$$

Then the natural map $K(\overline{V},\overline{F}) \ni \varphi \to [\varphi] \in H(V,F)$ is an isometry (Hodge).

Definition. Let $\|\cdot\|_{L^2,\lambda(F)}$ be the Hermitian metric on $\lambda(F)$ induced from the L^2-metric $(\cdot,\cdot)_{L^2}$ by the isomorphicm $H(V,F) \cong K(\overline{V},\overline{F})$. The Quillen metric on $\lambda(F)$ is defined by

$$\|\alpha\|^2_{Q,\lambda(F)} := \tau(\overline{V},\overline{F})\,\|\alpha\|^2_{L^2,\lambda(F)}, \qquad \alpha \in \lambda(F).$$

1.3. Relative versions.

Let us go back to the family $\pi\colon X \to S$ and the vector bundle $\xi \to X$. If the function on S, $t \mapsto h^q(X_t, \xi_t)$, is locally constant for all $q \geq 0$, all the direct image sheaves $R^q\pi_*\xi$ are locally free, and we set

$$\lambda(\xi) := \bigotimes_{q\geq 0} \det R^q\pi_*\xi.$$

Then $\lambda(\xi)_t = \lambda(\xi_t)$ for all $t \in S$.

N.B. In general, $R^q\pi_*\xi$ is not locally free. In this case, $\lambda(\xi)$ is defined as follows. For simplicity, we assume that $\pi\colon X \to S$ is projective. Then there exists a complex of holomorphic vector bundles on X

$$E_\bullet\colon E_0 \to E_1 \to \cdots \to E_N \to 0, \qquad N \gg 0$$

satisfying

(i) $E_\bullet$ is a resolution of ξ, i.e., $0 \longrightarrow \xi \longrightarrow E_\bullet \longrightarrow 0$ is an exact sequence;

(ii) $R^q\pi_*E_i = 0$ for all $q > 0$, $i \geq 0$.

Since we have a canonical isomorphism

$$R^q\pi_*\xi = H^q(\pi_*E_\bullet) = \ker(\pi_*E_q \to \pi_*E_{q+1})/\mathrm{Im}(\pi_*E_{q-1} \to \pi_*E_q),$$

we set

$$\lambda(\xi) := \det \pi_*E_\bullet = \bigotimes_{q\geq 0} \det \pi_*E_q^{(-1)^q},$$

which is independent of the choice of $E_\bullet$ satisfying (i), (ii).

Fact [BGS]. *The Quillen metric is a C^∞-Hermitian metric on $\lambda(\xi)|_{S^\circ}$.*

Remark. When the cohomology of ξ_t jumps, the L^2-metric on $\lambda(\xi)|_{S^\circ}$ is *not* continuous, while the Quillen metric is continuous.

§2. The Gauss map associated with the family $\pi\colon X \to S$.

It is crucial to consider the Gauss map for the study of the singularity of Quillen metrics.

2.1. The Gauss map.

Notation Let

$\varPi : \mathbb{P}(\Omega^1_X \otimes \pi^* TS) \to X$: the projective bundle associated with $\Omega^1_X \otimes \pi^* TS$,
$\varPi^\vee : \mathbb{P}(TX)^\vee \to X$: the dual of $\mathbb{P}(TX)$.
Then the fiber $\mathbb{P}(TX)^\vee_x$ is the set of all hyperplanes of $T_x X$ containing 0_x;

$$\mathbb{P}(\Omega^1_X \otimes \pi^* TS) = \mathbb{P}(\Omega^1_X) \cong \mathbb{P}(TX)^\vee.$$

Definition. Define the Gauss maps

$$\nu : X \setminus \Sigma_\pi \to \mathbb{P}(\Omega^1_X \otimes \pi^* TS), \qquad \mu : X \setminus \Sigma_\pi \to \mathbb{P}(TX)^\vee$$

by

$$\nu(x) := [d\pi_x] = \left[\sum_{i=0}^{n} \frac{\partial \pi}{\partial z_i}(x)\, dz_i \otimes \frac{\partial}{\partial t} \right], \qquad \mu(x) := [T_x X_{\pi(x)}].$$

Under the identification $\mathbb{P}(\Omega^1_X \otimes \pi^* TS) \cong \mathbb{P}(TX)^\vee$, one has $\nu = \mu$.

2.2. Description of the relative tangent bundle using Gauss maps.

Notation Let

$H = \mathcal{O}_{\mathbb{P}(T_X)^\vee}(1)$: the hypeplane bundle of $\mathbb{P}(T_X)^\vee$,
U : the universal hyperplane bundle of $(\varPi^\vee)^* TX$ satisfying the exact sequence

$$\mathcal{F} : 0 \longrightarrow U \longrightarrow (\varPi^\vee)^* TX \longrightarrow H \longrightarrow 0,$$

g_U : the Hermitian metric on U induced from $(\varPi^\vee)^* g_X$,
g_H : the Hermitian metric on H induced from $(\varPi^\vee)^* g_X$.

Lemma. *The following identity holds on $X \setminus \Sigma_\pi$*

$$(TX/S, g_{X/S}) = \mu^*(U, g_U).$$

Proof. The assertion follows from the identities

$$T_x X_{\pi(x)} = \{v \in T_x X;\, d\pi_x(v) = 0\}, \qquad (g_{X/S})_x = g_X|_{T_x X_{\pi(x)}}. \quad \square$$

Notation Let

$L := \mathcal{O}_{\mathbb{P}(\Omega^1_X \otimes \pi^* TS)}(-1)$: the tautological line bundle on $\mathbb{P}(\Omega^1_X \otimes \pi^* TS)$,
g_S : A Hermitian metric on S,
g_L : the Hermitian metric on L induced from $\varPi^*(g_{\Omega^1_X} \otimes \pi^* g_S)$ by the inclusion $L \subset \varPi^*(\Omega^1_X \otimes \pi^* TS)$.

Lemma. *The following identity holds on $X \setminus \Sigma_\pi$*

$$-dd^c \log \|d\pi\|^2 = \nu^* c_1(L, g_L).$$

Proof. The assertion follows from the fact that $d\pi$ is a nowhere vanishing holomorphic section of $\nu^* L|_{X \setminus \Sigma_\pi}$. $\square$

2.3. Resolution of the indeterminacy of the Gauss maps.

Proposition. *There exist:*
$\widetilde{X}$ *: a compact Kähler manifold,*
$E \subset X$ *: a divisor of normal crossing,*
$q\colon \widetilde{X} \to X$ *: a birational holomorphic map*
$\widetilde{\nu}\colon \widetilde{X} \to \mathbb{P}(\Omega^1_X \otimes \pi^* TS)$, $\widetilde{\mu}\colon \widetilde{X} \to \mathbb{P}(TX)^\vee$ *: holomorphic maps*
such that
(1) $q|_{\widetilde{X}\setminus q^{-1}(\Sigma_\pi)}\colon \widetilde{X} \setminus q^{-1}(\Sigma_\pi) \to X \setminus \Sigma_\pi$ *is an isomorphism;*
(2) $q^{-1}(\Sigma_\pi) = E$*;*
(3) $\widetilde{\nu} = \nu \circ q$, $\widetilde{\mu} = \mu \circ q$ *on* $\widetilde{X} \setminus E$*;*
(4) $\widetilde{\nu} = \widetilde{\mu}$ *under the natural isomorphism* $\mathbb{P}(\Omega^1_X \otimes \pi^* TS) \cong \mathbb{P}(TX)^\vee$. □

Proof. Since Σ_π is an analytic subset of X, the mappings

$$\nu\colon X \setminus \Sigma_\pi \to \mathbb{P}(\Omega^1_X \otimes \pi^* TS), \qquad \mu\colon X \setminus \Sigma_\pi \to \mathbb{P}(TX)^\vee$$

extend to meromorphic mappings

$$\nu\colon X \dashrightarrow \mathbb{P}(\Omega^1_X \otimes \pi^* TS), \qquad \mu\colon X \dashrightarrow \mathbb{P}(TX)^\vee,$$

respectively. Now the assertion follows from Hironaka's theorem. □

Notation Set

$$\widetilde{\pi} := \pi \circ q, \qquad \widetilde{X}_t := \widetilde{\pi}^{-1}(t), \qquad E_t := E \cap \widetilde{X}_t, \quad t \in S.$$

Then $E = \amalg_{s\in\Delta} E_s$.

Let $\mathcal{I}_{\Sigma_\pi}$ be the ideal sheaf of the critical locus Σ_π:

$$\mathcal{I}_{\Sigma_\pi} = \mathcal{O}_X\left(\frac{\partial \pi}{\partial z_0}(z), \cdots, \frac{\partial \pi}{\partial z_n}(z)\right), \qquad \forall\, p \in \Sigma_\pi.$$

Let $\mathcal{I}_E$ be the ideal sheaf of E. Then

$$\mathcal{I}_E = q^{-1}\mathcal{I}_{\Sigma_\pi}.$$

Lemma. *The following equation of currents on* $\widetilde{X}$ *holds*

$$dd^c(q^* \log \|d\pi\|^2) = \widetilde{\nu}^* c_1(L, g_L) - \delta_E.$$

Proof. Since $\widetilde{\nu}^* L = q^* \nu^* L$, $q^* d\pi$ extends to a holomorphic section of $\widetilde{\nu}^* L$ with zero divisor E. Hence the assertion follows from the Poincaré-Lelong formula. □

§3. The main results.

3.1. A function space.

Notation Let

$0 \in \Delta$,

$[0]$: the holomorphic line bundle $[0]$ on S defined by the divisor 0,

$\|\cdot\|$: a Hermitian metric on $[0]$,

$\sigma_{[0]}$: the canonical section of $[0]$ with zero divisor 0.

Define a functioin space $\mathcal{B}(S,0) \subset C^0(S)$ by

$$\mathcal{B}(S,0) := C^\infty(S) \oplus \bigoplus_{r \in \mathbb{Q}\cap(0,1]} \bigoplus_{k=0}^{n} \|\sigma_{[0]}\|^{2r} (\log \|\sigma_{[0]}\|)^k \, C^\infty(S).$$

For $F \in \mathcal{B}(S,0)$, there exist $r_1, \dots, r_m \in \mathbb{Q}\cap(0,1]$ and $f_0, f_{l,k} \in C^\infty(S)$, $l = 1, \dots, m$, $k = 0, \dots, n$, such that

$$F = f_0 + \sum_{l=1}^{m} \sum_{k=0}^{n} \|\sigma_{[0]}\|^{2r_l} (\log \|\sigma_{[0]}\|)^k \, f_{l,k}.$$

3.2. The main term of the singularity of Quillen metrics.

Theorem 1 [Y2]. *Let σ be a local holomorphic section of $\lambda(\xi)$ which does not vanish at $0 \in \Delta$. Then the following identity holds near $0 \in S$*

$$\log \|\sigma\|^2_{Q,\lambda(\xi)} \equiv \left(\int_{E_0} \widetilde{\mu}^* \left\{ \mathrm{Td}(U) \frac{\mathrm{Td}(H) - 1}{c_1(H)} \right\} q^* \mathrm{ch}(\xi) \right) \log \|\sigma_{[0]}\|^2 \mod \mathcal{B}(S,0).$$

Here $\mathrm{Td}(\cdot)$, $\mathrm{ch}(\cdot)$ *denotes the Todd genus and the Chern character, respectively.* $\square$

3.3. The Knudsen-Mumford section.

Identify X with the graph of $\pi \colon X \to S$

$$X = \Gamma = \{(x,s) \in X \times S;\, \pi(x) = s\}.$$

Notation Let

$[\Gamma]$: the holomorphic line bundle on $X \times S$ associated with the divisor Γ,

$s_\Gamma \in H^0(X \times S, [\Gamma])$: the defining section of $[\Gamma]$, i.e., $\mathrm{div}(s_\Gamma) = \Gamma$.

$i \colon \Gamma \hookrightarrow X \times S$: the embedding,

$p_1 \colon X \times S \to X$, $p_2 \colon X \times S \to S$: the projections.

On $X \times S$, one has the following exact sequence of coherent sheaves:

$$0 \longrightarrow \mathcal{O}_{X\times S}([\Gamma]^{-1} \otimes p_1^*\xi) \xrightarrow{\otimes s_\Gamma} \mathcal{O}_{X\times S}(p_1^*\xi) \longrightarrow i_*\mathcal{O}_\Gamma(p_1^*\xi) \longrightarrow 0.$$

Let
$\lambda(p_1^*\xi)$: the determinant of $R(p_2)_* p_1^*\xi$,
$\lambda([\Gamma]^{-1}\otimes p_1^*\xi)$: the determinant of $R(p_2)_*([\Gamma]^{-1}\otimes p_1^*\xi)$,
$\lambda(\xi)$: the determinant of $R\pi_*\xi$.

Definition. Under the identification $\xi \cong p_1^*\xi|_\Gamma$ induced by the isomorphism $\mathcal{X} \cong \Gamma$, the holomorphic line bundle over S

$$\lambda := \lambda\left([\Gamma]^{-1}\otimes p_1^*\xi\right)\otimes\lambda(p_1^*\xi)^{-1}\otimes\lambda(\xi)$$

has a nowhere vanishing holomorphic section σ_{KM}, called the *Knudsen-Mumford section* [KM], [BGS], [BL].

3.4. The constant term of the asymptotic expansion.

Notation Let
α : a local section of $\lambda(p_1^*\xi)\otimes\lambda([\Gamma]^{-1}\otimes p_1^*\xi)^{-1}$ without zeros near $0\in S$,
σ : a local section of $\lambda(\xi)$ without zeros near 0 defined by

$$\sigma := \sigma_{KM}\otimes\alpha$$

$\mathcal{V},\mathcal{U}$: small neiborhoods of 0 such that $\mathcal{V}\subset\mathcal{U}$,
$h_{[\Gamma]}$: a Hermitian metric on $[\Gamma]$ such that

$$h_{[\Gamma]}(s_\Gamma,s_\Gamma)(w,t)=\begin{cases}|\pi(w)-t|^2 & \text{if } (w,t)\in\pi^{-1}(\mathcal{V})\times\mathcal{V},\\ 1 & \text{if } (w,t)\in(X\setminus\mathcal{U})\times\mathcal{V}.\end{cases}$$

In what follows, we consider Quillen metrics with respect to g_X, $g_{X/S}$, h_ξ, $h_{[\Gamma]}$. Then $\log\|\alpha\|_Q^2$ and $\log\|\beta\|_Q^2$ are C^∞-functions near 0.

Theorem 2 [Y2]. *Let F be a generic fiber of $\pi\colon X\to S$. Then the following identity holds*

$$\lim_{s\to 0}\left[\log\|\sigma(s)\|^2_{Q,\lambda(\xi)}-\left(\int_{E_0}\widetilde{\mu}^*\left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\}q^*\mathrm{ch}(\xi)\right)\log\|\sigma_{[0]}(s)\|^2\right]$$
$$=\log\|\alpha(0)\|_Q^2-$$
$$\int_{X\times\{0\}}\frac{\mathrm{Td}(TX,g_X)\,\mathrm{ch}(\xi,h_\xi)}{\mathrm{Td}([\Gamma],h_{[\Gamma]})}\log\|s_\Gamma\|^2|_{X\times\{0\}}+$$
$$\int_{\widetilde{X}_0}\widetilde{\mu}^*\widetilde{\mathrm{Td}}(\mathcal{F};\,g_U,(\mathit{\Pi}^\vee)^*g_X,g_H)\,q^*\mathrm{ch}(\xi,h_\xi)+$$
$$\int_{\widetilde{X}}\pi^*(\log\|\sigma_{[0]}\|^2)\ \{\widetilde{\nu}^*\mathrm{Td}(-c_1(L,g_L))-\widetilde{\mu}^*\mathrm{Td}(U,g_U)\}\ q^*\mathrm{ch}(\xi,h_\xi)+$$
$$\int_{\widetilde{X}}(q^*\log\|d\pi\|^2)\,\pi^*c_1([0],\|\cdot\|)\,\widetilde{\mu}^*\mathrm{Td}(U,g_U)\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(L,g_L))-1}{-c_1(L,g_L)}\right\}q^*\mathrm{ch}(\xi,h_\xi)$$
$$-\int_X\mathrm{Td}(X)\,R(X)\,\mathrm{ch}(\xi)+\int_F\mathrm{Td}(F)\,R(F)\,\mathrm{ch}(\xi).$$

Here $\widetilde{\mathrm{Td}}(\mathcal{F};\, g_U, (\Pi^\vee)^* g_X, g_H)$ *denotes the Bott-Chern secondary class* [BGS] *associated with the exact sequence of holomorphic vector bundles*

$$\mathcal{F}:\; 0 \to U \to (\Pi^\vee)^* TX \to H \to 0$$

and the Hermitian metrics g_U*,* $(\Pi^\vee)^* g_X$*,* g_H *such that*

$$dd^c \widetilde{\mathrm{Td}}(\mathcal{F};\, g_U, (\Pi^\vee)^* g_X, g_H) = (\Pi^\vee)^* \mathrm{Td}(TX, g_X) - \mathrm{Td}(U, g_U)\, \mathrm{Td}(H, g_H),$$

and $R(\cdot)$ *denotes the Gillet-Soulé genus* [S] *associated with the formal power series*

$$R(x) := \sum_{m \text{ odd} \geq 1} \left(2\zeta'(-m) + \zeta(-m)(1 + \frac{1}{2} + \cdots + \frac{1}{m}) \right) \frac{x^m}{m!}, \qquad \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Remark. Theorem 2 is a generalization of [B, Th. 5.12].

3.5. Examples.

Example 1: Critical points defined by a quadric polynomial of rank 2.

Let $0 \in \Delta$. Assume that for every $x \in \Sigma_\pi \cap X_0$, there exists a system of coordinates $(z_0, \ldots, z_n)$ centered at x such that

$$\pi(z) = z_0 z_1.$$

Hence $\Sigma_\pi \subset X$ is a complex submanifold of codimension 2. Let $N_{\Sigma_\pi/X}$ be the normal bundle of Σ_π in X. In [B, Def. 5.1, Prop. 5.2], Bismut introduced the additive genus $E(\cdot)$ associated with the generating function

$$E(x) := \frac{\mathrm{Td}(x)\,\mathrm{Td}(-x)}{2x} \left(\frac{\mathrm{Td}^{-1}(x) - 1}{x} + \frac{\mathrm{Td}^{-1}(-x) - 1}{x} \right), \qquad \mathrm{Td}^{-1}(x) := \frac{1 - e^{-x}}{x}.$$

Theorem 3 [B, Th. 5.9]**.** *The following identity holds:*

$$\log \|\sigma(t)\|^2_{\lambda(\xi),Q} \equiv \frac{1}{2} \left(\int_{\Sigma_\pi \cap X_0} -\mathrm{Td}(T\Sigma_\pi)\, E(N_{\Sigma_\pi/X})\, \mathrm{ch}(\xi) \right) \log |t|^2 \mod \mathcal{B}(S, 0).$$

Example 2: Isolated critical points.

Let $0 \in \Delta$. Assume that $\Sigma_\pi \cap X_0$ consists of isolated critical points. Hence $\mathrm{Sing}(X_0)$ consists of isolated critical points.

Since Σ_π is discrete, we may identify $\mathbb{P}(\Omega^1_X)$ and $\mathbb{P}(TX)$ with the trivial projective bundle on a neighborhood of $\Sigma_\pi \cap X_0$ by fixing a system of coordinates near $\Sigma_\pi \cap X_0$. Under this trivialization, we consider the Gauss maps ν and μ only on a small neighborhood of $\Sigma_\pi \cap X_0$. Then we have the following on a neighborhood of each $p \in \Sigma_\pi \cap X_0$:

$$\mu(z) = \nu(z) = \left(\frac{\partial \pi}{\partial z_0}(z) : \cdots : \frac{\partial \pi}{\partial z_n}(z) \right).$$

For a formal power series $f(x) \in \mathbb{C}[[x]]$, denote by $f(x)|_{x^m}$ the coefficient of x^m. Let $\mu(\pi, p) \in \mathbb{N}$ be the Milnor number of the isolated critical point p of π.

Theorem 4 [Y1, Main Th.]. *The following identity holds:*

$$\log \|\sigma\|^2_{\lambda(\xi),Q} \equiv \frac{(-1)^n}{(n+2)!}\,\mathrm{rk}(\xi) \sum_{p\in \mathrm{Sing}(X_0)} \mu(\pi,p)\,\log|t|^2 \mod \mathcal{B}(S,0).$$

Remark. In [B], the constant term of the asymptotic expansion was compared with the Quillen metric on the determinant of cohomologies of the *normalization* of X_0. It seems to be interesting to consider the same problem for general semi-stable degenerations.

§4. Sketch of the proofs of Theorems 1 and 2.

Notation Let

$N_t = N_{X_t/X}$: the normal bundle of X_t in X,

$N_t^* = N_{X_t/X}^*$: the conormal bundle of X_t in X,

$d\pi|_{X_t} \in H^0(X_t, N_t^*)$: a holomorphic section generating N_t^* for $t \in S^o$,

$h_{N_t^*}$: the Hermitian metric on N_t^* defined by

$$h_{N_t^*}(d\pi|_{X_t}, d\pi|_{X_t}) = 1,$$

h_{N_t} : the Hermitian metric on N_t induced from $h_{N_t^*}$.

When $t \in S$ is sufficiently close to $0 \in \Delta$, one has the following identity by the embedding theorem of Bismut-Lebeau [BL]:

$$\begin{aligned}\log\|\sigma_{KM}(t)\|^2_{Q,\lambda} = &\int_{X\times\{t\}} -\frac{\mathrm{Td}(TX,g_X)\,\mathrm{ch}(\xi,h_\xi)}{\mathrm{Td}([\Gamma],h_{[\Gamma]})}\log\|s_\Gamma\|^2|_{X\times\{t\}}\\ &+\int_{X_t}\widetilde{\mathrm{Td}}(\mathcal{E}_t;\,g_{X_t},g_X,h_{N_t})\,\mathrm{ch}(\xi,h_\xi) \qquad \text{(BL)}\\ &-\int_X \mathrm{Td}(X)\,R(X)\,\mathrm{ch}(\xi)+\int_F \mathrm{Td}(F)\,R(F)\,\mathrm{ch}(\xi).\end{aligned}$$

Here $\mathcal{E}_t$ is an exact sequence of holomorphic vector bundles on X_t

$$\mathcal{E}_t : 0 \longrightarrow TX_t \longrightarrow TX|_{X_t} \longrightarrow N_t \longrightarrow 0,$$

and $\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t})$ is the Bott-Chern secondary class associated with the Todd genus, $\mathcal{E}_t$, and Hermitian metrics g_{X_t}, g_X, h_{N_t}.

The first term of the r.h.s. of (BL) is continuous in $t \in S$. The second term of the r.h.s. of (BL) can be represented as

$$\begin{aligned}&\int_{X_t}\widetilde{\mathrm{Td}}(\mathcal{E}_t;\,g_{X_t},g_X,h_{N_t})\,\mathrm{ch}(\xi,h_\xi)\\ &=\int_{X_t}\widetilde{\mu}^*\widetilde{\mathrm{Td}}(\mathcal{F};\,g_U,(\Pi^\vee)^*g_X,g_H)\,q^*\mathrm{ch}(\xi,h_\xi)\\ &\quad+\int_{X_t}\widetilde{\mu}^*\mathrm{Td}(U,g_U)\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(L,g_L))-1}{-c_1(L,g_L)}\right\}q^*\mathrm{ch}(\xi,h_\xi)\,(q^*\log\|d\pi\|^2).\end{aligned}$$

Hence Theorems 1 and 2 follow from the following:

Claim. *Let φ be a ∂-closed and $\bar{\partial}$-closed C^∞ (n,n)-form on $\widetilde{X}$. Then*

$$\widetilde{\pi}_*(q^*(\log\|d\pi\|^2)\,\varphi)^{(0,0)}(t) - \left(\int_{E_0}\varphi\right)\log\|\sigma_{[0]}\|^2 \in \mathcal{B}(S,0).$$

Moreover,

$$\lim_{s\to 0}\left\{\int_{\widetilde{X}_s} q^*(\log\|d\pi\|^2)\,\varphi - \left(\int_{E_0}\varphi\right)\log\|\sigma_{[0]}(s)\|^2\right\}$$
$$= -\int_{\widetilde{X}}(\pi^*\log\|\sigma_{[0]}\|^2)\,\widetilde{\nu}^*c_1(L,g_L)\wedge\varphi + \int_{\widetilde{X}} q^*(\log\|d\pi\|^2)\,\varphi\wedge\pi^*c_1([0],\|\cdot\|). \quad \square$$

To prove Claim, the following result is used, which itself seems to be of interest.

Lemma. *Let $F(z)$ be a holomorphic function defined on a neighborhood Ω of $0\in\mathbb{C}^n$. Let $\chi(z)\in C_0^\infty(\Omega)$ and set*

$$\psi(t) := \int_{\mathbb{C}^n}\log|F(z)-t|^2\,\chi(z)\,d\mu, \qquad t\in\mathbb{C},$$

where $d\mu$ is the Lebesgue measure on $\mathbb{C}^n$. Then there exist $r_1,\dots,r_m\in\mathbb{Q}\cap(0,1]$ and $f_0(t), f_{l,k}(t)\in C^\infty(\mathbb{C})$, $l=1,\dots,m$,$k=0,\dots,n$, such that

$$\psi(t) = f_0(t) + \sum_{l=1}^{m}\sum_{k=0}^{n}|t|^{2r_l}(\log|t|)^k\,f_{l,k}(t), \qquad |t|\ll 1.$$

To prove this last lemma, a theorem of Barlet [Ba] plays a crucial role.

§5. An application to mirror symmetry.

5.1 Mirror quintics.

Let $p\colon\mathcal{X}\to\mathbb{P}^1$ be the pencil of quintic threefolds in $\mathbb{P}^4$:

$$\mathcal{X} := \{([z],\psi)\in\mathbb{P}^4\times\mathbb{P}^1;\ F_\psi(z)=0\}, \qquad p=\mathrm{pr}_2.$$

$$F_\psi(z) := z_0^5+z_1^5+z_2^5+z_3^5+z_4^5-5\psi\,z_0z_1z_2z_3z_4.$$

Identify $\mathbb{Z}_5$ with the set of fifth roots of unity:

$$\mathbb{Z}_5 = \{\alpha\in\mathbb{C};\ \alpha^5=1\}.$$

The group of projective transformations

$$G := \left\{\begin{pmatrix} a_0 & 0 & 0 & 0 & 0\\ 0 & a_1 & 0 & 0 & 0\\ 0 & 0 & a_2 & 0 & 0\\ 0 & 0 & 0 & a_3 & 0\\ 0 & 0 & 0 & 0 & a_4\end{pmatrix}\in PSL(5);\ a_i\in\mathbb{Z}_5\right\}\cong\mathbb{Z}_5^3$$

acts projectively on $p\colon\mathcal{X}\to\mathbb{P}^1$ and preserves the fibers of p.

Fact (Batylev, Morrison). *There exists a resolution*

$$q\colon \mathcal{W} \to \mathcal{X}/G$$

such that $q_\psi\colon W_\psi \to X_\psi/G$ *is a crepant resolution for* $\psi^5 \neq 1, \infty$*, i.e.,*

$$K_{W_\psi} = q_\psi^* K_{X_\psi/G} \cong \mathcal{O}_{W_\psi}.$$

The choice of a resolution as above is not unique. In what follows, we fix such a resolution. We set $\pi := p \circ q$. The family of Calabi-Yau threefolds

$$\pi\colon \mathcal{W} \to \mathbb{P}^1$$

is called a family of *mirror quintics*.

5.2 Predictions of Bershadsky-Cecotti-Ooguri-Vafa.

Set

$$\Omega_\psi := 5\psi \frac{z_4\, dz_0 \wedge dz_1 \wedge dz_2}{\partial F_\psi(z)/\partial z_3} \in H^3(W_\psi, \Omega_{W_\psi}).$$

Let

$$F_1^{\mathrm{top}} := \left(\frac{\psi}{\varpi_0}\right)^{\frac{62}{3}} (\psi^5 - 1)^{-\frac{1}{6}} \frac{d\psi}{dt},$$

where

$$\varpi_0 := \int_{\gamma_0} \Omega_\psi$$

and where the mapping

$$\psi \mapsto t := \frac{\int_{\gamma_1} \Omega_\psi}{\int_{\gamma_0} \Omega_\psi}$$

with certain $\gamma_0, \gamma_1 \in H_3(W_\psi, \mathbb{Z})$ is the *mirror map*. Then the mirror map is given by the following explicit formula: For $|\psi| \gg 1$,

$$q = e^{2\pi\sqrt{-1}t} = \exp\left(\frac{y_1(\psi)}{y_0(\psi)}\right),$$

where

$$y_0(\psi) = \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}},$$

$$y_1(\psi) = \log\frac{1}{5\psi} + 5\sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5}\left[\sum_{j=n+1}^{5n} \frac{1}{j}\right] \frac{1}{(5\psi)^{5n}}.$$

Define the BCOV torsion of W_ψ by

$$\tau_{BCOV}(W_\psi) := \prod_{p\geq 0} \tau(W_\psi, \overline{\Omega}^p_{W_\psi})^{(-1)^p p},$$

where W_ψ is equipped with a Ricci-flat Kähler metric with volume 1.

Conjecture (Bershadsky-Cecotti-Ooguri-Vafa [BCOV1,2]**).**
(1) *The following identity holds*

$$F_1^{\mathrm{top}} = \mathrm{Const.} \left\{ \prod_{s=1}^{\infty} \eta(q^s)^{d_s} (1-q^s)^{\frac{n_s}{12}} \right\}^{-2}$$

where

$$d_r = \#\{\text{elliptic curves of degree } d \text{ in generic quintic} \subset \mathbb{P}^4\},$$

$$n_s = \#\{\text{rational curves of degree } d \text{ in generic quintic} \subset \mathbb{P}^4\}.$$

(2) *The following identity holds*

$$\begin{aligned}\tau_{BCOV}(W_\psi) &= \mathrm{Const.} \left\| \psi^{-\frac{62}{3}} (\psi^5 - 1)^{\frac{1}{6}} \, \Omega_\psi^{\frac{62}{3}} \otimes \frac{d}{d\psi} \right\|^2 \\ &= \mathrm{Const.} \left\| (F_1^{\mathrm{top}})^{-1} \left(\frac{\Omega_\psi}{\varpi_0} \right)^{\frac{62}{3}} \otimes \frac{d}{dt} \right\|^2,\end{aligned}$$

where the norm associated with the L^2-metric on the Hodge bundle and the Weil-Petersson metric on the tangent bundle of the moduli space are considered in the right hand side.

Remark. In Conjecture (1), the numbers d_r and n_s should be understood as the Gromov-Witten invariants of genus 1 and 0, respectively.

By BCOV's predictions (1), (2), the Gromov-Witten invariants of genus 0 and 1 for generic quintics in $\mathbb{P}^4$ and the BCOV torsion of the mirror quintics are expected to satisfy the following relation:

$$\tau_{BCOV}(W_\psi) = \mathrm{Const.} \left\| \left\{ \prod_{s=1}^{\infty} \eta(q^s)^{d_s} (1-q^s)^{\frac{n_s}{12}} \right\}^2 \left(\frac{\Omega_\psi}{\varpi_0} \right)^{\frac{62}{3}} \otimes \frac{d}{dt} \right\|^2,$$

where

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind η-function.

As an application of Theorem 1, we can prove:

Theorem 5 [FLY]**.** *BCOV's Conjecture* (2) *holds.*

Remark. BCOV's Conjecture (1) is studied by J. Li and A. Zinger [LZ], in which Conjecture (1) is verified when $s \leq 4$.

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On certain classes of nonlinear differential equations

Shun Shimomura *

1 Introduction

Consider the n-th order nonlinear differential equation

$$(1.1)_n \qquad y^{(n)} = R(x, y, y', ..., y^{(n-1)})$$

($' = d/dx$), where the right-hand member is a rational function of x, y and its derivatives with complex coefficients. It is well-known ([2, §§3.2, 3.3], [3, §12.5]) that, for each solution of first order equation $(1.1)_1$, every movable singularity is at most an algebraic branch point; where a movable singularity means one depending on initial data. In particular, $(1.1)_1$ admits the *Painlevé property*, namely, for each solution, every movable singularity is a pole, if and only if it is of Riccati type (see [2], [3], [5, Theorem 10.2]). Now let us say that equation $(1.1)_n$ admits the *quasi-Painlevé property* if, for each solution, every movable singularity is at most an algebraic branch point. For convenience, we regard a pole as a special case of algebraic branch point. As mentioned above, all the first order equations of the form $(1.1)_1$ admit the quasi-Painlevé property. However, for second order equations, the situation is different. For example, the equation

$$y'' = (1 - \kappa - \kappa y^\kappa)(y')^2/y, \qquad \kappa \in \mathbb{Z} \setminus \{0\}$$

has a general solution $y = (C_1 + \log(x - C_2))^{1/\kappa}$ with a logarithmic branch point $x = C_2$; and

$$y'' = (1 + i)(y')^2/y \qquad (i = \sqrt{-1}),$$

has a general solution $y = C_1(x - C_2)^i$ with an essential singularity $x = C_2$ (for movable essential singularities of second order equations, see [4]). The equation

$$(1.2) \qquad y'' = 6y^2 - g_2/2$$

*Department of Mathematics, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
e-mail: shimomur@math.keio.ac.jp

is a typical example admitting the Painlevé property. Indeed a general solution is expressed by $y = \wp(x - C_1, g_2, C_2)$ or by an exponential function, where $\wp(x, g_2, g_3)$ $(g_2^3 - 27g_3^2 \neq 0)$ is the Weierstrass $\wp$-function

$$\wp(x, g_2, g_3) = x^{-2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \big((x - \Omega_{mn})^{-2} - \Omega_{mn}^{-2}\big),$$

$\Omega_{mn} = m\omega_1(g_2, g_3) + n\omega_2(g_2, g_3), \mathrm{Im}(\omega_2(g_2, g_3)/\omega_1(g_2, g_3)) > 0$. Also it is well-known that the Painlevé equations admit the Painlevé property. The first and the second Painlevé equations are

$$\text{(PI)} \qquad y'' = 6y^2 + x,$$

$$\text{(PII)} \qquad y'' = 2y^3 + xy + \alpha.$$

Equation (1.2) is a special case of the following

$$\text{(1.3)} \qquad y'' = a_0 y^{2k} + b_0, \quad a_0 \neq 0, \quad k \in \mathbb{N},$$

which has a general solution expressed by the Abelian integral

$$x - C_0 = \int_{y_0}^{y} \frac{ds}{\sqrt{2a_0(2k+1)^{-1}s^{2k+1} + 2b_0 s + C}}.$$

Every movable singularity is an algebraic branch point; namely (1.3) admits the quasi-Painlevé property. We present a discrete class of second order equations with the quasi-Painlevé property which contains (PI) as a special case; and we show some basic properties of solutions.

2 Results

Consider an equation of the form

$$(\mathrm{E}_k) \qquad y'' = \frac{2(2k+1)}{(2k-1)^2} y^{2k} + x,$$

where $k \in \mathbb{N}$. Equation (E_1) is nothing but (PI).

Theorem 2.1. *For every $k \in \mathbb{N}$, equation (E_k) admits the quasi-Painlevé property.*

For each solution, an expression around a movable singularity is given by

Theorem 2.2. *Let $y(x)$ be an arbitrary solution of (E_k), and suppose that x_0 is a movable algebraic branch point of $y(x)$. Then, around $x = x_0$,*

$$\text{(2.1)} \quad y(x) = \xi^{-2/(2k-1)} - \frac{(2k-1)^2}{2(6k-1)} x_0 \xi^2 + c\xi^{4k/(2k-1)} + \frac{(2k-1)^2}{2(2k-3)(4k-1)} \xi^3 + \sum_{j \geq 6k-2} c_j \xi^{j/(2k-1)}, \quad \xi = x - x_0,$$

where c is an integration constant, c_j $(j \geq 6k-2)$ are polynomials of c and x_0, and $\xi^{1/(2k-1)}$ denotes an arbitrary branch of σ such that $\sigma^{2k-1} = \xi$.

Every solution of (PI) is transcendental, and there exists no entire solution. These facts are extended as follows:

Theorem 2.3. *Equation* (E_k) *admits no entire solution. Moreover, if* $k \geq 2$, *then equation* (E_k) *admits no meromorphic solution.*

Theorem 2.4. *Every solution of* (E_k) *is transcendental.*

As mentioned above, if $k \geq 2$, an arbitrary solution of (E_k) is a many-valued function, but is not an algebraic one. For the many-valuedness, we have

Theorem 2.5. *Suppose that* $k \geq 2$. *For any* $\nu \in \mathbb{N}$, *equation* (E_k) *admits a two-parameter family of solutions which are at least* ν*-valued.*

Furthermore the equations

$$y'' = \frac{k+1}{k^2} y^{2k+1} + xy^k + \alpha$$

constitute a class containing (PII) as a special case, and have analogous properties.

3 Lemmas

Consider the system of differential equations

$$dv_1/dt = F_1(t, v_1, v_2), \qquad dv_2/dt = F_2(t, v_1, v_2), \tag{3.1}$$

where $F_l(t, v_1, v_2)$ $(l = 1, 2)$ are analytic in a neighbourhood of $(a_0, b_0, c_0) \in \mathbb{C}^3$. Then we have the following lemma ([2, §3.2], [3, §12.3]).

Lemma 3.1. *Let* C $(\subset \mathbb{C})$ *be a curve with finite length terminating in* $t = a_0$. *Suppose that a solution* $(v_1, v_2) = (\varphi(t), \psi(t))$ *of* (3.1) *has the properties below:*

(i) *for each point* $\tau \in C \setminus \{a_0\}$, $\varphi(t)$ *and* $\psi(t)$ *are analytic at* $t = \tau$;

(ii) *there exists a sequence* $\{a_\nu\}_{\nu \in \mathbb{N}} \subset C \setminus \{a_0\}$, $a_\nu \to a_0$ $(\nu \to \infty)$ *such that* $(\varphi(a_\nu), \psi(a_\nu)) \to (b_0, c_0)$.

Then, $\varphi(t)$ *and* $\psi(t)$ *are analytic at* $t = a_0$.

The following lemma due to Clunie is useful in the study of nonlinear differential equations (see [5, Lemma 2.4.2]).

Lemma 3.2. *Suppose that the differential equation* $w^{p+1} = P(z, w)$, $p \in \mathbb{N}$ *admits a meromorphic solution* $w = f(z)$, *where* $P(z, w)$ *is a polynomial of* $z, w, w', ..., w^{(q)}$. *If the total degree of* $P(z, w)$ *with respect to* w *and its derivatives does not exceed* p, *then* $m(r, f) = O(\log T(r, f) + \log r)$ *as* $r \to \infty$, $r \notin E$, *where* $E \subset (0, \infty)$ *is an exceptional set with finite linear measure.*

4 Proofs of Theorems 2.1 and 2.2

Let $y(x)$ be an arbitrary solution of (E_k).
Theorem 2.2 is obtained by substituting a Puiseux series into (E_k) and determining its coefficients.

Around a movable singular point $x = x_0$, we write (2.1) in the form

$$(4.1)\quad y(x) = \xi^{-2/(2k-1)} - \frac{(2k-1)^2}{2(6k-1)} x\xi^2 + c\xi^{4k/(2k-1)} + \frac{(2k-1)^4}{(2k-3)(4k-1)(6k-1)}\xi^3 + \cdots,$$

and hence

$$(4.2)\quad y'(x) = -\frac{2}{2k-1}\xi^{-(2k+1)/(2k-1)} - \frac{(2k-1)^2}{6k-1}x\xi + \frac{4kc}{2k-1}\xi^{(2k+1)/(2k-1)} + \frac{(2k-1)^2(16k^2-10k+3)}{2(2k-3)(4k-1)(6k-1)}\xi^2 + \cdots.$$

Denote by $\pm u(x)$ each branch defined by $y(x) = u(x)^{-2}$ around $x = x_0$. From (4.1) and (4.2), we have

$$y'(x) = \mp \frac{2}{2k-1} u(x)^{-(2k+1)} \Big[1 + \frac{(2k-1)^2}{4} x u(x)^{4k} - \frac{6k+1}{2} c u(x)^{4k+2} + \cdots\Big] + \frac{(2k-1)^2}{2(2k-3)} u(x)^{4k-2}.$$

Viewing these identities, we define new unknowns u, v by

$$(4.3)\qquad y = u^{-2},$$

$$(4.4)\qquad y' = \mp\frac{2u^{-(2k+1)}}{2k-1}\Big[1 + \frac{(2k-1)^2}{4} x u^{4k} + u^{4k+2} v\Big] + \frac{(2k-1)^2}{2(2k-3)} u^{4k-2}.$$

Then, equation (E_k) is equivalent to the system

$$\frac{du}{dx} = \pm u^{-2k+2}\Phi_\pm(x,u,v), \qquad \frac{dv}{dx} = \mp u^{2k-1}\Psi_\pm(x,u,v),$$

with

$$\Phi_\pm(x,u,v) = \frac{1}{2k-1}\Big[1 + \frac{(2k-1)^2}{4} x u^{4k} + u^{4k+2} v \mp \frac{(2k-1)^3}{4(2k-3)} u^{6k-1}\Big],$$

$$\Psi_\pm(x,u,v) = \Big[\frac{(2k-1)^3}{4} x + (2k+1)u^2 v \mp \frac{(2k-1)^4}{2(2k-3)} u^{2k-1}\Big] \times \Big[\frac{2k-1}{4} x + \frac{u^2 v}{2k-1} \mp \frac{(2k-1)^2}{4(2k-3)} u^{2k-1}\Big].$$

For each solution $(u,v)=(u(x),v(x))$ corresponding to the solution $y(x)$ of (E_k), we regard (x,v) as a function of u; which is a solution of the system

$$\frac{dx}{du}=\pm\frac{u^{2k-2}}{\Phi_\pm(x,u,v)},\quad \frac{dv}{du}=-\frac{u^{4k-3}\Psi_\pm(x,u,v)}{\Phi_\pm(x,u,v)}. \tag{4.5}$$

From (4.3) and (4.4), we have

$$\Big[y'-\frac{(2k-1)^2}{2(2k-3)}y^{-(2k-1)}\Big]^2=\frac{4y^{2k+1}}{(2k-1)^2}\Big[1+\frac{(2k-1)^2}{4}xy^{-2k}+y^{-(2k+1)}v\Big]^2,$$

which is written in the form

$$\begin{aligned}V=\frac{4y^{-(2k+1)}}{(2k-1)^2}v^2+\Big(\frac{8}{(2k-1)^2}+2xy^{-2k}\Big)v\\ +\frac{(2k-1)^2}{4}x^2y^{-2k+1}-\frac{(2k-1)^4}{4(2k-3)^2}y^{-2(2k-1)}\end{aligned} \tag{4.6}$$

with

$$V=(y')^2-\frac{(2k-1)^2}{2k-3}y^{-(2k-1)}y'-\frac{4y^{2k+1}}{(2k-1)^2}-2xy. \tag{4.7}$$

Substituting the solution $y(x)$ of (E_k) into (4.7), we get the auxiliary function $V(x)$ with the following property.

Proposition 4.1. *If $y(x)^{-1}$ is bounded along a curve Γ, then $V(x)$ is also bounded along Γ.*

Derivation of Theorem 2.1. Suppose that $y(x)$ admits a singular point $x=a_0$, and let C be a segment terminating in $x=a_0$ such that each point on $C\setminus\{a_0\}$ is at most an algebraic branch point of $y(x)$. For each algebraic branch point on $C\setminus\{a_0\}$, replacing a part of C around it by a suitable small semi-circle, we get a curve Γ with finite length terminating in a_0 such that $y(x)$ is analytic along $\Gamma\setminus\{a_0\}$. According to the value $A=\liminf_{x\to a_0,\,x\in\Gamma}|y(x)|$, we divide into three cases: (i) $0<A<\infty$, (ii) $A=\infty$, (iii) $A=0$.

Case (i): $0<A<\infty$. By Proposition 4.1, the function $V(x)$ is bounded along Γ near $x=a_0$. Take a sequence $\{a_n\}_{n\in\mathbb{N}}\subset\Gamma$ such that $a_n\to a_0$ and that $y(a_n)\to y_0$ $(\neq 0,\infty)$. By (4.7), $\{y'(a_n)\}_{n\in\mathbb{N}}$ is also bounded, and hence there exists a subsequence $\{\tilde{a}_n\}_{n\in\mathbb{N}}\subset\Gamma$ satisfying $\tilde{a}_n\to a_0$, $y(\tilde{a}_n)\to y_0$, $y'(\tilde{a}_n)\to y_1$ $(\neq\infty)$. By Lemma 3.1, $y(x)$ is analytic at $x=a_0$.

Case (ii): $A=\infty$. Since $y(x)\to\infty$ as $x\to a_0$ along Γ, the function $V(x)$ is bounded along C near $x=a_0$. Note that (4.6) is a quadratic equation with respect to v, which admits two solutions $v_+(x)$ and $v_-(x)$ analytic along $\Gamma\setminus\{a_0\}$; one is bounded along Γ and the other tends to ∞ as $x\to a_0$ along Γ. Suppose that $v_-(x)$ is bounded along Γ, and let $u_-(x)$ be the branch of $u(x)=y(x)^{-1/2}$ corresponding to $v_-(x)$. By (4.7), $|u_-'(x)|=|y'(x)y(x)^{-3/2}/2|\sim(2k-1)^{-1}|y(x)^{k-1}|\neq 0,\infty$ along

$\Gamma \setminus \{a_0\}$. Denote by $x = x(u)$ the inverse function of $u = u_-(x)$. Then, $x = x(u)$ and $v = v_-(x(u))$ are analytic functions of u along $u_-(\Gamma)\setminus\{0\} = \{u = u_-(x) \,|\, x \in \Gamma \setminus \{a_0\}\}$ with the properties:

(ii.a) $x(u) \to a_0$ as $u \to u_-(a_0) = 0$ along $u_-(\Gamma)$;

(ii.b) $v_-(x(u))$ is bounded along $u_-(\Gamma)$.

Choose a sequence $\{b_n\}_{n\in\mathbb{N}} \subset u_-(\Gamma)$ satisfying $b_n \to u_-(a_0) = 0$, $x(b_n) \to a_0$, $v_-(x(b_n)) \to v_0$ ($\neq \infty$). By the fact that $(x(u), v_-(x(u)))$ is a solution of (4.5) along $u_-(\Gamma) \setminus \{0\}$, and by Lemma 3.1, the function $x(u)$ is analytic at $u = 0$, which implies that $x = a_0$ is at most an algebraic branch point of $y(x) = u(x)^{-2}$.

Case (iii): $A = 0$. By the same argument as in the proof of the Painlevé property of (PI), we can reduce this case to either (i) or (ii) (see [6]). Consequently $x = a_0$ is at most an algebraic branch point of $y(x)$.

5 Proof of Theorem 2.3

Suppose that (E_k) admits an entire solution $y_*(x)$. If $y_*(x)$ is a polynomial, then $y_*(x) = c_0x^d + o(x^{d-1})$, $d \in \mathbb{N}\cup\{0\}$, $c_0 \neq 0$ around $x = \infty$. Substituting this into (E_k), we get $2dk = 1$, which is a contradiction. Hence $y_*(x)$ is transcendental and entire. Observe that $m(r, y_*) = T(r, y_*)$. By Lemma 3.2, there exists a positive number K_0 such that $T(r, y_*) \leq K_0 \log r$ outside an exceptional set E_0 with total length $\mu_0 < \infty$. For each r, we may choose a number $r'(r) \geq r$ satisfying $r'(r) - r \leq 2\mu_0$ and $r'(r) \notin E_0$. Hence

$$T(r, y_*) \leq T(r'(r), y_*) \leq K_0 \log(r'(r)) \leq K_0 \log(r + 2\mu_0) = O(\log r)$$

for $r > 0$, which contradicts the transcendency of $y_*(x)$. This implies that (E_k) admits no entire solution. Furthermore, by Theorem 2.2, for $k \geq 2$, every solution of (E_k) has no pole, so that there exists no meromorphic solution.

6 Proof of Theorem 2.4

Suppose that (E_k) admits an algebraic solution $\tilde{y}(x)$. For each branch point $x_\iota \neq \infty$, the degree of ramification is $e_\iota - 1 = 2k - 2$. Furthermore, at $x = \infty$,

$$\tilde{y}(x) = \gamma_k x^{1/(2k)} + \sum_{j=0}^{\infty} b_j x^{-j/(2k)}, \quad \gamma_k \neq 0,$$

which implies that the degree of ramification for $x = \infty$ is $e_\infty - 1 = 2k - 1$. These facts contradict the Riemann-Hurwitz formula

$$2(1 - g) = 2n - \sum_{\iota\neq\infty}(e_\iota - 1) - (e_\infty - 1).$$

Therefore (E_k) admits no algebraic solution.

7 Proof of Theorem 2.5

Suppose that the solution $y(x)$ of (E_k) satisfies the initial condition $y(0) = y_0$, $y'(0) = y_1$. For $\varepsilon > 0$, by $y = ((2k-1)^2/4)^{1/(2k-1)}\varepsilon^{-1}Y$ and $x = \varepsilon^{(2k-1)/2}t$, equation ($\mathrm{E}_k$) and $y(x)$ are changed into, respectively,

$$\ddot{Y} = (k+1/2)Y^{2k} + (4(2k-1)^{-2})^{1/(2k-1)}\varepsilon^{3k-1/2}t \tag{7.1}$$

($\dot{} = d/dt$) and its solution $Y_\varepsilon(t)$ satisfying

$$\begin{aligned} Y_\varepsilon(0) &= \chi_0(\varepsilon) := (4(2k-1)^{-2})^{1/(2k-1)}\varepsilon y_0, \\ \dot{Y}_\varepsilon(0) &= \chi_1(\varepsilon) := (4(2k-1)^{-2})^{1/(2k-1)}\varepsilon^{(2k+1)/2}y_1. \end{aligned}$$

Putting $\varepsilon = 0$ in (7.1), we have

$$\ddot{Y} = (k+1/2)Y^{2k}. \tag{7.2}$$

The solution $Y_0(t)$ of (7.2) with $Y_0(0) = \chi_0(\varepsilon)$, $\dot{Y}_0(0) = \chi_1(\varepsilon)$ satisfies

$$\dot{Y}^2 = Y^{2k+1} + \chi_1(\varepsilon)^2 - \chi_0(\varepsilon)^{2k+1}. \tag{7.3}$$

If $\chi_1(\varepsilon)^2 - \chi_0(\varepsilon)^{2k+1} \neq 0$, then $Y_0(t)$ is expressible by the Abelian integral

$$t = \int_{\chi_0(\varepsilon)}^{Y_0(t)} \frac{ds}{\sqrt{s^{2k+1} + \chi_1(\varepsilon)^2 - \chi_0(\varepsilon)^{2k+1}}},$$

and hence $Y_0(t)$ is infinitely many-valued. Using this fact and an asymptotic property of (7.1), we derive the conclusion of the theorem.

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AN EXAMPLE RELATED TO BRODY'S THEOREM

JÖRG WINKELMANN

Abstract. We discuss an example related to the method of Brody.

1. Introduction

1.1. **Bloch principle.** In one-dimensional function theory there is a general philosophy which supposedly goes back to A. Bloch (see e.g. [12],[2]): If P is a sufficiently reasonable class of holomorphic maps or functions, then the following statements should be equivalent:

(1) Every map in class P defined on the complex line $\mathbb{C}$ is constant.
(2) The set of all maps in class P defined on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is a *normal family*.

(A family of maps is called a "normal family" if every sequence in it is either compactly divergent or contains a subsequence which converges uniformly on compact sets. A sequence of maps $f_n : X \to Y$ between topological spaces is "compactly divergent", if for every pair of compact subsets $K \subset X$, $C \subset Y$ there are only finitely many f_n with $f_n(K) \cap C \neq \{\}$.)

For example, every bounded holomorphic function on $\mathbb{C}$ is constant by Liouville's theorem and due to Montel's theorem the family of all bounded holomorphic functions on Δ is a normal family. Thus the Bloch principle is valid for the family P of all bounded holomorphic functions with values in $\mathbb{C}$.

1.2. **Brody's theorem.** Let Y be a complex manifold. It is called "taut" if the family of all holomorphic maps $f : \Delta \to Y$ is a normal family. Let us from now on assume that Y is compact. Then being "taut" is easily seen to be equivalent with hyperbolicity in the sense of Kobayashi. The theorem of Brody (see [3]) states that this is furthermore equivalent with the property that every holomorphic map from $\mathbb{C}$ to Y is constant. In other words: Brody's theorem states that the

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Bloch principle hold for the class of holomorphic maps with values in a (fixed) compact complex manifold Y.

Now we may raise the question: What about holomorphic maps to a compact complex manifold fixing some given base points? Given a compact complex manifold Y and a point $y \in Y$, let us consider the following two statements:

- Every holomorphic map $f : \mathbb{C} \to Y$ with $f(0) = y$ is constant.
- The family of all holomorphic maps $f : \Delta \to Y$ with $f(0) = y$ is a normal family.

Are they equivalent?

Using the notion of the infinitesimal Kobayashi-Royden pseudometric as introduced in [10] this can be reformulated into the following question: *"If the infinitesimal Kobayashi-Royden peusdometric on a compact complex manifold Y degenerates for some point $y \in Y$, does this imply that there exists a holomorphic map $f : \mathbb{C} \to Y$ with $y \in f(\mathbb{C})$?"*

Thanks to Brody's theorem it is clear that there exists some non-constant holomorphic map $f : \mathbb{C} \to Y$ if the Kobayashi-Royden pseudometric is degenerate at some point y of Y. But it is not clear that f can be chosen in such a way that y is in the image or at least in the closure of the image. Of course, at first it looks absurd that degeneracy of the Kobayashi-Royden pseudometric at one point y should only imply the existence of a non-constant holomorphic map to some part of Y far away of y and should not imply the existence of a non-constant map $f : \mathbb{C} \to Y$ whose image comes close to y.

Thus one is led to postulate

Conjecture. *Let X be a compact complex manifold, $x \in X$. Assume that the infinitesimal Kobayashi-Royden pseudometric is degenerate on T_xX.*

Then there exists a non-constant holomorphic map $f : \mathbb{C} \to X$ with $f(0) = x$.

1.3. **Bounded derivatives.** Let X be a complex manifold equipped with a hermitian metric h. For each holomorphic map $f : \mathbb{C} \to X$ and each point $z \in \mathbb{C}$ we may now calculate the norm of the derivatie Df at z with respect to the euclidean metric on $\mathbb{C}$ and h on X. Let P be a class of holomorphic maps $f : \mathbb{C} \to (X, h)$ with bounded derivatives (i.e. for every $f \in P$ there is a number $C > 0$ such that the inequality $||Df_z|| < C$ holds for all $z \in \mathbb{C}$). Let $f : \mathbb{C} \to X$ be a non-constant map in this class P. Via $f_n(x) = f(nx)$ this map f yields a non-normal family of maps $f_n : \Delta \to X$.

Now let P' denote the set of those maps in P for which the derivative (calculated with respect to the euclidean metric on $\mathbb{C}$ resp. Δ and the hermitian metric on X) is bounded. For each of the f_n defined above the derivative is clearly bounded, since Δ is relatively compact in $\mathbb{C}$, and $f_n : \mathbb{C} \to X$ extends through the boundary. Thus f_n is a non-normal family in P'. If the Bloch principle holds for P', this implies the existence of a non-constant holomorphic map $F : \mathbb{C} \to X$ in P.

Thus: *If the Bloch principle holds for P', the existence of a non-constant holomorphic map f in P implies the existence of a non-constant holomorphic map F in P with the additional property that $||DF||$ is bounded.*

Brody's theorem implies that this is indeed true if, given a compact complex manifold X, we consider the set P of all holomorphic maps with values in X.

However, we will give an example of a compact complex manifold X, an open subset Ω and a point $x \in \Omega$ such that this property does not hold if P is chosen as the family of all holomorphic maps f with image contained in Ω and $f(0) = x$.

1.4. **Reparametrization.** The key method for proving a Bloch principle is the following: Let $f_n : \Delta \to Y$ be a non-normal family. Then we look for an increasing sequence of disk Δ_{r_n} which exhausts $\mathbb{C}$ (i.e. $\lim r_n = +\infty$) and a sequence of holomorphic maps $\alpha_n : \Delta_{r_n} \to \Delta$ such that a subsequence of $f_n \circ \alpha_n$ converges (locally uniformly) to a non-constant holomorphic map from $\mathbb{C}$ to Y.

For the proof of his theorem Brody used this idea, taking combinations of affine-linear maps with automorphisms of the disk for the α_n.

Zalcman ([12]) investigated other reparametrizations where the α_n themselves are affin-linear maps, a concept which has the advantage that it can also be applied to harmonic maps.

1.5. **Subvarieties of abelian varieties.** Let A be a complex abelian variety (i.e. a compact complex torus which is simultaneously a projective algebraic variety) and X a subvariety. Let E denote the union of all translates of complex subtori of A which are contained in X. It is known that this union is either all of X or a proper algebraic subvariety ([6]).

Since A is a compact complex torus there is a flat hermitian metric on A induced by the euclidean metric on $\mathbb{C}^g$ via $A \simeq \mathbb{C}^g/\Gamma$. A holomorphic map $f : \mathbb{C} \to A$ has bounded derivative with respect to this metric if and only if it is induced by an affine-linear map from $\mathbb{C}$ to $\mathbb{C}^g$.

From this, one can deduce that $f(\mathbb{C}) \subset E$ for every holomorphic map $f : \mathbb{C} \to X$ with bounded derivative. Given the previous considerations about the Bloch principle, it is thus natural to conjecture:

Conjecture. *For every non-constant holomorphic map $f : \mathbb{C} \to X$ the image is contained in E. The Kobayashi-pseudodistance on X is a distance outside E.*

For example, this statement is a consequence of the more general conjecture VIII.I.4 by S. Lang in [9]. In the context of classification theory the above statement has also be conjectured by F. Campana ([4],§9.3).

In the spirit of the analogue between diophantine geometry and entire holomorphic curves as pointed out by Vojta [11], the conjecture above is also supported by the famous result of Faltings ([5]) with which he solved the Mordell conjecture. This result states the following: If we assume that A und X are defined over a number field K, then with only finitely many exceptions every K-rational point of X is contained in E.

1.6. **Our example.** We construct an example of the following type: There is a compact complex manifold X, equipped with some hermitian metric, an open subset Ω and a point $p \in \Omega$. There exists a non-constant holomorphic map $f : \mathbb{C} \to \Omega$ with $f(0) = p$. Via $f_n(z) = f(nz)$ this yields a non-normal family of holomorphic maps $f_n : \Delta \to \Omega$ with bounded derivatives such that $f_n(0) = p$.

But there is no non-constant holomorphic map $f : \mathbb{C} \to \Omega$ with $f(0) = p$ *and bounded derivative.*

2. The example

2.1. **Statement of main results.** We construct an example which shows that Brody reparametrization sometimes necessarily changes the image of the curve.

Theorem 1. *There exists a compact complex hermitian manifold (T, h) and open subsets $\Omega_2 \subset \Omega_1 \subset T$ such that:*

1. *Ω_2 is not dense in Ω_1 and neither is Ω_1 in T.*
2. *For every point $p \in \Omega_1$ there is a non-constant holomorphic map $f : \mathbb{C} \to \Omega_1$ with $p = f(0)$.*
3. *If $f : \mathbb{C} \to T$ is a non-constant holomorphic map with bounded derivative (with respect to the euclidean metric on $\mathbb{C}$ and h on T) and $f(\mathbb{C}) \subset \bar{\Omega}_1$, then $f(\mathbb{C}) \subset \bar{\Omega}_2$.*

Recall that Brody's method, starting from any holomorphic map from $\mathbb{C}$ to T, yields a holomorphic map from $\mathbb{C}$ to T with bounded derivative. Thus this examples provides a picture in which Brody's method really changes the properties of $f : \mathbb{C} \to T$ fundamentally.

Responding to some additional questions which may be asked, we prove a little bit more.

Theorem 2. *There exists a compact complex torus T, equipped with a flat hermitian metric h and open subsets $\Omega_2 \subset \Omega_1 \subset T$ such that:*

(1) *Ω_2 is not dense in Ω_1 and neither is Ω_1 in T.*
(2) *For every point $p \in \Omega_1$ and every $v \in T_p\Omega_1$ there is a non-constant holomorphic map $f : \mathbb{C} \to \Omega_1$ with $p = f(0)$, $v = f'(0)$ and $\bar{\Omega}_1 = \overline{f(\mathbb{C})}$.*
(3) *If $f : \mathbb{C} \to T$ is a non-constant holomorphic map with bounded derivative (with respect to the euclidean metric on $\mathbb{C}$ and h on T) and $f(\mathbb{C}) \subset \bar{\Omega}_1$, then $f(\mathbb{C}) \subset \bar{\Omega}_2$. Moreover f is affine-linear and $\overline{f(\mathbb{C})}$ is a closed analytic subset of T.*

We remark that this implies in particular that the infinitesimal Kobayashi-Royden pseudometric vanishes identically on Ω_1.

Furthermore, it provides examples of holomorphic maps from $\mathbb{C}$ into a compact complex torus with a rather "bad" image: The closure of the image with respect to the euclidean topology is $\overline{\Omega_2}$ and thus a set with non-empty interior such that the complement has also non-empty interior. This is in strong contrast to the Zariski-analytic closure: By the theorem of Green-Bloch-Ochiai for every holomorphic map f from $\mathbb{C}$ to a compact complex torus T the closure of the image $f(\mathbb{C})$ with respect to the analytic Zariski topology (i.e. the smallest closed analytic subset of T containing $f(\mathbb{C})$) is always a translated subtorus of T.

We will now describe our example.

We precede the construction with some elementary observations about tori: Let $T = \mathbb{C}^n/\Lambda$ be a torus, equipped with the flat euclidean metric and the corresponding distance function $d_T(\ ,\ )$. Let

$$\rho = \frac{1}{2} \min_{\gamma \in \Lambda \setminus \{0\}} ||\gamma||.$$

This is the *injectivity radius*, in other words ρ is the largest real number such that the natural projection $\pi : \mathbb{C}^n \to T$ induces a homeomorphism between the ball

$$B_\epsilon(\mathbb{C}^n; 0) = \{v \in \mathbb{C}^n : ||v|| < \epsilon\}$$

and

$$B_\epsilon(T; e) = \{x \in T : d_T(x, e) < \epsilon\}$$

for all $\epsilon < \rho$. Evidently, the injectivity radius ρ is a lower bound for the *diameter*

$$\rho \leq diam = \max_{x,y \in T} d_T(x,y)$$

If we pass from T to a subtorus $S \subset T$, the injectivity radius can only increase, while the diameter can only decrease. As a consequence we obtain:

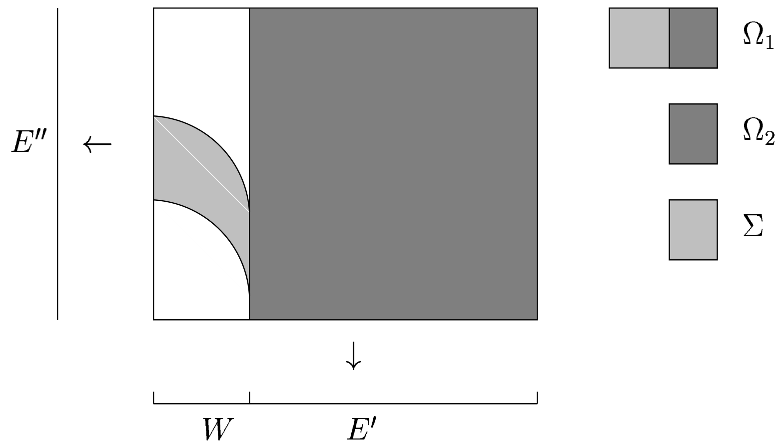
Lemma 1. *Let T be a compact (real or complex) torus with injectivity radius ρ. Then for every real positive-dimensional subtorus $S \subset T$ the diameter*

$$diam(S) = \max_{x,y \in S} d_T(x,y)$$

is at least ρ.

Furthermore, if $0 < \epsilon < \rho$ and $x \in T$, then the ball $B_\epsilon(T;x)$ contains no translate of any positive-dimensional real subtorus of T.

Before giving the details of the construction of our example, let us try to express its idea in a drawing:



Now let us start the precise construction of the example. Let $E' = \mathbb{C}/\Gamma'$ and $E'' = \mathbb{C}/\Gamma''$ be elliptic curves and $T = E' \times E''$. Let $\pi' : \mathbb{C} \to E'$, $\pi'' : \mathbb{C} \to E''$ and $\pi = (\pi', \pi'') : \mathbb{C}^2 \to T$ denote the natural projections. We assume that E' is not isogenous to E''. (For example, we might choose $E' = \mathbb{C}/\mathbb{Z}[i]$ and $E'' = \mathbb{C}/\mathbb{Z}[\sqrt{2}i]$.) Then $E' \times \{0\}$ and $\{0\} \times E''$ are the only non-trivial complex subtori of T.

Now $T = \mathbb{C}^2/\Gamma$ with $\Gamma = \Gamma' \times \Gamma''$. The compact complex torus T carries a hermitian metric h induced by the euclidean metric on $\mathbb{C}^2$ (i.e. $h = dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2$). The associated distance function is called d, the injectivity radius ρ is defined as explained above.

We choose numbers $0 < \rho' < \rho'' < \rho$ and define $W = B_{\rho'}(E', e)$.

Furthermore we choose $0 < \delta < \frac{1}{2}\rho$ and we choose a holomorphic map $\sigma : \mathbb{C} \to E''$ such that there exist complex numbers $t, t' \in B_{\rho'}(\mathbb{C}, 0)$ (i.e. $|t|, |t'| < \rho'$) and

$$d_{E''}(\sigma(t), \sigma(t')) > 2\delta.$$

(This is possible since 2δ is smaller than the injectivity radius ρ of T which in turn is a lower bound for the diameter of E'').

We denote by $s : \mathbb{C} \to \mathbb{C}$ a holomorphic function such that $\sigma = \pi'' \circ s$.

Since $\pi' : \mathbb{C} \to E'$ restricts to an isomorphism between $B_\rho(\mathbb{C}, 0)$ and $B_\rho(E', e)$, the holomorphic maps s and σ induce maps from $B_\rho(E', e)$ to $\mathbb{C}$ resp. E''. By abuse of notation these maps will also be denoted by s resp. σ.

Now define $\Omega_2 = (E' \setminus \bar{W}) \times E''$ and $\Omega_1 = \Omega_2 \cup \Sigma$ with

$$\Sigma = \{(x, y) : x \in \bar{W}, y \in E'', d_{E''}(y - \sigma(x)) < \delta\}$$

Let us now fix some point $p \in \Omega_1$ and $v = (v_1, v_2) \in T_p(T) = \mathbb{C}^2$. We have to show that there exists a holomorphic map f as stipulated in (2) of theorem 2.

Let $(p_1, p_2) \in \mathbb{C}^2$ be a point mapped on p by $\pi : \mathbb{C}^2 \to T$. If $p \in \Sigma$, we require $|p_1| \le \rho'$ and $|s(p_1) - p_2| < \delta$ and define $\delta' = \delta - |s(p_1) - p_2|$. If $p \notin \Sigma$, we require $|p_1| > \rho''$ and define $\delta' = \delta$.

As the next step, we will choose a pair of entire functions (Q, H).

Claim 1. *There is a pair of entire functions (Q, H) with the following properties:*

1. *Q is a non-constant polynomial,*
2. *$(Q(0), H(0)) = (p_1, p_2)$ and*
3. *$(Q'(0), H'(0))$ and v are parallel.*
4. *If $p \in \Sigma$, we require furthermore that $(Q(z), H(z) + y) \in \pi^{-1}(\Sigma)$ for all z and y with $|Q(z)| \le \rho'$ and $|y| \le \frac{1}{2}\delta'$.*

Let us first discuss the case where $p \notin \Sigma$.

Then it suffices to choose

$$Q(z) = z^2 + v_1 z + p_1$$

and

$$H(z) = v_2 z + p_2.$$

If $p \in \Sigma$, we proceed as follows: First, for $r, t \in \mathbb{C}$ we define

$$Q_t(z) = (z + t)^2 + p_1 - t^2$$

and

$$H_{r,t}(z) = p_2 - s(p_1) + s(Q_t(z)) + rz.$$

We will set $Q = Q_t$ and $H = H_{r,t}$ for appropriately chosen parameters r, t.

Evidently Q_t is a polynomial for any choice of t. Furthermore $(Q_t(0), H_{r,t}(0)) = (p_1, p_2)$ independent of the choice of r, t:

$$Q_t(0) = t^2 + p_1 - t^2 = p_1$$

and

$$H_{r,t}(0) = p_2 - s(p_1) + s(p_1) + 0 = p_2.$$

Let $\Phi_{r,t} = (Q_t, H_{r,t})$. We have

$$\Phi'_{r,t}(0) = (Q'_t(0), s'(Q_t(0))Q'_t(0) + r) = (2t, 2s'(p_1)t + r)$$

Observe that

$$(r, t) \mapsto \frac{2t}{2s'(p_1)t + r}$$

defines a meromorphic function on $\mathbb{C}^2$ with a point of indeterminacy at $(0, 0)$. This is true regardless of the value of $s'(p_1)$.

Thus every neighborhood of $(0, 0)$ contains a point $(r, t) \neq (0, 0)$ such that $\Phi'_{r,t}(0)$ is a non-zero multiple of v.

Next we note that $(t, z) \mapsto Q_t(z)$ defines a proper map from $\overline{B_1(\mathbb{C}, 0)} \times \mathbb{C}$ to $\mathbb{C}$. Therefore there is a constant $C > 0$ such that $|z| < C$, whenever there exists a parameter t such that $|t| \leq 1$ and $|Q_t(z)| \leq \rho$.

It is therefore possible to choose two numbers r, t in such a way that

(1) $\Phi'_{r,t}(0)$ is a non-zero multiple of v,
(2) $|t| < 1$ and
(3) $|2rC| < \delta'$.

Now assume that $z, y \in \mathbb{C}$ with $|Q_t(z)| \leq \rho'$ and $|y| < \frac{1}{2}\delta'$. By the definition of the constant C, this implies $|z| < C$. Let $(w_1, w_2) = \Phi_{r,t}(z) + (0, y)$. Then

$$|w_2 - s(w_1)| = |p_2 - s(p_1) + rz + y| < |p_2 - s(p_1)| + |rC| + \frac{1}{2}\delta' <$$
$$< (\delta - \delta') + \frac{1}{2}\delta' + \frac{1}{2}\delta' = \delta.$$

Now $|w_2 - s(w_1)| < \delta$ in combination with $|w_1| = |Q_t(z)| \leq \rho'$ implies $\pi(w_1, w_2) \in \Sigma$. Hence $\Phi_{r,t}(z) + (0, y) \in \pi^{-1}(\Sigma)$ under this assumption. Thus the claim is proved. Q.E.D.

Our next step is to construct a closed subset A of $\mathbb{C}$ to which we will apply Arakelyan approximation.

Let A_0 be the union of $\overline{B_{\rho''}(0)}$ and $\overline{B_{\rho'}(\gamma)}$ for all $\gamma \in \Gamma'$. If $p \notin \Sigma$, then $p_1 \notin A_0$. Hence in this case we can choose $\eta > 0$ such that $\overline{B_\eta(p_1)}$

is disjoint to A_0 and define A_1 as the union of A_0 with this closed ball $\overline{B_\eta(p_1)}$. If $p \in \Sigma$, we simply take $A_1 = A_0$.

Next we choose dense countable subsets $S_1 \subset int(\Sigma)$ (where $int(\Sigma)$ denotes the interior of Σ) and $S_2 \subset \Omega_2$. We observe that $\mathbb{C}\setminus A_1$ projects surjectively onto $E'\setminus\overline{W}$ and that the fibers of this projection are infinite discrete subsets of $\mathbb{C}$. For this reason we can find sequences a_n, b_n in $\mathbb{C}$ such that

$$S_2 = \{\pi(a_n, b_n) : n \in \mathbb{N}\}$$

and all the a_n are distinct elements of $\mathbb{C} \setminus A_1$ with $\lim_{n\to\infty} |a_n| = +\infty$. It follows that

$$\Theta = \{a_n : n \in \mathbb{N}\}$$

is a discrete subset of $\mathbb{C}$ which has empty intersection with A_1. We define $A_2 = A_1 \cup \Theta$.

We fix a bijection $\xi : \Gamma' \setminus \{0\} \xrightarrow{\sim} S_1$ and an enumeration $n \mapsto \gamma_n$ of $\Gamma' \setminus \{0\}$. Then we can choose sequences of complex numbers c_n, d_n such that the following properties hold for all $n \in \mathbb{N}$

(1) $\pi(c_n, d_n) = \xi(\gamma_n)$,
(2) $|c_n - \gamma_n| < \rho'$ and
(3) $|d_n - s(c_n)| < \delta$.

We define $A = Q^{-1}(A_2)$.

Claim 2. *Arakelyan approximation is applicable to A, i.e. $\{\infty\} \cup (\mathbb{C} \setminus A)$ is connected and locally connected.*

Observing that we can deform $B_{\rho''}(\mathbb{C}, 0)$ to $B_{\rho'}(\mathbb{C}, 0)$, we deduce from prop. 1 that $Q^{-1}(A_0)$ has the desired property. Now A and $Q^{-1}(A_0)$ differ only by removing the preimage of a closed disc and by removing a discrete countable set (namely $Q^{-1}(\Theta)$). This can not destroy connectivity, hence not only $\{\infty\} \cup (\mathbb{C} \setminus Q^{-1}(A_0))$ but also $\{\infty\} \cup (\mathbb{C} \setminus A)$ is connected and locally connected. Thus the claim is proved.

We will now define a continuous function h on A, which is holomorphic in its interior, and which we will then approximate by an entire function, using Arakelyan's theorem.

If $p \notin \Sigma$, we take $h(z) = H(z)$ on $Q^{-1}(\overline{B_\eta(p_1)})$ and $h = s$ on $Q^{-1}(\overline{B_{\rho''}(0)})$.

If $p \in \Sigma$, we define h on $Q^{-1}(\overline{B_{\rho''}(0)})$ as $H(z)$.

Next, for every $n \in \mathbb{N}$, we define $h(z)$ as

$$h(z) = s(Q(z) - \gamma_n) + d_n - s(c_n)$$

whenever $|Q(z) - \gamma_n| \le \rho'$.

Finally, we define h on $Q^{-1}(\Theta)$ by stipulating that $h(z) = b_n$ whenever $Q(z) = a_n$ for a number $n \in \mathbb{N}$.

By the construction of (Q, H) we know that $\pi(Q(0), h(0)) = p$ and that $(Q'(0), h'(0))$ is a multiple of v. The choice of h implies moreover that $S_1 \cup S_2$ is contained in the image of $z \mapsto \pi(Q(z), h(z))$.

Next we define a continuous positive function $\epsilon : A \to \mathbb{R}^+$ as follows:

- $\epsilon \equiv 1$ on $Q^{-1}(\overline{B_\eta(p_1)})$ if $p \notin \Sigma$.
- $\epsilon \equiv \frac{1}{2}\delta'$ on $Q^{-1}(\overline{B_{\rho''}(0)})$.
- $\epsilon(z) = \frac{1}{n}$ if $Q(z) = a_n$.
- $\epsilon(z) = \min\left\{\frac{1}{n}, \frac{1}{2}(\delta - |d_n - s(c_n)|)\right\}$ whenever $|Q(z) - \gamma_n| \leq \rho'$.

Using prop. 2, we deduce that there exists an entire function $F : \mathbb{C} \to \mathbb{C}$ such that

(1) $|F(z) - h(z)| < \epsilon(z)$ for all $z \in A$.
(2) $F(0) = h(0)$ and $F'(0) = h'(0)$.

By the second condition we obtain that $\pi(Q(0), F(0)) = p$ and that $(Q'(0), F'(0))$ is a multiple of v. The first condition ensures that $\pi(Q(z), F(z)) \in \Omega$ for all $z \in \mathbb{C}$. It also ensure that the image is dense: Indeed, let $w \in \Omega_2$. Then there is a sequence of points in S_2 converging to w. But $S_2 = \{\pi(a_n, b_n) : n \in \mathbb{N}\}$ and the construction of F implies that for every $n \in \mathbb{N}$ there exists a number $z_n \in \mathbb{C}$ such that $Q(z_n) = a_n$ and $|F(z_n) - b_n| < \frac{1}{n}$. It follows that there is a subsequence z_{n_k} such that $\lim_k \pi(Q(z_{n_k}), F(z_{n_k})) = w$. If $w \in \Sigma$, we argue similarily, with S_1 in the role of S_2. Thus the whole set Ω_1 is in the closure of the image of the map $z \mapsto \pi(Q(z), F(z))$ from $\mathbb{C}$ to T.

Finally, let μ be a complex number such that $\mu(Q'(0), F'(0)) = v$ and define

$$f(z) = \pi\left(Q(\mu z), F(\mu z)\right)$$

Then $f : \mathbb{C} \to \Omega_1$ is a holomorphic map with the desired properties.

3. Arakelyan Approximation with interpolation

We will need a slight improvement of Arakelyan's theorem.

We recall the theorem of Arakelyan (see [1]):

Theorem 3. *Let A be a closed subset of $\mathbb{C}$, $U = \mathbb{P}_1(\mathbb{C}) \setminus A$, $\epsilon : A \to \mathbb{R}^+$ a continuous function and $f_0 : A \to \mathbb{C}$ a continuous function which is holomorphic in the interior of A. Assume that U is connected and locally connected.*

Then there exists a holomorphic function $F : \mathbb{C} \to \mathbb{C}$ with $|F(z) - f(z)| < \epsilon(z)$ for all $z \in A$.

We want to verify that Arakelyan's theorem is applicable in our situation.

Proposition 1. *Let* Γ *be a lattice in* $\mathbb{C}$ *and* ρ' *a real number with*

$$0 < \rho' < \rho = \frac{1}{2} \min_{\gamma \in \Gamma \setminus \{0\}} |\gamma|$$

Let $A' = \{z \in \mathbb{C} : d(z, \Gamma) \leq \rho'\} = \cup_{\gamma \in \Gamma} \overline{B_{\rho'}(\mathbb{C}, \gamma)}$, $P : \mathbb{C} \to \mathbb{C}$ *a non-constant polynomial and* $U = \{\infty\} \cup (\mathbb{C} \setminus P^{-1}(A'))$.

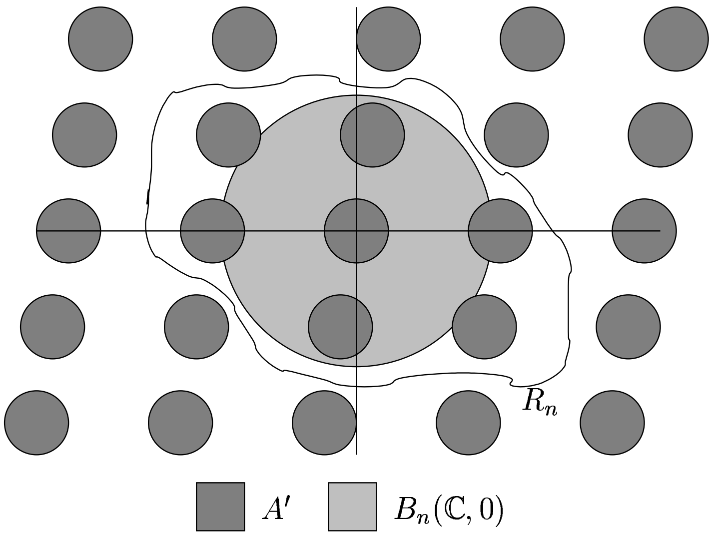
Then U *is connected and locally connected.*

Proof. First we want to verify that U contains no bounded connected component. Indeed, assume that there is such a connected component C. Its boundary ∂C is a connected set mapped into

$$\cup_{\gamma \in \Gamma} \overline{B_{\rho'}(\gamma)}$$

by P. This is a disjoint union due to the choice of ρ'. Hence continuity of P implies that there is one element $\gamma \in \Gamma$ such that $|P(z) - \gamma| \leq \rho'$ for all $z \in \partial C$.

But $C \subset U$ implies $|P(z) - \gamma| > \rho'$ for all $z \in C$, $\gamma \in \Gamma$. This is in contradiction with the maximum principle for the holomorphic function P. Hence there can not exist a bounded connected component $C \subset U$.



For each $n \in \mathbb{N}$ we choose a simple closed curve $R_n \subset \mathbb{C} \setminus A'$ such that the open bounded subset $V_n \subset \mathbb{C}$ which is enclosed by R_n has the property that $B_n(\mathbb{C}, 0) \subset V_n$.

The ramification locus

$$Z = \{z \in \mathbb{C} : \exists w \in \mathbb{C} : P(w) = z, P'(w) = 0\}$$

is a finite set. Let $N_0 = \max\{|z| : z \in Z\}$. Now the restriction of P to $P^{-1}(\mathbb{C} \setminus V_n) \to \mathbb{C} \setminus V_n$ is an unramified covering of degree $d = \deg(P)$

for all $n > N_0$. As a polynomial map, P extends to a proper map $\bar{P} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. For a suitably chosen local coordinate w at ∞ the map P near ∞ can be described as $w \mapsto w^d$. Using this fact and the fact that by construction each curve R_n defines a generator for

$$\pi_1(\mathbb{C}^*) \simeq \pi_1(\mathbb{C} \setminus \overline{B_n(\mathbb{C}, 0)})$$

we can conclude that $P^{-1}(R_n)$ is connected for all $n > N_0$. Then $P^{-1}(\Omega)$ is connected for every open subset $\Omega \subset \mathbb{C} \setminus \overline{B_n(\mathbb{C}, 0)}$ with $R_n \subset \Omega$.

In particular

$$W_n = U \setminus P^{-1}(V_n)$$

is connected for all $n > N_0$. The collection of all these open sets W_n constitutes an neighborhood basis of U at ∞, implying that U is locally connected at infinity. Furthermore, the connectedness of the sets W_n implies that U is only one unbounded connected component. Since we have already seen that U is no bounded connected component, this completes the proof that U is connected and locally connected. □

Proposition 2. *Let A be a closed subset in $\mathbb{C}$, $A \neq \mathbb{C}$, and suppose that for every function f on A which is holomorphic in its interior and every continuous map $\epsilon : A \to \mathbb{R}^+$ there is an entire function $F : \mathbb{C} \to \mathbb{C}$ with $|F(z) - f(z)| < \epsilon(z)$ for all $z \in A$.*

Let q be a point in the interior of A. Then we can find such an entire function F with the additional properties $F(q) = f(q)$ and $F'(q) = f'(q)$.

Proof. Let $U = \{\infty\} \cup (\mathbb{C} \setminus A)$. By assumption U is connected and locally connected at infinity. Let $p \in \mathbb{C} \setminus A$ and let W be a bounded connected open subset of $\mathbb{C}$ containing both p and q. Choose $\delta > 0$ such that

$$\delta < \min \{d(q, \partial W), d(q, \partial A), d(p, q)\}$$

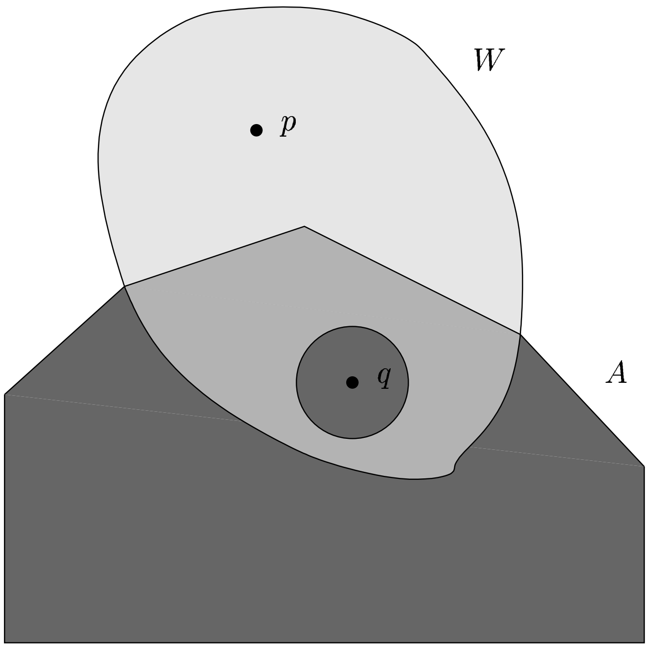
and define

$$\tilde{A} = \overline{B_\delta(q)} \cup A \setminus W$$

and

$$\tilde{U} = \{\infty\} \cup \left(\mathbb{C} \setminus \tilde{A}\right) = U \cup \left(W \setminus \overline{B_\delta(q)}\right).$$

Now both U and $\left(W \setminus \overline{B_\delta(q)}\right)$ are connected, and their intersection is non-empty, since it contains p. Therefore $\tilde{U}$ is connected. Moreover, $\tilde{U}$ is locally connected at infinity, because it coincides with U near ∞. Thus we have Arakelyan approximation for $\tilde{A}$.



We choose constants $\xi_0, \xi_1 \in \mathbb{C} \setminus \{0\}$ such that

$$|\xi_0| < \frac{1}{16}\epsilon(z)$$

and

$$|\xi_1(z-q)| < \frac{1}{16}\epsilon(z)$$

for all $z \in \overline{B_\delta(q)}$.

Then we define functions $g, h : \tilde{A} \to \mathbb{C}$ via

$$g(z) = \begin{cases} \xi_0 & \text{if } z \in \overline{B_\delta(q)} \\ 0 & \text{else} \end{cases}$$

and

$$h(z) = \begin{cases} \xi_1(z-q) & \text{if } z \in \overline{B_\delta(q)} \\ 0 & \text{else} \end{cases}$$

Clearly, g and h are continuous and holomorphic in the interior of $\tilde{A}$. The choice of ξ_0, ξ_1 implies that $|g(z)| < \frac{1}{16}\epsilon(z)$ and $|h(z)| < \frac{1}{16}\epsilon(z)$ for all $z \in A$.

By the Arakelyan property we find sequences of entire functions $g_n, h_n : \mathbb{C} \to \mathbb{C}$ such that

$$|g_n(z) - g(z)| < \frac{1}{8n}\epsilon(z)$$

and

$$|h_n(z) - h(z)| < \frac{1}{8n}\epsilon(z)$$

for all $n \in \mathbb{N}$, $z \in \tilde{A}$. Locally uniform convergence on $\tilde{A}$ implies that inside the interior of $\tilde{A}$ the derivatives converge as well. Hence we obtain

$$\lim_{n\to\infty} \begin{pmatrix} g_n(q) & h_n(q) \\ g_n'(q) & h_n'(q) \end{pmatrix} = \begin{pmatrix} g(q) & h(q) \\ g'(q) & h'(q) \end{pmatrix} = \begin{pmatrix} \xi_0 & 0 \\ 0 & \xi_1 \end{pmatrix}.$$

Thus, for n sufficiently large the vectors $(g_n(q), g_n'(q))$ and $(h_n(q), h_n'(q))$ are linearly independent.

Next we observe that $A \setminus \tilde{A}$ is relatively compact in A. Therefore, for sufficiently large numbers n, C the functions $\alpha = \frac{1}{C}g_n$ and $\beta = \frac{1}{C}h_n$ have the following properties:

(1) α, β are entire functions,
(2) $|\alpha(z)|, |\beta(z)| < \frac{1}{8}\epsilon(z)$ for all $z \in A$, and
(3) the vectors $(\alpha(q), \alpha'(q))$ and $(\beta(q), \beta'(q))$ are linearly independent.

By the approximation property for A there are sequences of entire functions $\alpha_n, \beta_n, f_n : \mathbb{C} \to \mathbb{C}$ such that

$$\max\{|\alpha_n(z) - \alpha(z)|, |\beta_n(z) - \beta(z)|, |f_n(z) - f(z)|\} < \frac{1}{n}\epsilon(z)$$

for all $n \in \mathbb{N}$, $z \in A$. The locally uniform convergence of $\lim \alpha_n = \alpha$, $\lim \beta_n = \beta$ and $\lim f_n = f$ on A implies that in the interior of A the respective derivatives converge as well. In particular, this happens at q. Hence the matrix

$$A_n = \begin{pmatrix} \alpha_n(q) & \beta_n(q) \\ \alpha_n'(q) & \beta_n'(q) \end{pmatrix}$$

converges to

$$\lim_{n\to\infty} A_n = A = \begin{pmatrix} \alpha(q) & \beta(q) \\ \alpha'(q) & \beta'(q) \end{pmatrix}.$$

Since A is invertible, it follows that A_n is likewise invertible for all sufficiently large n. Hence we can define (for sufficiently large n) sequences λ_n, μ_n via

$$\begin{pmatrix} \lambda_n \\ \mu_n \end{pmatrix} = A_n^{-1} \cdot \begin{pmatrix} f(q) - f_n(q) \\ f'(q) - f_n'(q) \end{pmatrix}.$$

Now $\lim f_n = f$, $\lim f_n' = f'$ and $\lim A_n^{-1} = A^{-1}$. Therefore $\lim \lambda_n = 0 = \lim \mu_n$.

Thus we can choose a natural number $N \in \mathbb{N}$ with the following properties:

(1) A_N is invertible,
(2) $|\lambda_N|, |\mu_N| < 1$,
(3) and $N > 4$.

We define

$$F(z) = f_N(z) + \lambda_N \alpha_N(z) + \mu_N \beta_N(z).$$

By the choice of λ_n, μ_n we have

$$\begin{pmatrix} F(q) \\ F'(q) \end{pmatrix} = \begin{pmatrix} f_N(q) + \lambda_N \alpha_N(q) + \mu_N \beta(q) \\ f'_N(q) + \lambda_N \alpha'_N(q) + \mu'_N \beta'(q) \end{pmatrix} =$$
$$= \begin{pmatrix} f_N(q) \\ f'_N(q) \end{pmatrix} + A_N \cdot \begin{pmatrix} \lambda_N \\ \mu_N \end{pmatrix} = \begin{pmatrix} f(q) \\ f'(q) \end{pmatrix}.$$

Furthermore

$$|F(z) - f(z)| \leq |f_N(z) - f(z)| + |\lambda_N| \cdot (|\alpha_N(z) - \alpha(z)| + |\alpha(z)|) +$$
$$+ |\mu_N| \cdot (|\beta_N(z) - \beta(z)| + |\beta(z)|)$$
$$\leq \frac{1}{N}\epsilon(z) + \frac{1}{N}\epsilon(z) + \frac{1}{8}\epsilon(z) + \frac{1}{N}\epsilon(z) + \frac{1}{8}\epsilon(z) < \left(\frac{3}{4} + \frac{2}{8}\right)\epsilon(z) = \epsilon(z)$$

for all $z \in A$. Thus F is an entire function with the desired properties. $\square$

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Jörg Winkelmann, Institut Elie Cartan (Mathématiques), Université Henri Poincaré Nancy 1, B.P. 239, F-54506 Vandœuvre-les-Nancy Cedex, France

E-mail address: `jwinkel@member.ams.org`

Webpage: `http://www.math.unibas.ch/~winkel/`

ARITHMETIC JET SPACES
(WORK IN PROGRESS)

Paul Vojta

University of California, Berkeley

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Abstract. When defined as iterated Kähler differentials, jet differentials are not suitable for working on varieties in positive characteristic or on schemes of mixed characteristic (such as arithmetic varieties). Instead, it is better to use Hasse-Schmidt divided differentials $d_n x$ (think: $(1/n!)d^n x$). These will be discussed briefly; for fuller details see [**V 2**]. These differentials allow one to define jet spaces for arbitrary scheme morphisms $X \to Y$; such spaces are analogous to generalizations of the relative tangent bundle, adding information on higher derivatives.

In his 1995 talk at Santa Cruz [**D**], J.-P. Demailly discussed compactified quotient jet spaces due originally to J. G. Semple and others. These correspond to certain closed subspaces of the iterated space of lines in the tangent bundle of a complex manifold: X, $\mathbb{P}(\Omega^1_{X/\mathbb{C}})$, $\mathbb{P}(\Omega^1_{\mathbb{P}(\Omega^1_{X/\mathbb{C}})/\mathbb{C}})$, etc.

I tried to generalize the Semple-Demailly jet spaces to arbitrary characteristic, but was not successful. Instead, though, I have found another definition of jet space, isomorphic to the Semple-Demailly jet spaces away from the "vertical" part, but generally nonisomorphic for 3-jets and higher. This definition will be discussed.

Let X be a complex manifold, and consider the sequence of manifolds $\widetilde{X}_0$, $\widetilde{X}_1$, ... defined inductively by letting $\widetilde{X}_0 = X$ and $\widetilde{X}_{n+1} = \mathbb{P}(\Omega_{\widetilde{X}_n/\mathbb{C}})$ for $n \ge 0$. Here, and in what follows, $\mathbb{P}(\mathscr{E})$ for a sheaf $\mathscr{E}$ is defined to be $\mathbf{Proj}\bigoplus_{n=0}^\infty S^n\mathscr{E}$, so that if $\mathscr{E}$ is a vector sheaf over X then a point on $\mathbb{P}(\mathscr{E})$ over a point $x \in X$ corresponds to a *hyperplane* in the fiber of $\mathscr{E}$ at x. For example, $\widetilde{X}_1 = \mathbb{P}(\Omega_{X/\mathbb{C}})$ is the space of lines in the tangent bundle of X. This is the definition used in [**EGA**].

The space $\widetilde{X}_n$ then incorporates information up to the n^{th} derivative. However, if X has dimension d, then $\widetilde{X}_n$ has dimension $2^n(d-1)+1$, which is growing far faster than the information that it is intended to capture. J. G. Semple [**S**], J.-P. Demailly [**D**], and others investigated certain proper closed subspaces X_n of $\widetilde{X}_n$ of dimension $d + n(d-1)$ having the property that they contain all canonical liftings to $\widetilde{X}_n$ of holomorphic curves $\mathbb{C} \to X$.

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For reasons described in Section 1, I have been trying to find a way to generalize this definition to give similar jet spaces for an arbitrary scheme morphism $X \to Y$. Of course, one can take $\widetilde{X}_0 = X$ and $\widetilde{X}_{n+1} = \mathbb{P}(\Omega_{\widetilde{X}_n/Y})$ as before, but the problem with this is that there is no lifting of a curve $C \to X$ to $\widetilde{X}_n$ in characteristic $p > 0$ when $n \ge p$, since n^{th} derivatives vanish in this situation. A definition using divided differentials (see [**V 2**]) may work, but the combinatorics of such a definition have eluded me so far.

However, I did find a different type of jet space, isomorphic to the Semple-Demailly jet spaces away from the "vertical" part, but nonisomorphic in general. This talk describes work in progress on this definition, as well as possible applications. More complete notes will appear elsewhere, when the work is finished.

§1. The Quest

The motivation for this work stems from a search for an answer to a question that has plagued number theorists for a long time:

How does one "differentiate" in number theory?

Derivatives are ubiquitous in Nevanlinna theory, and a major obstacle to translating proofs from Nevanlinna theory to number theory comes from the fact that there is no known way to translate the concept of derivative. Here, for a variety X over a number field k, we are looking for a "derivative" in the *relative* tangent bundle

$$\mathbb{V}(\Omega_{X/k}) = \mathbf{Spec} \bigoplus_{n \ge 0} S^n \Omega_{X/k} ,$$

so the notion of derivative desired here is different from that occurring in the function field case. In fact, there is no known counterpart to this derivative in the function field case unless there is a canonical projection from the absolute tangent bundle to the relative tangent bundle (e.g., in the split case).

As a possible partial answer to this question, we could ignore the magnitude of the derivative and just look for its direction; this would involve looking for a point in $\mathbb{P}(\Omega_{X/k})$. Similarities between Ahlfors' proof of Cartan's theorem on approximation to hyperplanes in projective space, and Schmidt's proof of his Subspace Theorem ([**V 1**], Ch. 6), seem to suggest that one should look at successive minima. Therefore, a candidate for the direction of the derivative might be given by extending X to an arithmetic variety $\mathscr{X}$ over $\mathscr{Y} := \operatorname{Spec} \mathscr{O}_k$. A rational point on X would then correspond to a section of the map $\mathscr{X} \to \mathscr{Y}$; the restriction of the relative cotangent bundle $\Omega_{\mathscr{X}/\mathscr{Y}}$ to this section gives a vector bundle over $\mathscr{Y}$. Together with Arakelovized information at infinity, this gives something that looks more classically like a lattice in $\mathbb{R}^d$ (with $d = \dim X$) together with a length function. The first successive minimum corresponds to a line subsheaf of $\Omega_{\mathscr{X}/\mathscr{Y}}$ of maximal degree; this may be the desired derivative (or, rather, at least its direction).

In the case of curves over k, though, we have $\mathbb{P}(\Omega_{X/k}) = X$, so there is nothing to work with. We have to work in higher dimensions. Work in Nevanlinna theory then strongly suggests that we have to work with higher jets. (This is also in keeping with the general dictum that if you're working on something and can't get anywhere, look at a special case, and if you're still stuck, look at a generalization.)

§2. Just what *is* a jet space, anyway?

There are (at least) two types of jet spaces in common use. Both are generalizations of spaces associated with Kähler differentials.

Jets of arcs:

- generalize the Zariski tangent space $\operatorname{Hom}_k(\operatorname{Spec} k[\epsilon]/\epsilon^2, X)$ for schemes X/k;
- parametrize (infinitesimal) arcs $\operatorname{Spec} k[t]/t^{m+1} \to X$ over k;
- are often studied via (higher) jet differentials.

Jets of functions:

- generalize the construction of Kähler differentials as $\mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is the sheaf of ideals defining the diagonal in $X \times X$;
- parametrize elements of the completed local ring at points of X;
- are often studied via Grothendieck's theory of principal parts.

This talk considers jets of arcs.

§3. Arithmetic Jets

This section briefly summarizes some things from [**V 2**] for the convenience of the reader, then concludes with a brief discussion of various types of quotients of jets.

Throughout this paper, all rings (and algebras) are assumed to be commutative. Moreover, $\mathbb{N} = \{0, 1, 2, \dots\}$.

Recall from the Introduction that if $X \to Y$ is a morphism of schemes, then $\mathbb{P}(\Omega_{X/Y})$ parametrizes the set of all lines in the relative tangent bundle.

Also recall that if B is an A-algebra, then $\Omega_{B/A}$ is the B-module with generators $\{db : b \in B\}$ and relations:

$$d(b_1 + b_2) = db_1 + db_2; \qquad da = 0; \qquad d(b_1 b_2) = b_1 db_2 + b_2 db_1$$

for all $b_1, b_2 \in B$ and all $a \in A$.

If we formally allow d to be iterated, then we get

$$d^n(xy) = \sum_{i+j=n} \binom{n}{i} d^i x \, d^j y \,.$$

Letting $d_n x = \dfrac{1}{n!} d^n x$, this becomes

$$d_n(xy) = \sum_{i+j=n} d_i x \, d_j y \,.$$

Differentials satisfying this identity are called **Hasse-Schmidt differentials**.

Definition. Let $m \in \mathbb{N}$ and let B be an A-algebra. The **Hasse-Schmidt algebra** is the B-algebra $\mathrm{HS}^m_{B/A}$ given as the quotient of the polynomial algebra

$$B[x^{(i)}]_{x\in B,\; i=1,\dots,m}$$

by the ideal I generated by the union of the sets

$$\begin{gathered}\{(x+y)^{(i)} - x^{(i)} - y^{(i)} : x, y \in B;\; i = 1,\dots,m\}\;, \\ \{f(a)^{(i)} : a \in A;\; i = 1,\dots,m\}\;, \quad \text{and} \\ \Big\{(xy)^{(k)} - \sum_{i+j=k} x^{(i)}y^{(j)} : x, y \in B;\; k = 0,\dots,m\}\;,\end{gathered}$$

where we identify $x^{(0)}$ with x for all $x \in B$. The image of $x^{(i)}$ in $\mathrm{HS}^m_{B/A}$ is denoted $d_i x$; we also write $d_1 x = dx$. The resulting algebra $\mathrm{HS}^m_{B/A}$ is an algebra over B; it can also be viewed as an algebra over A via f. It is also a graded algebra (either over B or over A) in which $d_i x$ has degree i.

If $X \to Y$ is an arbitrary morphism of schemes, then we get a graded sheaf $\mathrm{HS}^m_{X/Y}$ of $\mathscr{O}_X$-algebras, and we define the **jet space**

$$J_m(X/Y) = \mathbf{Spec}\,\mathrm{HS}^m_{X/Y}\;.$$

Theorem (Jet desideratum). *For all A-algebras R, we have a natural bijection*

$$\mathrm{Hom}_A(\mathrm{HS}^m_{B/A}, R) \to \mathrm{Hom}_A(B, R[t]/(t^{m+1}))$$

in which a map $\phi \in \mathrm{Hom}_A(\mathrm{HS}^m_{B/A}, R)$ is associated to the map $B \to R[t]/(t^{m+1})$ or $B \to R[[t]]$ given by

$$x \mapsto \phi(d_0 x) + \phi(d_1 x)t + \cdots + \phi(d_m x)t^m \pmod{(t^{m+1})}\;.$$

In the context of schemes, this becomes

$$\mathrm{Hom}_Y(\mathbf{Spec}\,\mathscr{O}_Z[[t]]/(t^{m+1}), X) \xrightarrow{\sim} \mathrm{Hom}_Y(Z, J_m(X/Y))\;.$$

From now on assume $m > 0$.

For all $a \in R^*$, the map $t \mapsto at$ gives an automorphism of $R[t]/(t^{m+1})$, hence an action of $\mathbb{G}_{\mathrm{m},X}$ on $J_m(X/Y)$. A quotient of this action can be defined; it is the **Green-Griffiths projectivized jet space**

$$P_m(X/Y) := \mathbf{Proj}\,\mathrm{HS}^m_{X/Y}\;.$$

But, why not mod out by *all* of $\mathrm{Aut}_R(R[t]/(t^{m+1}))$? The rationale for doing so is that, like throwing out the magntude information for vectors in the relative tangent bundle, it focuses on the information that is most likely to be useful.

§4. Groups, Group Actions, and Quotients

The definitions of group schemes and group actions are well known, so the definitions are only briefly summarized here. For a more complete reference, see [**M-F**] or [**SGA 3**].

Definition. Let S be a scheme. A **group scheme** Γ/S is a morphism of schemes $\pi\colon \Gamma \to S$, together with morphisms $\mu\colon \Gamma \times_S \Gamma \to \Gamma$, $\beta\colon \Gamma \to \Gamma$, and $e\colon S \to \Gamma$ over S (expressing the group operation, inverse, and the identity element, respectively), satisfying the following conditions.

(a). (Associativity) The following diagram commutes:

$$\begin{array}{ccc} \Gamma \times_S \Gamma \times_S \Gamma & \xrightarrow{\mathrm{Id}_\Gamma \times \mu} & \Gamma \times_S \Gamma \\ \big\downarrow{\scriptstyle \mu\times\mathrm{Id}_\Gamma} & & \big\downarrow{\scriptstyle \mu} \\ \Gamma \times_S \Gamma & \xrightarrow{\mu} & \Gamma \end{array},$$

(b). (Inverse) The diagrams

$$\begin{tikzcd} \Gamma \arrow[r, "\Delta"] \arrow[drr, "\pi"'] & \Gamma \times_S \Gamma \arrow[r, "\mathrm{Id}_\Gamma \times \beta"] & \Gamma \times_S \Gamma \arrow[r, "\mu"] & \Gamma \\ & & S \arrow[ur, "e"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} \Gamma \arrow[r, "\Delta"] \arrow[drr, "\pi"'] & \Gamma \times_S \Gamma \arrow[r, "\beta\times\mathrm{Id}_\Gamma"] & \Gamma \times_S \Gamma \arrow[r, "\mu"] & \Gamma \\ & & S \arrow[ur, "e"'] & \end{tikzcd}$$

commute. (Here Δ is the diagonal morphism.)

(c). (Identity) The compositions

$$\Gamma \xrightarrow{\sim} S \times_S \Gamma \xrightarrow{e\times\mathrm{Id}_\Gamma} \Gamma \times_S \Gamma \xrightarrow{\mu} \Gamma$$

and

$$\Gamma \xrightarrow{\sim} \Gamma \times_S S \xrightarrow{\mathrm{Id}_\Gamma \times e} \Gamma \times_S \Gamma \xrightarrow{\mu} \Gamma$$

both equal the identity on Γ.

Group actions are defined in a similar vein. Details are left as an exercise for the reader.

As for quotients, the definition can be quite delicate. We use here the definition of Mumford and Fogarty [**M-F**]. The details of this definition do not matter for this application, though, because we only use the fact that the following is an example of a quotient morphism:

Example. Let Γ be a group scheme over S, let X' be an S-scheme with trivial Γ-action, and let $X = \Gamma \times_S X'$, with Γ acting on itself via left translation. Then the projection $X = \Gamma \times_S X' \to X'$ is a quotient morphism.

Lemma ("Quotient lemma"). *Let Γ/S be a group scheme, let X_0 be an S-scheme with an action of Γ (not necessarily trivial), and let $X = \Gamma \times_S X_0$. Then a quotient of X by its product action exists, and is isomorphic (as an S-scheme) to X_0. If $\sigma_0 \colon \Gamma \times_S X_0 \to X_0$ expresses the action of Γ on X_0, then the quotient morphism is given by $\sigma_0 \circ (\beta \times \mathrm{Id}_{X_0})$.*

More generally, if $T \to X$ is a Γ-**torsor** (over X) (i.e., a principal fibre bundle), then X is a quotient of T.

§5. Semple-Demailly Quotient Jet Spaces

Throughout this section, let B be an A-algebra, and let $m \in \mathbb{Z}_{>0}$.

Let

$$\Gamma_{m,B} = \operatorname{Spec} B[z_1, z_1^{-1}, z_2, \dots, z_m]$$

be the group scheme of automorphisms of $B[t]/(t^{m+1})$, where $z_1, \dots, z_m$ corresponds to the automorphism

$$t \mapsto z_1 t + z_2 t^2 + \dots + z_m t^m$$

and the group law is given by function composition modulo t^{m+1}. The group operation $\Gamma_m \times \Gamma_m \to \Gamma_m$ corresponds to a ring homomorphism

$$B[z_1, z_1^{-1}, z_2, \dots, z_m] \longrightarrow B[y_1, y_1^{-1}, y_2, \dots, y_m] \otimes_B B[z_1, z_1^{-1}, z_2, \dots, z_m]$$

given by

$$z_i \mapsto p_{i1}(z_1) y_1 + p_{i2}(z_1, z_2) y_2 + \dots + p_{ii}(z_1, \dots, z_i) y_i ,$$

with $p_{ij} \in \mathbb{Z}[z_1, \dots, z_j]$. For example:

$$\begin{gathered} p_{11} = z_1 ; \\ p_{21} = z_2 , \qquad p_{22} = z_1^2 ; \\ p_{31} = z_3 , \qquad p_{32} = 2 z_1 z_2 , \qquad p_{33} = z_1^3 ; \\ p_{41} = z_4 , \qquad p_{42} = z_2^2 + 2 z_1 z_3 , \qquad p_{43} = 3 z_1^2 z_2 , \qquad p_{44} = z_1^4 . \end{gathered}$$

Via the "jet desideratum," we also get a *right* action of Γ_m on $\mathrm{HS}^m_{B/A}$; this satisfies:

$$d_i x \mapsto \sum_{j \in \mathbb{N}} p_{ij}(z_1, \dots, z_j) d_j x$$

So, similarly, we have:

$$\begin{aligned} (d_1 x)\gamma &= a_1\, d_1 x ; \\ (d_2 x)\gamma &= a_1^2\, d_2 x + a_2\, d_1 x ; \\ (d_3 x)\gamma &= a_1^3\, d_3 x + 2 a_1 a_2\, d_2 x + a_3\, d_1 x ; \\ (d_4 x)\gamma &= a_1^4\, d_4 x + 3 a_1^2 a_2\, d_3 x + (a_2^2 + 2 a_1 a_3) d_2 x + a_4\, d_1 x . \end{aligned}$$

where γ is the automorphism

$$t \mapsto a_1 t + a_2 t^2 + \cdots + a_m t^m \ .$$

We define $Q_m^\circ(B/A)$ to be the quotient of $J_m(B/A)$ by the action of Γ_m, and likewise $Q_m^\circ(X/Y)$.

Example. If $m = 1$ then $\Gamma_m = \mathbb{G}_\mathrm{m}$, and

$$Q_1^\circ(X/Y) = P_1(X/Y) = \mathbb{P}(\Omega_{X/Y}) \ .$$

If B is a polynomial algebra $B = A[x_i]_{i \in I}$, then the "quotient lemma" applies over open affines $D_+(dx_i)$ in $\mathbb{P}(\Omega_{B/A})$, and these pieces patch together to give schemes $Q_m^\circ(B/A)$. For general B, write B as a polynomial algebra modulo an ideal; this ideal is preserved by the group action and allows us to define the quotient $Q_m^\circ(B/A)$ also in this case. Further patching yields $Q_m^\circ(X/Y)$ for arbitrary scheme morphisms $X \to Y$.

(In other words, $J_m(B/A)$ is a torsor, trivialized over the Zariski-open subsets $D_+(dx_i)$ of $\mathbb{P}(\Omega_{B/A})$.)

We get morphisms

$$Q_m^\circ(B/A) \longrightarrow Q_{m-1}^\circ(B/A) \longrightarrow \ldots \longrightarrow Q_1^\circ(B/A) = \mathbb{P}(\Omega_{B/A}) \ .$$

One should note that $Q_m^\circ(X/Y)$ are *affine* over $Q_1^\circ(X/Y) = \mathbb{P}(\Omega_{X/Y})$, but the jet spaces of Semple and Demailly are *proper* over $\mathbb{P}(\Omega_{X/Y})$ (and therefore proper over X).

§6. Completing $Q_m^\circ(B/A)$

($A \to B$ and m are as before.)

We want to define a "completion" $Q_m(B/A)$ such that:

(i). if B is of finite type over A, then $Q_m(B/A)$ is proper over $\operatorname{Spec} B$; and
(ii). $Q_m^\circ(B/A)$ embeds as an open subset of $Q_m(B/A)$.

Recall that the Semple-Demailly quotient jet spaces are defined by taking

$$\mathbb{P}(\Omega_{X/\mathbb{C}}), \qquad \mathbb{P}(\Omega_{\mathbb{P}(\Omega_{X/\mathbb{C}})/\mathbb{C}}), \qquad \text{etc.}$$

and passing to certain closed subsets.

For $x_1, \ldots, x_n \in B$ and $n \in \mathbb{Z}_{>0}$ define the Wronskian-like determinant

$$\mathscr{D}(x_1, \ldots, x_n) = \begin{vmatrix} d_1 x_1 & \ldots & d_n x_1 \\ \vdots & \ddots & \vdots \\ d_1 x_n & \ldots & d_n x_n \end{vmatrix} \ .$$

A group element $\gamma \colon t \mapsto a_1 t + a_2 t^2 + \cdots + a_m t^m$ acts on the above matrix by column operations, in which each column is replaced by a linear combination of itself and columns to its left. Also, the same-column coefficient for the i^{th} column is a_1^i. Therefore, the effect of γ on the determinant is to multiply it by $a_1^{n(n+1)/2}$. Thus

$$(dg)^{-n(n+1)/2}\mathscr{D}(x_1,\dots,x_n)$$

is *invariant* under the action of Γ_m.

We then define $Q_m(B/A)$ inductively on m by giving an open covering by affines $U = U_{g_1,\dots,g_m}$ ($g_1,\dots,g_m \in B$) with

$$\mathscr{O}(U) \subseteq \left(\mathrm{HS}^m_{B/A}\right)_{(\omega)} ;$$

here $\omega = \mathscr{D}(g_1)\mathscr{D}(g_1,g_2)\cdots\mathscr{D}(g_1,\dots,g_m)$.

Let $\mathscr{S}$ be the graded quasi-coherent sheaf on $Q_m(B/A)$ determined by

$$\begin{aligned}\mathscr{S}(U_{g_1,\dots,g_m}) &= \mathscr{O}(U_{g_1,\dots,g_m})\bigl[\mathscr{D}(x_1,\dots,x_{m+1}) : x_1,\dots,x_{m+1}\in B\bigr]\\ &\subseteq \left(\mathrm{HS}^{m+1}_{B/A}\right)_\omega .\end{aligned}$$

Then we define

$$Q_{m+1}(B/A) = \mathbf{Proj}\,\mathscr{S} .$$

Let $g_1,\dots,g_{m-1},x_1,\dots,x_m \in B$. Then

$$\begin{aligned}&\begin{vmatrix}\mathscr{D}(x_1) & \mathscr{D}(g_1,x_1) & \dots & \mathscr{D}(g_1,\dots,g_{m-1},x_1)\\ \vdots & \vdots & \ddots & \vdots\\ \mathscr{D}(x_m) & \mathscr{D}(g_1,x_m) & \dots & \mathscr{D}(g_1,\dots,g_{m-1},x_m)\end{vmatrix}\\ &\qquad = \mathscr{D}(g_1)\mathscr{D}(g_1,g_2)\dots\mathscr{D}(g_1,\dots,g_{m-1})\cdot\mathscr{D}(x_1,\dots,x_m) .\end{aligned}$$

Therefore

$$\mathscr{S}(U_{g_1,\dots,g_m}) = \mathscr{O}(U_{g_1,\dots,g_m})\bigl[\mathscr{D}(g_1,\dots,g_m,x) : x\in B\bigr] .$$

By induction, we then have:

$$\mathscr{O}(U_{g_1,\dots,g_m}) = B\left[\frac{\mathscr{D}(x)}{\mathscr{D}(g_1)},\dots,\frac{\mathscr{D}(g_1,\dots,g_{m-1},x)}{\mathscr{D}(g_1,\dots,g_m)}\right] .$$

(Of course the above discussion omits the technical details of showing that $\mathscr{S}$ indeed glues over the whole scheme.)

§7. Comparison with Semple-Demailly Jet Spaces

We can canonically embed $Q^\circ_m(B/A)$ into $Q_m(B/A)$. Its image is the open subset

$$\bigcup_{g\in B} U_{g,g^2,\dots,g^m} .$$

This uses the identity $\mathscr{D}(g,g^2,\dots,g^m)=(dg)^{m(m+1)/2}$.

If $A=\mathbb{C}$, then we can also embed $Q^\circ_m(B/A)$ as an open subset of the Semple-Demailly jet space.

If $m\le 2$ and $m!$ is invertible in B , then $Q_m(B/A)$ coincides with the Semple-Demailly jet space. If $m=0$ or $m=1$ this is trivially so, because they're both equal to $\operatorname{Spec} B$ or $\mathbb{P}(\Omega_{B/A})$. If $m=2$ this is because of the identity

$$d\left(\frac{dx}{dg}\right)=\frac{d^2x\,dg-d^2g\,dx}{(dg)^2}=\frac{2\mathscr{D}(g,x)}{(dg)^2} .$$

If $m\ge 3$, though, the two types of jet spaces are different.

Among the relative benefits of the two types of jet spaces are the following:

- I know how to define $Q_m(X/Y)$.
- Semple-Demailly jet spaces have nice intuitive properties.
- Semple-Demailly jet spaces are nonsingular if X/Y is smooth. It is unclear whether the jet spaces defined here are smooth, even if X is smooth over Y .

§8. Possible Applications

Once the work on these jet spaces themselves is finished, directions for further research include:

1. The original motivation for looking at jets stemmed from a search for a new proof of the following theorem.

Theorem (Faltings). *Let X be a closed subvariety of an abelian variety A over a number field k . Assume that X has trivial Ueno fibration (i.e., it is not invariant under a positive-dimensional subgroup of A). Then $X(k)$ is not Zariski dense.*

I'd like to find another proof of this theorem using methods similar to those of Schmidt's Subspace Theorem. It has been known for some time that this proof is similar to Ahlfors' proof of the corresponding theorem for holomorphic curves. Where Schmidt used successive minima, though, Ahlfors used derivatives. Past attempts to extend Schmidt's methods to give a proof of the above theorem failed, due to an inability to find a version of Davenport's lemma valid in the context of abelian varieties. Working with jets, though, may allow me to get around this problem.

[*Added after the talk:* It seems likely, however, that this problem specifically needs to use Semple-Demailly jet spaces, due to the way they would be used in the problem. This task may therefore be put on hold.]

2. It may be possible to extend work of Demailly and El Goul [**D-E**] to higher jets using these jet spaces. Their work extended work of Bogomolov from the cotangent bundle (i.e., 1-jets) to 2-jets, but were unable to work with higher jets due to combinatorial difficulties with working with Schur functors. Since this definition of jets given here arose as an alternative to combinatorial difficulties associated with Semple-Demailly jet spaces, and since this definition agrees with the Semple-Demailly definition only up to 2-jets, it may be the case that the work of Demailly and El Goul can be extended using the jets proposed here. It was this possibility, in fact, that led me to view the definition given here as a viable alternative to Semple-Demailly jet spaces. See also [**R**].

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Department of Mathematics, University of California, 970 Evans Hall #3840, Berkeley, CA 94720-3840

HOLOMORPHIC CURVES INTO ALGEBRAIC VARIETIES

Min Ru

This manuscript is based on the talk given at the Hayama Symposium on several complex variables 2004 @ Shonan Village Center, Japan, December 18-21. In this manuscript, we will outline the proof given by the author which establishes a defect relation for algebraically nondegenerate holomorphic mappings into an arbitrary non-singular complex projective variety V (rather than just the projective space) intersecting possible non-linear hypersurfaces, extending H. Cartan's result. Our method consists of embedding V into a linear variety by means of a suitable Veronese map and then apply Cartan's defect relation. In doing so, we first derive, for a projective variety X, an explicit lower bound of the m-th normalized Hilbert weight of X in terms of the normalized Chow weight of X (or Mumford's degree of contact). The full manuscript has been written (see [Ru4]) and was submitted elsewhere.

1. Introduction and statements

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map, and $H_j, 1 \leq j \leq q$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. In 1933, H. Cartan [Ca] proved the defect relation(or a Second Main Theorem) $\sum_{j=1}^{q} \delta_f(H_j) \leq n+1$. Since then, researches of higher dimensional Nevanlinna theory have been carried out along these two directions: (i) study the algebraically nondegenerate holomorhic mappings into an arbitrary non-singular complex projective variety V; (ii) replace targets of the hyperplanes appearing in Cartan's result by curvilinear divisors. However, over nearly 70 years, substantial progress along these two directions is still limited. Known results are mainly restricted to either $V = \mathbb{P}^n$ or V is an abelian variety.

This paper studies the defect relation for holomorphic curves $f : \mathbb{C} \to V$ intersecting hyperpsurfaces, where V is an arbitrary non-singular complex projective variety. Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n \geq 1$. Let

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$D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$. $D_1, \ldots, D_q$ are said to be *in general position in* V if for every subset $\{i_0, \ldots, i_n\} \subset \{1, \ldots, q\}$,

$$V \cap \mathrm{supp} D_{i_0} \cap \cdots \cap \mathrm{supp} D_{i_n} = \emptyset, \tag{1.1}$$

where supp(D) means the support of the divisor D. In this manuscript, we will outline the proof of the following theorem, claimed in the talk given at the Hayama Symposium on several complex variables 2004 @ Shonan Village Center, December 18-21. The full manuscript has been written (see [Ru4]) and was submitted elsewhere.

Main Theorem. *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree d_j, located in general position in V. Let $f : \mathbb{C} \to V$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon) T_f(r),$$

where the inequality holds for all $r \in (0, +\infty)$ except for a possible set E with finite Lebesgue measure.

Define the defect, with respect to a hypersurface D of degree d,

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r, D)}{dT_f(r)}.$$

The main theorem then implies that

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \dim V + 1.$$

The case when $V = \mathbb{P}^n$ was obtained by the author [Ru3] earlier, which completely settles a long-standing conjecture of B. Shiffman.

2. Motivations

The result is motivated by the analogy between Nevanlinna theory and Diophantine approximation, discovered by C. Osgood, P. Vojta and S. Lang. etc.. It is now known that Cartan's Second Main Theorem corresponds to Schmidt's subspace theorem. In 1994, G. Faltings and G. Wüstholz [FW] extended Schmidt's result to the systems of Diophantine inequalities to be solved in algebraic points of an arbitrary projective variety. Whereas Schmidt's proof of his subspace theorem is based on techniques from Diophantine approximation and geometry of numbers, Faltings and G. Wüstholz developed a totally different method, based on Faltings' Product Theorem (cf. [FW], Theorem 3.1, 3.3). Moreover they introduced a probability measure on $\mathbb{R}$ whose expected value is the crucial tool in the proof of their main result. R.G. Ferretti (see [F1], [F2]) later observed that their expected value can be reformulated in terms of the Chow weight of X (or Mumford's degree of contact). In fact, for every N-tuple $\mathbf{c} = (c_0, \ldots, c_N)$ where $c_0, \ldots, c_N$ are integers with $c_0 \geq \cdots \geq c_N$, R.G. Ferretti observed that $E_{\mathbf{c},\infty} = \frac{e_{\mathbf{c}}(X)}{(\dim(X)+1)\deg(X)}$, where $E_{\mathbf{c},\infty}$ is the Faltings-Wüstholz expected value with respect to $\mathbf{c}$ and $e_{\mathbf{c}}(X)$ is the Chow weight of X with respect to $\mathbf{c}$. Ferretti's observation brought the geometric invariant theory (Mumford's degree of contact is a birational invariant often considered in Geometric Invariant Theory (see [Mu], [Mo])) into the study of Diophantine approximation. Later, J.H. Evertse and R. Ferretti (cf. [EF1], [EF2]) further developed this technique and derived a quantitative version of Faltings and Wüstholz's result directly from Schmidt's (quantitative) subspace theorem. They also extended Schmidt's subspace theorem with polynomials of arbitrary degree (see also [CZ]). This paper is inspired by these developments.

3. Chow Weights and Hilbert Weights

Chow Form: Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a projective subvariety of dimension n and degree

$\triangle$. To X we can associate, up to a constant scalar, a unique irreducible polynomial

$$F_X(\mathbf{u}_0, \dots, \mathbf{u}_n) = F_X(u_{00}, \dots, u_{0N}; \dots; u_{n0}, \dots, u_{nN})$$

in $n+1$ blocks of variables $\mathbf{u}_i = (u_{i0}, \dots, u_{iN})$ $(i = 0, \dots, n)$, which is called the *Chow form* of X, with the following properties: F_X is homogeneous of degree $\triangle$ in each block $\mathbf{u}_i(i = 0, \dots, n)$; and $F_X(\mathbf{u}_0, \dots, \mathbf{u}_n) = 0$ if and only if $X \cap H_{\mathbf{u}_0} \cap \dots \cap H_{\mathbf{u}_n} \neq \emptyset$, where $H_{\mathbf{u}_i}$ $(i = 0, \dots, n)$ are the hyperplanes given by $\mathbf{u}_i \cdot \mathbf{x} = u_{i0}x_0 + \dots + u_{iN}x_N = 0$.

Chow Weight: Let $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$. Consider

$$\begin{aligned} &F_X(t^{c_0}u_{00}, \dots, t^{c_N}u_{0N}; \dots; t^{c_0}u_{n0}, \dots, t^{c_N}u_{nN}) \\ &= \sum_{k=0}^{r} t^{e_k} G_k(\mathbf{u}_0, \dots, \mathbf{u}_n) \end{aligned}$$

with $G_k \mathbb{C}[u_{00}, \dots, u_{0N}; \dots; u_{n0}, \dots, u_{nN}], k = 0, \dots, r$ and $e_0 > e_1 > \dots > e_r$. We define the *Chow weight* of X with respect to $\mathbf{c}$ by

$$e_X(\mathbf{c}) := e_0.$$

Hilbert function and Hilbert Weight: Let $I = I_X$ be the prime ideal in $\mathbb{C}[x_0, \dots, x_N]$ defining X, and let $\mathbb{C}[x_0, \dots, x_N]_m$ be the vector space of homogeneous polynomials of degree m (including 0). Put $I_m := \mathbb{C}[x_0, \dots, x_N]_m \cap I$ and define the *Hilbert function* H_I of I by, for $m = 1, 2, \dots$,

$$H_I(m) := \dim \left(\mathbb{C}[x_0, \dots, x_N]_m / I_m \right).$$

Then

$$H_I(m) = \triangle \cdot \frac{m^n}{n!} + O(m^{n-1}).$$

We define the *m-th Hilbert weight* $S_I(m, \mathbf{c})$ of I with respect to $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$ by

$$S_I(m, \mathbf{c}) := \max \left(\sum_{i=1}^{H_I(m)} \mathbf{a}_i \cdot \mathbf{c} \right),$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_I(m)}}$ whose residue classes modulo I form a basis of $\mathbb{C}[x_0, \ldots, x_N]_m/I_m$. According to Mumford, we have

$$S_I(m, \mathbf{c}) = e_I(\mathbf{c}) \cdot \frac{m^{n+1}}{(n+1)!} + O(m^n),$$

which implies that

$$\lim_{m\to\infty} \frac{1}{mH_I(m)} \cdot S_I(m, \mathbf{c}) = \frac{1}{(n+1)\triangle} \cdot e_I(\mathbf{c}).$$

Write $H_X(m) = H_{I_X}(m), S_X(m, \mathbf{c}) = S_{I_X}(m, \mathbf{c})$, where $I = I_X$ be the prime ideal in $\mathbb{C}[x_0, \ldots, x_N]$ defining X. We can prove the following theorem, which gives a lower bound of $S_X(m, \mathbf{c})$ in terms of $e_X(\mathbf{c})$ with the explict coefficients (see [Ru4] for the proof).

Theorem 3.1. *Let X be a subvariety of $\mathbb{P}^N$ of dimension n and degree $\triangle$. Let $m > \triangle$ be an integer and let $\mathbf{c} = (c_0, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geq 0}$. Then*

$$\frac{1}{mH_X(m)} S_X(m, \mathbf{c}) \geq \frac{1}{(n+1)\triangle} e_X(\mathbf{c}) - \frac{(2n+1)\triangle}{m} \cdot \left(\max_{i=0,\ldots,N} c_i \right).$$

We also need the following lemma (see [Ru4] for the proof).

Lemma 3.1. *Let Y be a subvariety of $\mathbb{P}^{q-1}$ of dimension n and degree $\triangle$. Let $\mathbf{c} = (c_1, \ldots, c_q)$ be a tuple of reals. Let $\{i_0, \ldots, i_n\}$ be a subset of $\{1, \ldots, q\}$ such that*

$$Y \cap \{y_{i_0} = 0, \ldots, y_{i_n} = 0\} = \emptyset.$$

Then

$$e_Y(\mathbf{c}) \geq (c_{i_0} + \cdots + c_{i_n}) \cdot \triangle.$$

4. Vojta's extension of Cartan's theorem

We first introduce some standard notations in Nevanlinna theory: Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic map. Let $\tilde{f} = (f_0, \ldots, f_N)$ be a reduced representative of f, where $f_0, \ldots, f_N$ are entire functions on $\mathbb{C}$ and have no common zeros. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log ||\tilde{f}(re^{i\theta})|| d\theta$$

where

$$\|\tilde{f}(z)\| = \max\{|f_0(z)|, \ldots, |f_N(z)|\}.$$

The above definition is independent, up to an additive constant, of the choice of the reduced representation of f. Let D be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ of degree d. Let Q be the homogeneous polynomial (form) of degree d defining D. The proximity function $m_f(r, D)$ is defined as

$$m_f(r, D) = \int_0^{2\pi} \log \frac{\|\tilde{f}(re^{i\theta})\|^d \|Q\|}{|Q(\tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi},$$

where $\|Q\|$ is the maximum of the absolute values of the coefficients of Q. To define the counting function, let $n_f(r, D)$ be the number of zeros of $Q \circ \tilde{f}$ in the disk $|z| < r$, counting multiplicity. The counting function is then defined by

$$N_f(r, D) = \int_0^r \frac{n_f(t, D) - n_f(0, D)}{t} dt + n_f(0, D) \log r.$$

The Poisson-Jensen formula implies:

First Main Theorem. *Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic map, and let D be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ of degree d. If $f(\mathbb{C}) \not\subset D$, then for every real number r with $0 < r < \infty$*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

where $O(1)$ *is a constant independent of* r.

The following Second Main Theorem on holomorphic curves intersecting hyperplanes extends H. Cartan's result. It is due to P. Vojta (see [V2]). The theorem is also stated and proved in [Ru1] (see Theorem 2.1 in [Ru1]).

Theorem 4.1. *Let* $f = [f_0 : \cdots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *be a holomorphic map whose image is not contained in any proper linear subspace. Let* $H_1, ..., H_q$ *be arbitrary hyperplanes in* $\mathbb{P}^n(\mathbb{C})$. *Let* $L_j, 1 \leq j \leq q$, *be the linear forms defining* $H_1, \ldots, H_q$. *Then, for every* $\epsilon > 0$,

$$\int_0^{2\pi} \max_K \log \prod_{j \in K} \frac{\|f(re^{i\theta})\| \|L_j\|}{|L_j(f)(re^{i\theta})|} \frac{d\theta}{2\pi} \leq (n + 1 + \epsilon) T_f(r),$$

where the inequality holds for all r *outside of a set* E *with finite Lebesgue measure, the maximum is taken over all subsets* K *of* $\{1, \ldots, q\}$ *such that* $\#K = n + 1$ *and the linear forms* $L_j, j \in K$, *are linearly independent, and* $\|L_j\|$ *is the maximum of the absolute values of the coefficients in* L_j.

5. Proof of the Main Theorem

We now prove our Main Theorem.

Proof of the Main Theorem. Let $D_1, \ldots, D_q$ be the hypersurfaces in $\mathbb{P}^N(\mathbb{C})$, located in general position on V. Let $Q_j, 1 \leq j \leq q$, be the homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ of degree d_j defining D_j. Replacing Q_j by Q_j^{d/d_j} if necessary, where d is the l.c.m of $d_j' s$, we can assume that $Q_1, \ldots, Q_q$ have the same degree of d. For every $\mathbf{b} = [b_0 : \cdots : b_N] \in \mathbb{P}^N(\mathbb{C})$, consider the function

$$\|\mathbf{b}, D_j\| = \frac{|Q_j(\mathbf{b})|}{\|\mathbf{b}\|^d \|Q_j\|}, \tag{5.1}$$

where $\|\mathbf{b}\| = \max_{0 \leq j \leq N} |b_j|$ and $\|Q_j\|$ is the maximum of the absolute values of the coefficients of Q_j. By the "in general position" condition, at each point $\mathbf{b} \in V$, there

$\|\mathbf{b}, D_j\|$ can be zero for no more than n indicies $j \in \{1, \dots, q\}$. For the remaining indices j, we have $\|\mathbf{b}, D_j\| > 0$ and by the continuity of these functions and the compactness of V, there exists $C > 0$ such that $\|\mathbf{b}, D_j\| > C$ for all $\mathbf{b} \in V$ and all D_j, except for at most n of them. Hence, for any holomorphic map $f : \mathbb{C} \to V$,

$$\begin{aligned}
\sum_{j=1}^{q} m_f(r, D_j) &= \int_0^{2\pi} \sum_{j=1}^{q} \log \frac{1}{\|f(re^{i\theta}), D_j\|} \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log \prod_{j=1}^{q} \frac{\|f(re^{i\theta})\|^d \|Q_j\|}{|Q_j(f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&\leq \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \left\{ \log \prod_{k=0}^{n} \frac{\|f(re^{i\theta})\|^d \|Q_{i_k}\|}{|Q_{i_k}(f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1).
\end{aligned} \tag{5.2}$$

Define a map $\phi : \mathbf{x} \in V \mapsto [Q_1(\mathbf{x}) : \cdots : Q_q(\mathbf{x})] \in \mathbb{P}^{q-1}(\mathbb{C})$ and let $Y = \phi(V)$. By the "in general position" assumption, ϕ is a finite morphism on V and Y is a complex projective subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$. We also have $\dim Y = n$ and $\deg Y := \triangle \leq d^n D$, where $D = \deg V$. For every $\mathbf{a} = (a_1, \dots, a_q) \in \mathbb{Z}^q_{\geq 0}$, denote by $\mathbf{y}^{\mathbf{a}} = y_1^{a_1} \cdots y_q^{a_q}$. Let m be a positive integer. Put

$$n_m := H_Y(m) - 1, \ q_m := \binom{q+m-1}{m} - 1. \tag{5.3}$$

Consider the Veronese embedding

$$\phi_m : \mathbb{P}^{q-1}(\mathbb{C}) \hookrightarrow \mathbb{P}^{q_m}(\mathbb{C}) : \mathbf{y} \mapsto (\mathbf{y}^{\mathbf{a}_0}, \dots, \mathbf{y}^{\mathbf{a}_{q_m}}), \tag{5.4}$$

where $\mathbf{y}^{\mathbf{a}_0}, \dots, \mathbf{y}^{\mathbf{a}_{q_m}}$ are the monomials of degree m in $y_1, \dots, y_q$, in some order. Denote by Y_m the smallest linear variety of $\mathbb{P}^{q_m}$ containing $\phi_m(Y)$. Then, clearly, a linear form $\sum_{i=0}^{q_m} \gamma_i z_i$ vanishes identically on Y_m if and only if $\sum_{i=0}^{q_m} \gamma_i \mathbf{y}^{\mathbf{a}_i}$, as a polynomial of degree m, vanishes identically on Y. In other words, there is an isomorphism

$$\mathbb{C}[y_1, \dots, y_q]/(I_Y)_m \simeq Y_m^{\vee} : \mathbf{y}^{\mathbf{a}_i} \mapsto z_i, \ i = 0, \dots, q_m,$$

where I_Y is the prime ideal in $\mathbb{C}[y_1, \ldots, y_q]$ defining Y, $(I_Y)_m$ is the vector space of homogeneous polynomials of degree m in I_Y, and $Y_m^\vee$ is the vector space of linear forms in $\mathbb{C}[z_0, \ldots, z_{q_m}]$ modulo the linear forms vanishing identically on Y_m. Hence Y_m is an n_m-dimensional linear subspace of $\mathbb{P}^{q_m}(\mathbb{C})$ where $n_m = H_Y(m)-1$. Since Y_m is an n_m-dimensional linear subspace of $\mathbb{P}^{q_m}(\mathbb{C})$, there are linear forms $L_0, \ldots, L_{q_m} \in \mathbb{C}[w_0, \ldots, w_{n_m}]$ such that the map

$$\psi_m : \mathbf{w} \in \mathbb{P}^{n_m}(\mathbb{C}) \mapsto (L_0(\mathbf{w}), \ldots, L_{q_m}(\mathbf{w})) \in Y_m \tag{5.5}$$

is a linear isomorphism from $\mathbb{P}^{n_m}(\mathbb{C})$ to Y_m. Thus $\psi_m^{-1}\phi_m$ is an injective map from Y into $\mathbb{P}^{n_m}$. Let $f : \mathbb{C} \to V$ be the given holomorphic map and let $F = \psi_m^{-1} \circ \phi_m \circ \phi \circ f : \mathbb{C} \to \mathbb{P}^{n_m}$. Then F is a holomorphic map. Furthermore, since f is algebraically non-degenerate, F is linearly non-degenerate. For every $z \in \mathbb{C}$, let $\mathbf{c}_z = (c_{1,z}, \ldots, c_{q,z})$ where

$$c_{j,z} := \log \frac{\|f(z)\|^d \|Q_j\|}{|Q_j(f)(z)|}, \; j = 1, \ldots, q. \tag{5.6}$$

Obviously, $c_{j,z} \geq 0$ for $j = 1, \ldots, q$. For every $z \in \mathbb{C}$ there is a subset I_z of $\{0, \ldots, q_m\}$ of cardinality $n_m+1 = H_Y(m)$ such that $\{\mathbf{y}^{\mathbf{a}_i} : i \in I_z\}$ is a basis of $\mathbb{C}[y_1, \ldots, y_q]/(I_Y)_m$ and

$$S_Y(m, \mathbf{c}_z) = \sum_{i \in I_z} \mathbf{a}_i \cdot \mathbf{c}_z. \tag{5.7}$$

Note that, for every $\mathbf{w} \in \mathbb{P}^{n_m}$, we have $L_i(\mathbf{w}) = \mathbf{y}^{\mathbf{a}_i}$, $i = 0, \ldots, q_m$. Hence, using (5.7), we have

$$\begin{aligned}
\log \prod_{i \in I_z} \frac{\|L_i\|}{|L_i(F)(z)|} &= \log \prod_{i \in I_z} \frac{1}{|Q_1(f)(z)|^{a_{i,1}} \cdots |Q_q(f)(z)|^{a_{i,q}}} + O(1) \\
&= \log \prod_{i \in I_z} \left[\left(\frac{\|f(z)\|^d \|Q_1\|}{|Q_1(f)(z)|} \right)^{a_{i,1}} \cdots \left(\frac{\|f(z)\|^d \|Q_q\|}{|Q_q(f)(z)|} \right)^{a_{i,q}} \right] \\
&\quad - dH_Y(m) m \log \|f(z)\| + O(1) \\
&= \sum_{i \in I_z} \mathbf{a}_i \cdot \mathbf{c}_z - dmH_Y(m) \log \|f(z)\| + O(1) \\
&= S_Y(m, \mathbf{c}_z) - dmH_Y(m) \log \|f(z)\| + O(1).
\end{aligned} \tag{5.8}$$

Hence

$$\begin{aligned} S_Y(m,\mathbf{c}_z) &\le \max_J \log \prod_{j\in J} \frac{\|L_j\|}{|L_j(F)(z)|} + dmH_Y(m)\log\|f(z)\| + O(1) \\ &= \max_J \log \prod_{j\in J} \frac{\|F(z)\|\|L_j\|}{|L_j(F)(z)|} + dmH_Y(m)\log\|f(z)\| - (n_m+1)\log\|F(z)\| + O(1), \end{aligned} \tag{5.9}$$

where the maximum is taken over all $J \subset \{0,\ldots,q_m\}$ such that $\#J = n_m+1$ and $L_j, j\in J$, are linearly independent. By Theorem 3.1

$$\frac{1}{mH_Y(m)} S_Y(m,\mathbf{c}_z) \ge \frac{1}{(n+1)\triangle} e_Y(\mathbf{c}_z) - \frac{(2n+1)\triangle}{m}\left(\max_{1\le i\le q} c_{i,z}\right), \tag{5.10}$$

and by Lemma 3.1, for any $\{i_0,\ldots,i_n\} \subset \{1,\ldots,q\}$, since $D_1,\ldots,D_q$ are in general position in V ,

$$e_Y(\mathbf{c}_z) \ge (c_{i_0,z} + \cdots + c_{i_n,z}) \cdot \triangle. \tag{5.11}$$

Combining (5.6), (5.10) and (5.11) gives

$$\begin{aligned} \frac{1}{mH_Y(m)} S_Y(m,\mathbf{c}_z) &\ge \frac{1}{(n+1)}(c_{i_0,z} + \cdots + c_{i_n,z}) - \frac{(2n+1)\triangle}{m}\left(\max_{1\le i\le q} c_{i,z}\right) \\ &= \frac{1}{(n+1)} \log\left(\frac{\|f(z)\|^d\|Q_{i_0}\|}{|Q_{i_0}(f)(z)|}\cdots\frac{\|f(z)\|^d\|Q_{i_n}\|}{|Q_{i_n}(f)(z)|}\right) \\ &\quad - \frac{(2n+1)\triangle}{m}\left(\max_{1\le j\le q} \log\frac{\|f(z)\|\|Q_j\|}{|Q_j(f)(z)|}\right). \end{aligned} \tag{5.12}$$

By (5.9) and (5.12) we have

$$\begin{aligned} &\max_{i_0,\ldots,i_n} \log\left(\frac{\|f(z)\|^d\|Q_{i_0}\|}{|Q_{i_0}(f)(z)|}\cdots\frac{\|f(z)\|^d\|Q_{i_n}\|}{|Q_{i_n}(f)(z)|}\right) \\ &\le \frac{(n+1)}{mH_Y(m)}\left(\max_J \log\prod_{j\in J}\frac{\|F(z)\|\|L_j\|}{|L_j(F)(z)|} - (n_m+1)\log\|F(z)\|\right) \\ &+ d(n+1)\log\|f(z)\| + \frac{(2n+1)(n+1)\triangle}{m}\left(\max_{1\le j\le q}\log\frac{\|f(z)\|\|Q_j\|}{|Q_j(f)(z)|}\right) + O(1). \end{aligned} \tag{5.13}$$

Applying integration on the both sides of (5.13) and using the First Main theorem yield

$$\int_0^{2\pi} \max_{i_0,\dots,i_n} \log\left(\frac{\|f(re^{i\theta})\|^d\|Q_{i_0}\|}{|Q_{i_0}(f)(re^{i\theta})|}\cdots\frac{\|f(re^{i\theta})\|^d\|Q_{i_n}\|}{|Q_{i_n}(f)(re^{i\theta})|}\right)\frac{d\theta}{2\pi}$$
$$\le \frac{(n+1)}{mH_Y(m)}\left(\int_0^{2\pi}\max_J \log\prod_{j\in J}\frac{\|F(re^{i\theta})\|\|L_j\|}{|L_j(F)(re^{i\theta})|}\frac{d\theta}{2\pi} - (n_m+1)T_F(r)\right)$$
$$+ d(n+1)T_f(r) + \frac{(2n+1)(n+1)\triangle}{m}\sum_{1\le j\le q} m_f(r, D_j) + O(1)$$
$$\le \frac{(n+1)}{mH_Y(m)}\left(\int_0^{2\pi}\max_J \log\prod_{j\in J}\frac{\|F(re^{i\theta})\|\|L_j\|}{|L_j(F)(re^{i\theta})|}\frac{d\theta}{2\pi} - (n_m+1)T_F(r)\right)$$
$$+ d(n+1)T_f(r) + \frac{(2n+1)(n+1)q\triangle}{m}T_f(r) + O(1), \tag{5.14}$$

here we note that various constants in the "O(1)" term above depend only on $Q_1,\dots,Q_q$, not on f and z. For the $\epsilon > 0$ given in the Main Theorem, take m large enough so that

$$\frac{(n+1)}{H_Y(m)} < \epsilon/3d, \text{ and } \frac{(2n+1)(n+1)q\triangle}{m} < \epsilon/3. \tag{5.15}$$

Fix such an m. Applying Theorem 5.1 with $\epsilon = 1$ to holomorphic map F and linear forms $L_0,\dots,L_{q_m}$, we obtain that

$$\int_0^{2\pi}\max_J \log\prod_{j\in J}\frac{\|F(re^{i\theta})\|\|L_j\|}{|L_j(F)(re^{i\theta})|}\frac{d\theta}{2\pi} \le (n_m+2)T_F(r) \tag{5.16}$$

holds for all r outside of a set E with finite Lebesgue measure. By combining (5.2), (5.14), (5.15) and (5.16), we get

$$\sum_{j=1}^q m_f(r, D_j) \le \frac{\epsilon}{3dm}T_F(r) + d(n+1)T_f(r) + (\epsilon/3)T_f(r) + O(1) \tag{5.17}$$

where the inequality holds for all r outside of a set E with finite Lebesgue measure. By the definition of the characteristic function, we have $T_F(r) \le dmT_f(r)$. Hence (5.17) becomes

$$\sum_{j=1}^q m_f(r, D_j) \le (d(n+1) + 2\epsilon/3)T_f(r) + C \tag{5.18}$$

where the inequality holds for for all r outside of a set E with finite Lebesgue measure, and where C is a constant, independent of r. Take r big enough so we can make $C \leq (\epsilon/3)T_f(r)$. Thus we have

$$\sum_{j=1}^{q} m_f(r, D_j) \leq (d(n+1)+\epsilon)T_f(r)$$

where the inequality holds for for all r outside of a set E with finite Lebesgue measure. This proves the Main Theorem. $\square$

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Department of Mathematics, University of Houston, Houston, TX 77204,
E-mail address: `minru@math.uh.edu`

RELATIVE ENDS AND PROPER HOLOMORPHIC MAPPINGS TO RIEMANN SURFACES

TERRENCE NAPIER

This talk describes joint work with Mohan Ramachandran and is arranged as follows:

1. The main result.
2. Definitions.
3. History.
4. The theorem of Gromov and Schoen.
5. The proof.

1. The main result

We will consider the following generalization of a recent result of [Delzant-Gromov 2004]:

Theorem 1.1 ([N-Ramachandran 2004]). *Let X be a connected noncompact complete Kähler manifold which has bounded geometry, which is weakly 1-complete, or which admits a positive symmetric Green's function G that vanishes at infinity. If $\tilde{e}(X) \geq 3$, then X admits a proper holomorphic mapping onto a Riemann surface (i.e X is holomorphically convex with 1-dimensional Cartan-Remmert reduction).*

2. Definitions

Definition 2.1. Let M be a connected noncompact manifold.

(a) Depending on the context, by an *end* of M we will mean either a component E of $M \setminus K$ with noncompact closure, where K is a given compact subset of M, or an element of

$$\varprojlim \pi_0(M \setminus K)$$

where the limit is taken as K ranges over the compact subsets of M whose complement $M \setminus K$ has no relatively compact components. The cardinality of the above set is denoted by $e(M)$. For a compact set K such that $M \setminus K$ has no relatively compact components, we get an *ends decomposition*

$$M \setminus K = E_1 \cup \cdots \cup E_m,$$

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where $E_1, \dots, E_m$ are the distinct components of $M \setminus K$.

(b) For $\Upsilon : \tilde{M} \to M$ the universal covering of M, following [Kropholler-Roller 1989] (and language due to Geoghegan), we will call the set

$$\lim_{\leftarrow} \pi_0[\Upsilon^{-1}(M \setminus K)],$$

where the limit is taken as K ranges over the compact subsets of M whose complement $M \setminus K$ has no relatively compact components, the set of *filtered ends* (or *relative ends* or *KRends*) for the pair $\widetilde{M}$ and M. We will denote the cardinality of this set of relative ends by

$$\tilde{e}(M) = e(\widetilde{M}, M).$$

Clearly, $\tilde{e}(M) \geq e(M)$. In fact, for $k \in \mathbb{N}$, we have $\tilde{e}(M) \geq k$ if and only if there exists an ends decomposition $M \setminus K = E_1 \cup \cdots \cup E_m$ for M such that, for $\Gamma_j = \operatorname{im}(\pi_1(E_j) \to \pi_1(M))$ for $j = 1, \dots, m$, we have

$$\sum_{j=1}^{m} [\pi_1(M) : \Gamma_j] \geq k.$$

Definition 2.2. A complete Kähler manifold X *has bounded geometry* (*of order* 2) if there exists a constant $C > 0$ and, for each point $p \in X$, a biholomorphism Ψ of a neighborhood U of p onto $B(0;1) \subset \mathbb{C}^n$ such that $\Psi(p) = 0$, $C^{-1}\Psi^* g_{\mathbb{C}^n} \leq g \leq C\Psi^* g_{\mathbb{C}^n}$, and $|D^k g|_{g_{\mathbb{C}^n}} \leq C$ for $k = 1, 2$.

For example, a connected covering manifold of a compact Kähler manifold has bounded geometry.

3. History

Theorem 3.1 ([Li 1990], [Gromov 1991], N-Ramachandran [1995[). *Let X be a connected noncompact complete Kähler manifold which has bounded geometry, which is weakly 1-complete, or which admits a positive symmetric Green's function G that vanishes at infinity. If $e(X) \geq 3$, then X admits a proper holomorphic mapping onto a Riemann surface.*

Sketch of the proof. For an ends decomposition $X \setminus K = E_1 \cup E_2 \cup E_3$, there exist pluriharmonic functions ρ_1, ρ_2 such that, the limit inferior at infinity is ∞ or 1 for ρ_1 and $-\infty$ or 0 for ρ_2 along E_1, $-\infty$ or 0 for ρ_1 and ∞ or 1 for ρ_2 along E_2, and $-\infty$ or 0 for both ρ_1 and ρ_2 along E_3 ([Nakai 1962], [Nakai 1970], [Sario-Nakai 1970], [Sario-Noshiro 1966], [Sullivan 1981]). These conditions guarantee that ρ_1, ρ_2 are independent. On the other hand, Gromov's cup product lemma [Gromov 1991] and other considerations imply that $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$. Stein factorization of the map $\dfrac{\partial\rho_1}{\partial\rho_2} : X \to \mathbb{P}^1$ gives the required mapping. $\square$

Theorem 1.1 is a generalization of the following:

Theorem 3.2 ([Delzant-Gromov 2004]). *Let X be a connected noncompact complete Kähler manifold which has bounded geometry and which admits a positive symmetric Green's function G that vanishes at infinity. If $\tilde{e}(X) \geq 3$, then X admits a proper holomorphic mapping onto a Riemann surface*

Their proof relies on the existence of Pluriharmonic maps into trees ([Gromov-Schoen 1992], [Sun 2003]). The proof in [N-Ramachandran 2004] relies only on the existence and properties of pluriharmonic functions.

Counter-examples (for example, the example of [Cousin 1910] of a covering space X of an Abelian variety A such that $\tilde{e}(X) = e(X) = 2$ but $\mathcal{O}(X) = \mathbb{C}$) demonstrate that, even with much stronger conditions, Theorem 1.1 fails for 2 (filtered) ends.

4. The theorem of Gromov and Schoen

A closely related topic is the study of Kähler groups; i.e. fundamental groups of compact Kähler manifolds. Theorem 1.1 can be used to prove the following:

Theorem 4.1 ([Gromov-Schoen 1992]). *Let X be a connected compact Kähler manifold whose fundamental group admits an amalgamated product decomposition*

$$\pi_1(X) = \Gamma_1 *_\Gamma \Gamma_2$$

where the index of Γ in Γ_1 is at least 3 and the index of Γ in Γ_2 is at least 2. Then some finite (unramified) covering of X admits a surjective holomorphic mapping onto a curve of genus $g \geq 2$.

Like Delzant and Gromov's proof of Theorem 3.2, Gromov and Schoen's proof of Theorem 4.1 relies on the existence of Pluriharmonic maps into trees. Theorem 1.1 also gives the following:

Theorem 4.2 ([N-Ramachandran 2001], [N-Ramachandran 2004],). *Let X be a compact Kähler manifold such that $\pi_1(X)$ is a proper ascending HNN extension. Then X admits a surjective holomorphic mapping onto a curve of genus $g \geq 2$.*

5. The proof

We will illustrate the main ideas of the proof of Theorem 1.1 by considering the following special case:

Sketch of the proof for X of bounded geometry and $e(X) = 2$ with both ends nonparabolic. In this case, X admits an ends decomposition $X \setminus K = E_0 \cup E_1$ such that $\Gamma_1 = \mathrm{im}\,[\pi_1(E_1) \to \pi_1(X)]$ is a proper subgroup. There exists a pluriharmonic function $\rho_1 : X \to (0,1)$ with finite energy (i.e. $\int |\nabla\rho|^2 < \infty$) such that

$$\liminf_{x\to\infty} \rho_1|_{\overline{E}_0}(x) = 0 \qquad \text{and} \qquad \limsup_{x\to\infty} \rho_1|_{\overline{E}_1}(x) = 1.$$

Moreover, if $\Upsilon : \widehat{X} \to X$ is a connected covering space with $\Upsilon_*\pi_1(\widehat{X}) = \Gamma_1$, then Υ maps a component Ω_1 of $\Upsilon^{-1}(E_1)$ isomorphically onto E_1 and $\Omega_2 = \Upsilon^{-1}(E_1) \setminus \Omega_1 \neq \emptyset$. Let $\Omega_0 = \Upsilon^{-1}(E_0)$ and let $\hat{\rho}_1 = \rho_1 \circ \Upsilon$. Fix r with $\max_K \rho_1 < r < 1$ and let $V = \{\, x \in E_1 \mid \rho_1(x) > r \,\}$.

The set Ω_1 is D-massive in the sense of [Grigor'yan 1999] (see Section 6) with the finite energy admissible subharmonic function

$$\alpha \equiv \begin{cases} \max(\hat{\rho}_1 - r, 0) & \text{on } \Omega_1 \\ 0 & \text{on } \widehat{X} \setminus \Omega_1 \end{cases}$$

Similarly, Ω_0 and Ω_2 are plurimassive sets. Hence there exists a pluriharmonic function $\rho_2 : \widehat{X} \to (0,1)$ with finite energy such that

$$\liminf_{x\to\infty} \rho_2|_{\overline{\Omega}_0 \cup \overline{\Omega}_2}(x) = 0 \qquad \text{and} \qquad \limsup_{x\to\infty} \rho_2|_{\overline{\Omega}_1}(x) = 1.$$

If $\Omega_2 \to E_1$ is a finite cover, then $e(\widehat{X}) \geq 3$ and $\widehat{X}$ admits a proper holomorphic mapping onto a Riemann surface (Theorem 3.1). If an infinite cover, then $d\hat{\rho}_1|_{\Omega_2} \notin L^2$ but $d\rho_2|_{\Omega_2} \in L^2$. Hence $d\hat{\rho}_1$ and $d\rho_2$ are linearly independent. A version of Gromov's cup product lemma in $\Upsilon^{-1}(V) \cap \Omega_1 \cong V$ implies that $\partial\hat{\rho}_1 \wedge \partial\rho_2 \equiv 0$. Stein factorization of the map $\dfrac{\partial\hat{\rho}_1}{\partial\rho_2} : \widehat{X} \to \mathbb{P}^1$ gives a proper holomorphic mapping of V onto a Riemann surface Z. From this, one concludes that the required mapping on X must exist. □

6. Appendix: Massive sets

Definition 6.1 ([Grigor'yan 1999[). Let Ω be an open subset of a Riemannian manifold M.

- If there exists a continuous subharmonic function $\alpha : M \to [0,1]$ such that $\alpha \equiv 0$ on $M \setminus \Omega$ and $\sup_\Omega \alpha > 0$, then we call Ω a *massive set* and α an *admissible subharmonic function* for Ω.
- If α may be chosen to have finite energy, then we call Ω a *D-massive set*.
- If M is a complex manifold and we may choose α to be plurisubharmonic, then we call Ω *plurimassive*.

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Analytic aspects in the local and global theory of ideals of holomorphic functions

Henri Skoda

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By this talk I have chosen to describe how some analytical tools of complex analysis contributed to make decisive progress in local and global analytic geometry for the past thirty years.
Let me briefly recall that the **Oka-Cartan theorem** states that on a Stein manifold X, a holomorphic function f belongs to the ideal generated by a finite number of holomorphic functions $g_1, g_2, \ldots, g_p$ if and only if it is locally true in the ring of germs of holomorphic functions at every point $P \in X$.
In 1965, **L. Hörmander**, in his book [Hör66], proved a similar result in the space of entire functions of exponential type and for **polynomial** generators g_j using for the first time the fundamental $\boldsymbol{L^2}$ **estimates** for the $\boldsymbol{\bar\partial}$ operator.
Let me now state the following theorem [Sko72a] giving in terms of L^2 estimates a **sufficient** condition about f so that $\boldsymbol{f}$ **belongs to the ideal** $\boldsymbol{I}$ generated by the holomorphic functions $g_1, g_2, \ldots, g_p$.
As usual, let us call: $|g|^2 := |g_1|^2 + |g_2|^2 + \ldots + |g_p|^2$.

Theorem 1 (1972) .
Let Ω be an open pseudoconvex set in $\mathbb{C}^{\mathbf{n}}$ and ϕ be a plurisubharmonic function on Ω. Let g_j, $1 \leq j \leq p$, be holomorphic functions on Ω. Let q be the integer: $\boldsymbol{q = \min(p-1, n)}$ and $\boldsymbol{k > q}$ be a given number.
Let f be an holomorphic function on Ω satisfying the estimate:

$$\int_\Omega |f|^2 \, |g|^{-2k-2} \, e^{-\phi} \, d\lambda < +\infty \tag{1}$$

Then there exists holomorphic functions h_j in Ω such that:

$$\boldsymbol{f = gh = g_1h_1 + g_2h_2 + \ldots + g_ph_p} \tag{2}$$

and:

$$\int_\Omega \boldsymbol{|h|^2 \, |g|^{-2k} \, e^{-\phi} \, d\lambda \leq \frac{k}{k-q} \int_\Omega |f|^2 \, |g|^{-2k-2} \, e^{-\phi} \, d\lambda} \tag{3}$$

where $d\lambda$ is the usual Lebesgue measure on $\mathbb{C}^{\mathbf{n}}$.

Let's observe that the choice of the integer q and of the real number k is the best because the local finiteness of the integral has to imply the vanishing of f on the common zeros-set of the functions g_j.
But, of course, application of the theorem needs more information than the vanishing of f on the common zeros-set. You need to know that f is small enough at every point where $|g|$ is small.
Let me just give the main idea of the proof. We solve a very specific $\bar\partial$-equation:

$$\bar\partial u = v$$

where v is a (0,1) differential form closely connected with g and for that, we use Hörmander's L^2 identity for (0,1) form with weight, involving the Levi form of the plurisubharmonic function $k \log |g|^2$.
This result can be easily **iterated** and provides a similar result for the **powers $\boldsymbol{I^l}$** of the ideal I. If $l \geq 1$ is a given integer, one can substitute $k+l-1$ to k. One obtains the more general estimation:

$$\int_\Omega |h|^2 \, |g|^{-2k-2l+2} \, e^{-\phi} \, d\lambda \leq \frac{k+l-1}{k+l-1-q} \int_\Omega |f|^2 \, |g|^{-2k-2l} \, e^{-\phi} \, d\lambda \tag{4}$$

By induction we can iterate the estimation (4) and the relation (2). We obtain the existence of coefficients h_l such that:

$$(5) \qquad \boldsymbol{f = g^l\, h_l := \sum_{|J|=l} g^J h_{l,J} = \sum_{j_1,j_2,\ldots,j_l} g_{j_1} g_{j_2} \ldots g_{j_l}\ h_{l,\,j_1,j_2,\ldots,j_l}}$$

where $J = (j_1, j_2, \ldots, j_l) \in \{1, p\}^l$ is a multi-index of length l (with repetition), with the estimate:

$$(6) \qquad \boldsymbol{\int_\Omega |h_l|^2\ |g|^{-2k}\ e^{-\phi}\ d\lambda \leq (\frac{k}{k-q})^l \int_\Omega |f|^2\ |g|^{-2k-2l}\ e^{-\phi}\ d\lambda}$$

The **basic fact** is that in the estimate:

$$(7) \qquad \int_\Omega |h_l|^2\ |g|^{-2k}\ e^{-\phi}\ d\lambda < +\infty$$

the weight $\boldsymbol{|g|^{-2k} e^{-\phi}}$ **does not depend on** $\boldsymbol{l}$, so that the behaviour of the functions h_l is, roughly speaking, independent on l.

At that time (1973), the first main motivation for such a result was the search of a very precise control of the global behaviour of the functions h_j using the weight ϕ to develop the theory of entire functions in $\mathbb{C}^{\mathbf{n}}$ in the same vein as L. Hörmander and E. Bombieri [Bom70]. It was an amazing fact that such a theorem also implies very strong results in the local theory of ideals.

At first let me recall some basic definitions and results about ideals.

Definition
A germ of homorphic function f is said to ***integral over the ideal I*** *if and only if there exist an integer l and germs of functions $\boldsymbol{a_j \in I^j}$ such that we have the following relation:*

$$(8) \qquad \boldsymbol{f^l + a_1 f^{l-1} + \ldots + a_j f^{l-j} + \ldots + a_l = 0}$$

The set of functions f which are integral over I is called the ***integral closure of I*** *denoted by $\bar{I}$*

The following result gives classical characterizations of the integral closure.

Proposition
The three following statements are equivalent:
1) $\boldsymbol{f}$ *is* ***integral*** *over* $\boldsymbol{I}$.
2) For every germ of holomorphic curve γ*:* $\mathbb{C}, \mathbf{0} \to \mathbf{C^n}, \mathbf{0}$ *the pullback* $f \circ \gamma$ *belongs to the ideal* $\gamma^\star I$ *of* $\mathcal{O}$ *generated by* $g_1 \circ \gamma, g_2 \circ \gamma, \ldots, g_p \circ \gamma$.
3) In some neigbourhood U *of 0, there exists a constant* C *such that in* U, f *verifies the inequality:*
$|\boldsymbol{f}| \leq \boldsymbol{C}|\boldsymbol{g}|$.

First of all we need that the third point is a consequence of the first. The proof of that is an easy application of the Cauchy-Schwarz inequality in $\mathbb{C}^{\mathbf{l}}$.

Of course if f is integral over I, f vanishes on the set of common zeros of the generators $g_1, g_2, \ldots, g_p$ of I and therefore, as a consequence of the Nullstellensatz theorem, some power f^N belongs to I.
The following result [BS74] provides a very simple and optimal value for the integer N.

Theorem 2*, J. Briançon-H. Skoda 1973.*
Let r *be the integer defined by* $\boldsymbol{r} = \mathbf{min}(\boldsymbol{p}, \boldsymbol{n})$. *Then we have the following inclusion:*

$$\bar{\boldsymbol{I}}^{\boldsymbol{r}} \subset \boldsymbol{I} \tag{9}$$

Proof:
Using the third characterization of $\bar{I}$, a function $f \in \bar{I}$ verifies an inequality: $|f| \leq C|g|$ in some neigbourhood U of 0, so that in U:

$$|f^{q+1}| \leq C^{q+1}|g|^{q+1}$$

or

$$|f^{q+1}|^2|g|^{-2q-2} \leq C^{2q+2}$$

Therefore we can apply theorem 1 to f^{q+1} with $\Omega = U$, $\phi = 0$, $k = q + \epsilon$ where $\epsilon > 0$ is taken small enough such that the integral $\int_U |g|^{-2\epsilon} d\lambda < +\infty$ is finite so that theorem 1 implies: $f^{q+1} \in I$.
For $p > n$, we only obtain $f^{n+1} \in I$. We have to work a little more to obtain $f^n \in I$ (using some algebraic results). But we're not giving details today (cf.

[BS74]).

Using the iteraded version of theorem 1 for the ideal I^l, we obtain by the same method, for all integer l, the inclusion:

$$\textbf{(10)} \qquad (\bar{\boldsymbol{I}})^{\boldsymbol{r+l-1}} \subset \boldsymbol{I^l}$$

Application to the Jacobian Ideal:

Let $f \in \mathcal{O}_{\mathbb{C}^n}$ be such that $\boldsymbol{f(0) = 0}$. We consider the Jacobian ideal

$$(11) \qquad J_f = (\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n})$$

generated by the partial derivatives of f at 0.
Of course for $n = 1$, $f \in (\frac{\partial f}{\partial z})$. For $n > 1$, It is not always true that:

$$f \in (\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n})$$

Netherveless, we have:

Corollary
If f is a germ of holomorphic function at 0 vanishing at the origin, then f^n belongs to the Jacobian ideal:

$$\textbf{(12)} \qquad \boldsymbol{f^n \in (\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n})}$$

The result is sharp taking into account the following example:

$$(13) \qquad f = (z_1 z_2 \dots z_n)^{3n} + z_1^{3n-1} + z_2^{3n-1} + \dots + z_n^{3n-1}$$

It is easy to see that: $f^{n-1} \notin J_f$, invoking homogeneity reasons.

Proof of the corollary.
The second characterization of $\bar{I}$ using holomorphic curves γ running through the origin proves that f is in the integral closure of the Jacobian ideal because the result is true for $f \circ \gamma$ in the case of one variable.

Therefore $f^n \in J_f$.

In 1973, no algebraic proof of these results was known except for $n = 2$. The first algebraic proofs were obtained by **B. Tessier, J. Lipman and A. Sataye in 1981** [LS81] and [LT81] and these analytical results were a powerful motivation for further closely related developments in homological algebra as it appears in the R. Lazarsfeld and Hochster's works (for instance the Hochster's work on the "tight closure of an ideal" [WB96]).

Multiplier ideal sheaves and the Briançon-Skoda theorem

In this section, we'll show how the theorem 1 can be useful to study the properties of multiplier ideal sheaves. We briefly recall the definition and main properties of multiplier ideal sheaves. These have been originally explicitly introduced by **A. Nadel (1989)** [Nad89] for the study of the existence of Kähler-Einstein metrics although these ideals were implicitly used in the **E. Bombieri and Y.T Siu 's works in 1970** [Bom70], [Siu74].

Definition
Let φ be a plurisubharmonic function on an open subset $\Omega \subset X$. The following ***ideal sheaf $I(\varphi)$*** *is associated to φ. It is the subsheaf of the sheaf O_Ω of germs of holomorphic functions $f \in O_{\Omega,x}$ such that $|f|^2 e^{-2\varphi}$ is* ***integrable*** *with respect to the Lebesgue measure in some local coordinates system near x.*

The zero variety $V(I(\varphi))$ is thus the set of points in a neighbourhood of which $e^{-2\varphi}$ is not integrable. Such points occur only if φ has logarithmic poles, according to the following basic lemma due to H. Skoda (1972) [Sko72b].
Let's recall the definition of the Lelong's number of the plurisubharmonic function ϕ at the point x:

$$\nu(\varphi, x) := \lim_{r \to 0}[\frac{1}{\log r} \max_{|z-x| \leq r} \phi(z)] \tag{14}$$

Lemma (H. Skoda)
Let φ be a plurisubharmonic function on an open set Ω and let $x \in \Omega$.
a) If $\nu(\varphi, x) < 1$, then $e^{-2\varphi}$ is integrable in a neighbourhood of x, particularly $I(\varphi)_x = O_{\Omega,x}$.
b) If $\boldsymbol{\nu(\varphi, x) \geq n + s}$ for some integer $s \geq 0$, then $e^{-2\varphi} \geq C|z - x|^{-2n-2s}$ in a neighbourhood of x and $\boldsymbol{I(\varphi)_x \subset m_{\Omega,x}^{s+1}}$, where $m_{\Omega,x}$ is the maximal ideal of $O_{\Omega,x}$.

In fact, the ideal sheaf $I(\varphi)$ is always a coherent ideal sheaf and therefore its zero variety is an analytic set. This result is due to Nadel (1989).

Proposition (A. Nadel)
For any psh function φ on $\Omega \subset X$, the sheaf $\boldsymbol{I(\varphi)}$is a ***coherent sheaf*** *of ideals over Ω.*

Proof:
It is a consequence of the classical Oka's coherence theorem for the sheaf $\mathcal{O}$ and of the L. Hörmander's L^2 estimates for $\bar{\partial}$.
The importance of multiplier ideal sheaves comes from the following basic vanishing theorem due to Nadel and also J.P. Demailly [Dem90] et [Dem94], which is a direct consequence of the Andreotti-Vesentini-Hörmander's L^2 estimates. If (L, h) is an hermitian line bundle, we denote $I(h) = I(\varphi)$ where φ is the weight function of h relatively to any trivialization of L over an open set.

Nadel's vanishing theorem 3
Let (X, ω) be a compact Kähler manifold and let L be an holomorphic line bundle over X with a ***singular*** *hermitian metric h such that its curvature form $\Theta_h(L)$ verifies in the sense of currents $\boldsymbol{\Theta_h(L) \geq \varepsilon\omega}$ for some continuous strictly positive function ε on X. Then:*

$$\textbf{(15)} \qquad \boldsymbol{H^q(X, K_X \bigotimes L \bigotimes I(h)) = 0}$$

for all $q \geq 1$.

Now we are trying to understand somewhat better the behaviour of the multiplier ideal sheaf $I(l\varphi)$ as l goes to $+\infty$. Our feeling is that the ideal grows more or less "linearly" with l.

The following result [Dem9] provides a natural inclusion for mutiplier ideal sheaves in the same vein as the Briançon-Skoda's theorem. It is a local result.

Theorem 4 (J.P. Demailly)
Let X be a complex manifold of dimension n and let ϕ and ψ be plurisubharmonic functions on X. Then for any integer l, we have the following inclusions:

$$\boldsymbol{I(l\phi+\psi)\subset I(\phi)^{l-n-1}\,I(\psi)} \tag{16}$$

Proof.
It is equivalent to prove that:

$$I((l+n+1)\phi+\psi)\subset I(\phi)^{l}\,I(\psi) \tag{17}$$

for all $l\in\mathbb{N}$. Since the result is local, we can assume that $X=\Omega$ is a bounded pseudoconvex open set. In that case, after shrinking a little Ω if necessary, the coherence of the multiplier ideal sheaf $I(\varphi)$ shows that $I(\varphi)$ is generated by a finite number of elements $g=(g_1,\ldots,g_p)\in O(\Omega)$ such that:

$$\int_\Omega |g_j|^2 e^{-2\varphi}d\lambda<+\infty. \tag{18}$$

It's a consequence of J.P. Demailly's estimation [Dem99] using the Ohsawa-Takegoshi's extension theorem that:

$$\boldsymbol{\varphi\le \log|g|+C} \tag{19}$$

for some constant $C>0$.
Now, let $f\in I((l+n+1)\varphi+\psi)_{z_0}$ be a germ of holomorphic function defined on a neighbourhood V of $z_0\in\Omega$. If V is small enough, the inequality (19) implies:

$$\int_V |f|^2|g|^{-2l-2n-2}e^{-2\psi}d\lambda\le C'\int_V |f|^2 e^{-2(l+n+1)\varphi-2\psi}d\lambda<+\infty. \tag{20}$$

By the iterated form of the division theorem 1 with $q = n$, $k = n + 1$, this implies that f can be written as:

$$f = g^l . h_l = \sum g_{j_1} g_{j_2} \cdots g_{j_l} h_{l, j_1 j_2 \ldots j_l} \tag{21}$$

for a multi-indexed collection $h_l = (h_{l, j_1 j_2 \ldots j_l})$ of holomorphic functions on V such that:

$$\int_V |h_l|^2 |g|^{-2n-2} e^{-2\psi} d\lambda \leq (n+1)^l \int_V |f|^2 |g|^{-2l-2n-2} e^{-2\psi} d\lambda \tag{22}$$

The last L^2 inequality shows that $\boldsymbol{h_l} \in \boldsymbol{I(\psi)_{z_0}}$. The theorem follows.

J.P. Demailly [Dem99] used this result to give a new proof of Fujita's theorem about the global structure of a big line bundle L on a projective manifold.
If L is a line bundle over X we define the volume of L as:

$$v(L) = \limsup_{k \to +\infty} \frac{n!}{k^n} h^0(X, kL). \tag{23}$$

L is big if and only if $v(L) > 0$ or $h^0(X, kL) = ck^n + o(k^n)$ for some constant $c > 0$ as $k \to +\infty$. One can prove that L is big if and only if L can be written $m_0 L = E + A$ where m_0 is an integer, E is an effective divisor and A is ample.

Fujita's decomposition theorem 6.
Let L be a ***big*** *line bundle. Then for every $\varepsilon > 0$, there exists a* ***modification $\mu : \tilde{X} \to X$*** *and a decomposition* $\boldsymbol{\mu^\star L = E + A}$ *, where E is an effective $\mathbb{Q}$-divisor and A an ample $\mathbb{Q}$-divisor, such that $A^n > v(L) - \epsilon$.*

Another strong geometrical consequence of the theorem 1 is the following Siu's result. It was first observed by Y.T. Siu [Siu98] for his proof of the invariance of plurigenera:

Theorem 7 (Y.T. Siu 97).
Let $\boldsymbol{E}$ *be an* ***ample*** *holomorphic line bundle over an n-dimensional compact complex manifold* X *such that for every point* P *of* X *there is a finite number of global sections of* E *which all vanish to order at least* $n+1$ *at* P *and do'nt simultaneously vanish outside* P.
Then for every holomorphic line bundle $\boldsymbol{L}$ *over* X *with an Hermitian metric locally of the form* $\mathbf{e}^{-\boldsymbol{\phi}}$, ϕ *being plurisubharmonic, of associated ideal sheaf* I_ϕ, *the space of* ***global sections*** *of the sheaf* $I_\phi \otimes (L+E+K_X)$ ***generates the stalk of the sheaf*** $\boldsymbol{I_\phi \otimes (L+E+K_X)}$ *at every point of* X.

It will be easier to read the following formulas if we also denote by I_ϕ the multiplier ideal $I(\phi)$.

Proof:
Let us fix arbitrarly $P \in X$ and take an arbitrary germ s of $(I_\phi)_P$. Let $z = (z_1, z_2, \ldots, z_n)$ be a local coordinates system on some open neighbourhood U of P with $z(P) = 0$ such that $L_{|U}$ is trivial. Let ρ be a cutoff function centered at P so that ρ is a smooth, non-negative-valued function with compact support in U which is identically 1 on some Stein open neighbourhood Ω of P. Let us choose global sections $u_1, u_2, \ldots, u_N$ of E whose common zero-set consists of the single point P and which all vanish to order at least $n+1$ at P.
We suppose that E has a given smooth hermitian metric whose **curvature form is strictly positive** at every point of X

By the standard Hörmander's method of L^2 estimates for $\bar{\partial}$, we can solve the $\bar{\partial}$-equation:

$$\bar{\partial}\sigma = s\ \bar{\partial}\rho \tag{24}$$

for a smooth section σ of the line bundle $L+E+K_X$ which is L^2 with respect to the singular Hermitian metric of $L+E$ defined by multiplying the given metric of E by the weight:

$$\frac{1}{(\sum_{j=1}^{N} |u_j|^2)^{1-\eta}} \tag{25}$$

where $\eta > 0$ is chosen small enough such that

$$(n+1)(1-\eta) > n, \tag{26}$$

that is $\eta < \frac{1}{n+1}$. This new metric has still strictly positive curvature and we have the classical L^2 estimate:

$$\int_X \frac{|\sigma|^2}{(\sum_{j=1}^{N} |u_j|^2)^{1-\eta}} d\lambda \leq \frac{C^{te}}{\eta} \int_X \frac{|s\ \bar{\partial}\rho|^2}{(\sum_{j=1}^{N} |u_j|^2)^{1-\eta}} d\lambda < +\infty \tag{27}$$

The crucial point is: we'll use the **division** theorem 1 to prove that σ has to be in **the right ideal**.
Since $s\ \bar{\partial}\rho$ is identically 0 on Ω, σ is holomorphic on Ω. We now apply the division theorem 1 on Ω to the case $g_j = z_j$, $q = n-1$ and $k+1 = (n+1)(1-\eta) > n$, $k > n-1$ such that the assumptions of theorem 1 are valid. It follows that σ can be written:

$$\sigma = \sum_{j=1}^{n} z_j \tau_j \tag{28}$$

on Ω for some holomorphic $\tau_1, \tau_2, \ldots, \tau_N$.
and:

$$\int_\Omega |\tau_U|^2\ |z|^{-2k}\ e^{-\phi}\ d\lambda \leq \frac{k}{k-n+1} \int_\Omega |\sigma_U|^2\ |z|^{-2k-2}\ e^{-\phi}\ d\lambda \tag{29}$$

where σ_U and $\tau_{j,U}$ are the trivialization of σ and τ_j over U ($|\sigma|^2 = |\sigma_U|^2 e^{-\phi}$).

Since $u_1, u_2, \ldots, u_N$ vanish to order at least $n+1$ the integral:

$$\int_\Omega |\sigma|^2 \, |z|^{-2k-2} \, e^{-\phi} \, d\lambda = \int_\Omega |\sigma|^2 \, |z|^{-2(n+1)(1-\eta)} \, e^{-\phi} \, d\lambda \leq$$

(30) $$C^{te} \int_\Omega |\sigma|^2 \frac{e^{-\phi}}{(\sum_{j=1}^N |u_j|^2)^{1-\eta}} < +\infty$$

is finite (the first equality comes from the definition of k and the integral is finite because of (28)).
The **key point** is now the following: because of the estimate (29), $\boldsymbol{\tau_1, \tau_2, \ldots, \tau_N}$ are in $\boldsymbol{(I_\phi)_P}$ and because of the division identity (28), $\boldsymbol{\sigma \in m_P \, (I_\phi)_P}$, where m_P is the maximum ideal of X at P.
Let J be the ideal at P generated by the global sections over X of the sheaf $I_\phi \bigotimes (L + E + K_X)$ over $(O_X)_P$. It follows from:

(31) $$\rho s - \sigma \in \Gamma(I_\phi \bigotimes (L + E + K_X)) \subset J$$

that:

(32) $$\boldsymbol{s \in J + m_P \, (I_\phi)_P}$$

(because $\rho = 1$ in a neighbourhood of P).
Since s is an arbitrary element of $(I_\phi)_P$, it follows that:

(33) $$(I_\phi)_P \subset J + m_P \, (I_\phi)_P$$

By Nakayama's lemma, this implies that:

(34) $$(I_\phi)_P \subset J$$

Clearly we have: $J \subset (I_\phi)_P$ so that:

(35) $$J = (I_\phi)_P$$

This last theorem together with the use of Nadel's multiplier ideal sheaves and of the Ohsawa-Takegoshi-Manivel's extension theorem [OT87], [Ohs88] et [Man93] is an essential tool in Siu's proof of the invariance of the plurigenera [Siu98]:

Theorem 8 (Y.T. Siu)
Let $\pi : X \to \Delta$ be a ***smooth projective family of compact complex manifolds*** *parametrized by the unit disk Δ. Let's assume that the fibers $X_t = \pi^{-1}(t)$, $t \in \Delta$, are of general type (the canonical bundle of X_t is ample). Then for every integer m the* ***plurigenus*** $\mathbf{dim}\,\boldsymbol{\Gamma(X_t, mK_{X_t})}$*is* ***independent of*** $\boldsymbol{t \in \Delta}$ *, where K_{X_t} is the canonical line bundle of X_t.*

By the family $\pi : X \to \Delta$ being projective, we mean that there exists a positive holomorphic line bundle on the total space X of the family.

Let me give some hints about the proof:
The theorem is equivalent to the statement that for every $t \in \Delta$ and every positive integer m, every global section in $\Gamma(X_t, mK_{X_t})$ can be extended to a global section of $\Gamma(X, mK_X)$.
As we will use the Oshawa-Takegoshi-Manivel's extension theorem, we need to construct a suitable metric with positive curvature on K_X. Using global sections of mK_X and mK_{X_0}, Y.T Siu constructs singular metrics on mK_X and mK_{X_0} and he has to compare these two metrics. But this is essentially equivalent to prove that their multiplier ideal sheaves are very close one together. He considers a germ of the multiplier ideal sheaf defined on the fiber X_0. The **first crucial step** is: he substitutes to this germ a **global** section of the same multiplier ideal sheaf on the fiber X_0 using the last theorem about **global generation of mutiplier ideal** sheaf. The **second crucial step** is to use **Oshawa-Takegoshi extension theorem** to extend this global section on the fiber X_0 to a global section of the multiplier ideal sheaf defined on the total space X and this allows him to achieve the comparison between the two metrics. Y.T. Siu uses for the last time Oshawa-Takegoshi-Manivel's theorem to end the proof.

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Analytic compactifications of $\mathbb{C}^2/\mathbb{Z}_n$

Makoto ABE, Mikio FURUSHIMA and Tadashi SHIMA

1 Introduction

Let G be the cyclic group of order n generated by the 2×2-matrix $\begin{pmatrix} \rho & 0 \\ 0 & \rho^q \end{pmatrix}$, where n and q are integers with $(n,q)=1$ and $0<q<n$ and ρ is a primitive n-th root of 1. Let $X_{n,q} := \mathbb{C}^2/G$ be the geometric quotient surface and let $\pi : \mathbb{C}^2 \to X_{n,q}$ be the quotient map, which is a proper surjective morphism. Then $X_{n,q}$ is a normal affine algebraic surface with an isolated singularity $x := \pi(0,0)$ and $\pi : \mathbb{C}^2 \setminus \{(0,0)\} \to X \setminus \{x\}$ is an unramified cyclic covering of order n. The fundamental group $\pi_1(X \setminus \{x\}\,;x_0)$, where $x_0 \neq x$, is isomorphic to $G \cong \mathbb{Z}_n$. Let $\psi : \hat{X}_{n,q} \to X_{n,q}$ be the minimal resolution with the exceptional set $E = \bigcup_{i=1}^r E_i = \psi^{-1}(x)$. Then each irreducible component E_i of E is a smooth rational curve with the self-intersection number $(E_i^2) = -b_i \leq -2$ and the (weighted dual) graph $\Gamma(E)$ of E is as in Figure 1, where

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$$

is the continued fractional expansion of n/q.

Figure 1.

Let (M,C) be a minimal normal analytic compactification of $\hat{X}_{n,q}$ (see Morrow [4]), i.e., M is a smooth compact complex analytic surface and $C = \bigcup_{i=1}^s C_i$ is a compact analytic curve satisfying the following three conditions:

(i) $M \setminus C$ is biholomorphic to $\hat{X}_{n,q}$,

(ii) any singular point of C is an ordinary double point, and

(iii) no non-singular rational irreducible component of C with the self-intersection number -1 has at most two intersection points with the other components of C.

Then Abe-Furushima-Yamasaki [1] proved that M is a rational surface and the graph $\Gamma(C)$ of C is a linear tree of smooth rational curves as in Figure 2, where $n_i := (C_i^2)$ for $1 \leq i \leq s$ and $\max_{1\leq i\leq s}\{n_i\} \geq 0$.

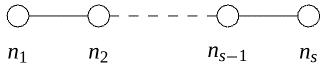

Figure 2.

2 Main results

We use the term of *semi-stable* compactification to indicate the minimal normal compactification (M^*, C^*) of $\hat{X}_{n,q}$ with the graph $\Gamma(C^*)$ as in Figure 3, where $m \neq -1$ and $m_i \geq 2$ $(1 \leq i \leq k)$.

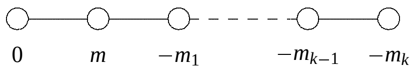

Figure 3.

Theorem 1 *There exists a semi-stable compactification (M^*, C^*) of the (minimal resolution of) the cyclic quotient affine surface $X_{n,q}$. In particular $\{m_j\}_{1\leq j\leq k}$ satisfy*

$$\frac{n}{n-q} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{\ddots - \cfrac{1}{m_k}}} \qquad \textit{or} \qquad \frac{n}{n-p} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{\ddots - \cfrac{1}{m_k}}},$$

where $0 < p < n$ and $pq \equiv 1$ (mod n).

Theorem 2 *Let (M, C) be the minimal normal compactification of the minimal resolution $\hat{X}_{n,q}$ of the cyclic quotient affine surface $X_{n,q}$. Then the possible types of the graph $\Gamma(C)$ are classified according to the following cases.*

(i) $\underset{n}{\bigcirc}$ $\quad (n > 1)$

(ii) $\underset{m_1}{\bigcirc}\!-\!\!\!-\!\underset{-m_2}{\bigcirc}$ $\quad (m_1 > 0,\ m_2 \geq 2,\ m_1m_2+1 = n)$

(iii) $\underset{0}{\bigcirc}\!-\!\!\!-\!\underset{m}{\bigcirc}\!-\!\!\!-\!\underset{-n}{\bigcirc}$ $\quad (m \neq -1)$

(iv) $\underset{m}{\bigcirc}\!-\!\!\!-\!\underset{0}{\bigcirc}\!-\!\!\!-\!\underset{-(m+n)}{\bigcirc}$ $\quad (m \neq -1, 0)$

(v) $\underset{m_1}{\bigcirc}\!-\!\!\!-\!\underset{-m_2}{\bigcirc}\!-\!\!\!-\!\underset{-m_3}{\bigcirc}$ $\quad (m_1 > 0,\ m_2, m_3 \geq 2,\ m_1m_2m_3 - m_1 + m_3 = n)$

(vi) $\underset{-m_1}{\bigcirc}\!-\!\!\!-\!\underset{m_2}{\bigcirc}\!-\!\!\!-\!\underset{-m_3}{\bigcirc}$ $\quad (m_2 > 0,\ m_1, m_3 \geq 2,\ m_1m_2m_3 + m_1 + m_3 = n)$

(vii) $\underset{0}{\bigcirc}\!-\!\!\!-\!\underset{m}{\bigcirc}\!-\!\!\!-\!\underset{-m_1}{\bigcirc}\cdots\underset{-m_{s-2}}{\bigcirc}$ $\quad (s \geq 4,\ m \neq -1,\ m_i \geq 2,\ [m_1, m_2, \ldots, m_{s-2}] = n)$

(viii) $\underset{m}{\bigcirc}\!-\!\!\!-\!\underset{0}{\bigcirc}\!-\!\!\!-\!\underset{-m_2}{\bigcirc}\cdots\underset{-m_{s-1}}{\bigcirc}$ $\quad (s \geq 4,\ m \neq -1, 0,\ m_i \geq 2$ for $i \geq 2)$

(ix) $\underset{m}{\bigcirc}\!-\!\!\!-\!\underset{-m_1}{\bigcirc}\!-\!\!\!-\!\underset{-m_2}{\bigcirc}\cdots\underset{-m_{s-1}}{\bigcirc}$ $\quad (s \geq 4,\ m > 0,\ m_i \geq 2$ for $i \geq 1)$

(x) $\underset{-m_1}{\bigcirc}\!-\!\!\!-\!\underset{m}{\bigcirc}\!-\!\!\!-\!\underset{-m_2}{\bigcirc}\cdots\underset{-m_{s-1}}{\bigcirc}$ $\quad (s \geq 4,\ m > 0,\ m_i \geq 2$ for $i \geq 1)$

(xi) $\underset{-m_1}{\bigcirc}\cdots\underset{-m_{i_0-1}}{\bigcirc}\!-\!\!\!-\!\underset{0}{\bigcirc}\!-\!\!\!-\!\underset{m}{\bigcirc}\!-\!\!\!-\!\underset{-m_{i_0+2}}{\bigcirc}\cdots\underset{-m_s}{\bigcirc}$

$(s \geq 4,\ 2 \leq i_0 \leq s-2,\ m \neq -1,\ m_i \geq 2$ for $i \neq i_0, i_0+1)$

(xii) $\underset{-m_1}{\bigcirc}\cdots\underset{-m_{i_0-1}}{\bigcirc}\!-\!\!\!-\!\underset{m_{i_0}}{\bigcirc}\!-\!\!\!-\!\underset{-m_{i_0+1}}{\bigcirc}\!-\!\!\!-\!\underset{-m_{i_0+2}}{\bigcirc}\cdots\underset{-m_s}{\bigcirc}$

$(s \geq 4,\ 3 \leq i_0 \leq s-2,\ m_{i_0} > 0,\ m_i \geq 2$ for $i \neq i_0)$

When every self-intersection number of all irreducible components of C is non-zero, we say the minimal normal compactification (M, C) is of the *non-zero* type. For given (n, q), there exist finitely many minimal normal compactifications of $\hat{X}_{n,q}$ of the non-zero type. On the other hand, the surface $\hat{X}_{n,q}$ possesses countably many minimal normal compactifications of which C has a irreducible component with its self-intersection number 0.

Each minimal normal compactification of $\hat{X}_{n,q}$ of the non-zero type belongs to one of the types (i), (ii), (v), (vi), (ix), (x), (xii) of Theorem 2, and we can count how many minimal normal compactifications of each type exist. In the following table, we collect the numbers of minimal normal compactifications of each type.

3 Table

We only treat the case that the sequence $(b_1, b_2, \cdots, b_r)$ is not *symmetric*, that is $(b_1, b_2, \cdots, b_r) \neq (b_r, b_{r-1}, \cdots, b_1)$, and $c = c(n/q)$ indicates the number of $-b_i = (E_i^2)$ less than -2, $1 \leq i \leq r$.

Type	$q = 1$	$q > 1$								
		$\sum(b_i - 2) = 0$	$\sum(b_i - 2) = 1$		$\sum(b_i - 2) > 1$					
			A	B	C	D	E	otherwise		
								F	G	H
i	1	0	0	0	0	0	0	0	0	0
ii	0	1	1	0	1	0	0	0	0	0
v	0	0	1	1	0	0	$1\ (b_k = 3)$ $0 (b_k > 3)$	0	0	0
vi	0	0	0	1	0	1	0	0	0	0
ix	0	0	0	0	1	2	$1 (b_k = 3)$ $2 (b_k > 3)$	2	2	2
x	0	0	0	0	0	0	1	2	1	0
xii	0	0	0	0	0	0	0	$c - 2$	$c - 2$	$c - 2$

- A : b_1 or $b_r = 3$.
- B : $b_j = 3$, $1 < j < r$.
- C : $\sum_{i=1}^{r}(b_i - 2) = b_1 - 2$ or $b_r - 2$.
- D : $\sum_{i=1}^{r}(b_i - 2) = b_j - 2$, $1 < j < r$.
- E : $\sum_{i=1}^{r}(b_i - 2) = b_j - 2 + b_k - 2$, $j = 1$ or r, $1 < k < r$.
- F : b_1, $b_2 = 2$.
- G : $b_1 = 2$, $b_r \neq 2$ or $b_1 \neq 2$, $b_r = 2$.
- H : b_1, $b_2 \neq 2$.

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