The $\bar\partial_b$-problem on convex domains of finite type

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1 The $\bar\partial_b$-complex

Let D be a smoothly bounded domain in $\mathbb{C}^n$. The Cauchy-Riemann operators $\bar\partial$ on $\mathbb{C}^n$ induce in a natural way a complex of differential operators on ∂D, the tangential Cauchy-Riemann complex or $\bar\partial_b$-complex. The $\bar\partial_b$-complex was first formulated by Kohn-Rossi [KR] in the mid 1960s to study the holomorphic extension of CR functions from the boundary of a complex manifold. Since then, CR manifolds and the $\bar\partial_b$-complex have been extensively studied for their intrinsic interest and because of their application to other fields of study as partial differential equations and mathematical physics.

Let $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a bounded domain in $\mathbb{C}^n$ with C^∞ boundary. Let $\bar\omega = \frac{\bar\partial\rho}{|\bar\partial\rho|}$ be the complex unit normal (0,1)-form defined on ∂D. Locally, in an open neighborhood $U \cap \partial D$, we choose $\bar\omega_1, \ldots, \bar\omega_{n-1}, \bar\omega_n$ to be an orthonormal basis for (0,1)-forms. For each s with $1 \leq s \leq \infty$, we define $\tilde{L}_{(p,q)}(\partial D)$ to be the space of (p,q)-forms in $\mathbb{C}^n$ which has L^s boundary values on ∂D. Thus $f \in \tilde{L}^s_{(p,q)}(\partial D)$ if we can write $f = \sum'_{I,J} f_{I,J} dz^I \wedge d\bar z^J$ where $f_{I,J}|_{\partial D} \in L^s(\partial D)$ for each I, J. Let

$$\mathcal{N} = \frac{1}{|\bar\partial\rho|}\sum_{j=1}^n \frac{\partial\rho}{\partial z_j}\frac{\partial}{\partial \bar z_j}.$$

The space $L^s_{(p,q)}(\partial D)$ is defined to be the subspace of $\tilde{L}_{(p,q)}(\partial D)$ such that

$$\text{if } f \in L^s_{(p,q)}(\partial D), \text{ then } f \in \tilde{L}^s_{(p,q)}(\partial D) \text{ and } \mathcal{N} \lrcorner f = 0 \text{ on } \partial D.$$

Since $\mathcal{N}$ is the dual to the form $\bar\omega_n$, locally, we can express

$$f = \sum'_{|I|=p,|J|=q,n\notin J} f_{I,J}\omega^I \wedge \bar\omega^J + \sum'_{|I|=p,|J|=q,n\in J} f_{I,J}\omega^I \wedge \bar\omega^J,$$

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where $f_{I,J}$'s are $L^s(\partial D\cap U)$ functions and $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$ are multiindices in $\{1, \dots, n\}$. We also use the notation $\omega^I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}, \bar{\omega}^J = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{i_q}$. Let τ denote the projection map

$$\tau : \tilde{L}^s_{(p.q)}(\partial D) \to L^s_{(p,q)}(\partial D)$$

defined by

$$\tau(f) = \sideset{}{'}\sum_{|I|=p,|J|=q,n\notin J} f_{I,J}\omega^I \wedge \bar{\omega}^J.$$

The projection τ is well-defined since it is independent of the choice of $\{\bar{\omega}_1, \dots, \bar{\omega}_{n-1}\}$.

From definition, $f \in L^2_{(p,q)}(\partial D)$, if and only if

$$f = \tau(F) \quad \text{on} \quad \partial D \tag{1.1}$$

where F is a (p, q)-form in $\tilde{L}^2_{(p,q)}(\partial D)$. Condition (1.1) is also equivalent to the following condition: for any smooth $(n-p, n-q-1)$-form ϕ in $\mathbb{C}^n$,

$$\int_{\partial D} f \wedge \phi = \int_{\partial D} F \wedge \phi.$$

We define the $\bar{\partial}_b$-complex on $L^2_{(p,q-1)}(\partial D)$ as follows:

Definition 1.1. For any $u \in L^2_{(p,q-1)}(\partial D)$, if for some $f \in L^2_{(p,q)}(\partial D)$, we have

$$\int_{\partial D} u \wedge \bar{\partial}\phi = (-1)^{p+q} \int_{\partial D} f \wedge \phi \quad \text{for any} \quad \phi \in C^\infty_{(n-p,n-1-q)}(\mathbb{C}^n),$$

then u is said to be in $\mathrm{Dom}(\bar{\partial}_b)$ and $\bar{\partial}_b u = f$. The $\bar{\partial}_b$ is a closed, densely defined linear operator.

In the 1960s, Kohn [Ko1] introduced L^2 approach to construct solution to the tangential Cauchy-Riemann complex on the boundary of a strictly pseudoconvex domain. Later, Henkin [He] developed integral kernels to represent solutions to the tangential Cauchy-Riemann equations. A closely related topic is the nonsolvability of certain systems of partial differential equations. In the 1950s, Hans Lewy [Le] constructed an example of a partial differential equation with smooth coefficients that has no locally defined smooth solution. In particular, he showed that C^∞ cannot replace *real analytic* in the statement of the Cauchy-Kowalevsky theorem. Lewy's example is a kind of the tangential Cauchy-Riemann equations on the Heisenberg group in $\mathbb{C}^2$. His example illustrates that the tangential Cauchy-Riemann complex on a strictly pseudoconvex boundary is not always solvable at the top degree. Later, Henkin [He] developed a criterion for solvability of the tangential Cauchy-Riemann complex at the top degree.

2 Known results

(1) *Strictly pseudoconvex case.* In 1965, Kohn [Ko1] proved Sobolev estimates for $\bar{\partial}_b$ by using subelliptic estimates for $\Box_b$. Hölder and L^p-estimates for $\bar{\partial}_b$ were proved by Folland-Stein [FS] in 1974. In 1976 through 1977, Skoda [Sk], Henkin [He], and Romanov [Rom] introduced integral kernel method for $\bar{\partial}_b$, independently.

(2) *Weakly pseudoconvex case.* In 1982, Rosay [Ros] proved C^∞ solvability for $\bar{\partial}_b$. In 1985 through 1986, some results on L^2 and Sobolev estimates for $\bar{\partial}_b$ have been proved by Shaw [Sh1], Boas-Shaw [BS], and Kohn [Ko2], independently.

(3) *A case of pseudoconvex domains of finite type.* In 1988, Hölder estimates for $\bar{\partial}_b$ on the boundaries of pseudoconvex domains of finite type in $\mathbb{C}^2$ were proved by Fefferman-Kohn [FK] by using the L^2 estimates and microlocal analysis. Shaw ([Sh2], [Sh3]) proved Hölder and L^p estimates for $\bar{\partial}_b$ on the boundaries of real ellipsoids in $\mathbb{C}^n$ and weakly pseudoconvex domains of uniform strict type M in $\mathbb{C}^2$. Let me state the Shaw's results more precisely.

Theorem 2.1. *Let $D = \{z \in \mathbb{C}^n : \rho(z) = \sum_{j=1}^n (x_j^{2n_j} + y_j^{2m_j}) - 1 < 0\}$, where $z_j = x_j + iy_j, n_j, m_j \in \mathbb{N}, n \geq 2$. Let $M = \max_{1\leq j\leq n}\{2n_j, 2m_j\}$. Let $f \in L^p_{(0,1)}(\partial D)$, $1 \leq p \leq \infty$, and f satisfy the compatibility conditions:*

(1) *If $n = 2$, $\int_{\partial D} f \wedge \varphi = 0$ for every $\bar{\partial}$-closed $(2,0)$-form φ whose coefficients are in $C^\infty(\bar{D})$.*

(2) *If $n > 2$, $\bar{\partial}_b f = 0$ in the distribution sense.*

Then there exists a solution u of $\bar{\partial}_b u = f$ on ∂D in the distribution sense which satisfies the following estimates:

(i) *For any $1 \leq p, r \leq \infty$ with $1/r > 1/p - 1/((n-1)M+2)$ we have $\|u\|_{L^r(\partial D)} \leq C\|f\|_{L^p_{(0,1)}(\partial D)}$.*

(ii) *For $p > (n-1)M + 2$ we have $\|u\|_{\Lambda_\alpha(\partial D)} \leq C\|f\|_{L^p_{(0,1)}(\partial D)}$ for $\alpha = 1/M - ((n-1)M+2)/(Mp)$. Here $\|u\|_{\Lambda_\alpha(\partial D)}$ is the Hölder norm of order α on ∂D.*

The constants in (i) *and* (ii) *depend only on p, M, and D.*

Remark 2.2. (i) Chen-Ma [CM] proved the optimal case $1/r = 1/p - 1/((n-1)M+2)$ in (i) of Theorem 2.1 on the boundaries of real ellipsoids by using weak type estimates for the solution operator of $\bar{\partial}_b$. The result is an improvement of a theorem of Shaw [Sh3].

(ii) The Hölder exponent $1/M - ((n-1)M+2)/(Mp)$ in (ii) of Theorem 2.1 is optimal. In [Sh3] Shaw gave an example on a real ellipsoid of finite type M which shows the exponent $1/M - ((n-1)M+2)/(Mp)$ is the best possible.

3 Non-isotropic support functions

Let D be a smoothly bounded convex domain in $\mathbb{C}^n$. A point $z \in \partial D$ is said to be of finite type if the order of contact of complex lines with ∂D at z is finite. The domain D is said to of finite if every point on ∂D is of finite type. We denote by M the maximum of the types of points on ∂D.

From now on we always denote by $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ a bounded convex domain with C^∞-smooth boundary of finite type M. We also define $D_\delta := \{z \in \mathbb{C}^n : \rho(z) < \delta\}$ for small absolute values $|\delta|$. The defining function ρ can be chosen in such a way that there exists a neighborhood U of ∂D such that $|d\rho(z)| > 1/2$ for all $\zeta \in U$ and all the domains $D_{\rho(\zeta)}$ are convex domains of finite type M.

If n_ζ is the unit outward normal vector at ζ on the hypersurface $\{z : \rho(z) = \rho(\zeta)\}$ we define $w = \Phi(\zeta)(z-\zeta)$, where the unitary matrix $\Phi(\zeta)$ satisfies $\Phi(\zeta)n_\zeta = (1, 0, \ldots, 0)$ for all $\zeta \in U$. The following definitions are in [DF]:

$$\rho_\zeta(w) := \rho(\zeta + (\overline{\Phi}(\zeta))^T w),$$

$$S_\zeta(w) := 3w_1 + Kw_1^2 - c\sum_{j=2}^{M} N^{2^j}\sigma_j \sum_{\substack{|\alpha|=j\\ \alpha_1=0}} \frac{1}{\alpha!}\frac{\partial^j \rho_\zeta}{\partial w^\alpha}(0)w^\alpha$$

for $N > 0$ suitably large, $c > 0$ suitably small (all independent of ζ). We define

$$Q_\zeta^j(w) := \int_0^1 \frac{\partial S_\zeta}{\partial w_j}(tw)dt, \quad j = 1, \ldots, n,$$

and

$$\begin{aligned} Q(z,\zeta) &= (Q_1(z,\zeta), \ldots, Q_n(z,\zeta)) \\ &:= \Phi(\zeta)^T(Q_\zeta^1(\Phi(\zeta)(z-\zeta)), \ldots, Q_\zeta^n(\Phi(\zeta)(z-\zeta))). \end{aligned}$$

We put $S(z,\zeta) := S_\zeta(\Phi(\zeta)(z-\zeta))$. Then $S(z,\zeta)$ is a non-isotropic support function on D, holomorphic in $z \in \bar{D}$ and C^∞ in $\zeta \in U$ with the following estimates. Let v be a unit vector complex tangential to the level set $\{\rho = \rho(\zeta)\}$ at ζ. Define

$$a_{\alpha\beta}(\zeta, v) := \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha\partial\bar{\lambda}^\beta}\rho(\zeta + \lambda v)|_{\lambda=0}.$$

Then there are constants $K, c, d > 0$, such that one has for all points z written as $z = \zeta + \mu n_\zeta + \lambda v$ with $\mu, \lambda \in \mathbb{C}$ the estimate

$$\begin{aligned} 2\text{Re } S(z,\zeta) \le& -|\text{Re } \mu| - K(\text{Im } \mu)^2 \\ & - c\sum_{j=2}^{m}\sum_{\alpha+\beta=j}|a_{\alpha\beta}(\zeta,v)||\lambda|^j + d\sup\{0, \rho(z)-\rho(\zeta)\}. \end{aligned} \tag{3.1}$$

The non-isotropic support function S is the key factor in the kernels of the solution operators for $\bar{\partial}_b$.

4 Our results

Using above (1,0)-form $Q(z,\zeta)$ and support function $S(z,\zeta)$ we define two kernels

$$K(z,\zeta) = \sum_{j=0}^{n-q-2} K_j(z,\zeta), \quad K^*(z,\zeta) = \sum_{j=0}^{q} K_j^*(z,\zeta),$$

where $0 \le q \le n-2$ and

$$\begin{aligned} K_j(z,\zeta) &= c_j\frac{Q \wedge b \wedge (\bar{\partial}_\zeta Q)^j \wedge (\bar{\partial}_\zeta b)^{n-q-2-j} \wedge (\bar{\partial}_z b)^q}{S(z,\zeta)^{j+1}|z-\zeta|^{2(n-j-1)}} \\ K_j^*(z,\zeta) &= d_j\frac{Q^* \wedge b \wedge (\bar{\partial}_\zeta b)^{n-q-2} \wedge (\bar{\partial}_z Q^*)^j \wedge (\bar{\partial}_z b)^{q-j}}{S^*(z,\zeta)^{j+1}|z-\zeta|^{2(n-j-1)}}. \end{aligned}$$

Here $b = \sum_{j=1}^{n}(\bar{z}_j - \bar{\zeta}_j)d\zeta_j$, c_j, d_j are suitably chosen constants for our purpose, $Q^* = Q^*(z,\zeta) := Q(\zeta, z)$ and $S^*(z,\zeta) := S(\zeta, z)$.

Next we introduce two integral operators. Let $f \in L^1_{(0,q+1)}(\partial D)$, $0 \le q \le n-2$. Define

$$R^+ f(z) := \int_{\partial D} K(z,\zeta) \wedge f(\zeta), \quad z \in D$$

$$R^- f(z) := \int_{\partial D} K^*(z,\zeta) \wedge f(\zeta), \quad z \in \bar{D}^c := \mathbb{C}^n \setminus \bar{D}.$$

Theorem 4.1 ([AC]). *Let $f \in L^p_{(0,1)}(\partial D)$ for $p > (n-1)M + 2$. Then we have*

$$\|R^+ f\|_{\Lambda_\alpha(D)} + \|R^- f\|_{\Lambda_\alpha(\bar{D}^c)} \le C\|f\|_{L^p_{(0,1)}(\partial D)} \quad for \quad \alpha = \frac{1}{M} - \frac{(n-1)M+2}{Mp}.$$

By Theorem 4.1, for $p > (n-1)M + 2$ the integrals $R^+ f$ and $R^- f$ are continuously extended up to the boundary. So we can define

$$Tf(z) := \int_{\partial D} K(z,\zeta) \wedge f(\zeta), \quad Sf(z) := \int_{\partial D} K^*(z,\zeta) \wedge f(\zeta), \quad z \in \partial D.$$

for $f \in L^p_{(0,1)}(\partial D)$, $p > (n-1)M+2$. Moreover, if f satisfies the compatibility conditions, then

$$\bar{\partial}_b(Tf - Sf) = f$$

in the distribution sense (see Theorem (2.13) of [Sh3] for details). Now we consider the case $1 \le p \le (n-1)m+2$. Even though the integrals $R^+ f$ and $R^- f$ have boundary values almost everywhere for $z \in \partial D$, we do not know whether they are continuously extended up to boundary. In this case, by approximation argument [Rom], we can assume that $f \in C^\infty_{(1,0)}(\partial D)$.

Theorem 4.2 ([AC]). *Let D and M be as above. Let $f \in L^p_{(0,1)}(\partial D)$, $1 \le p \le \infty$, and f satisfy the compatibility conditions:*

(1) *If $n = 2$, $\int_{\partial D} f \wedge \varphi = 0$ for every $\bar{\partial}$-closed $(2,0)$-form φ whose coefficients are in $C^\infty(\bar{D})$.*

(2) *If $n > 2$, $\bar{\partial}_b f = 0$ in the distribution sense.*

Then there exists a solution u of $\bar{\partial}_b u = f$ on ∂D in the distribution sense which satisfies the following estimates:

(i) *For any $1 \le p, r \le \infty$ with $1/r > 1/p - 1/((n-1)M+2)$ we have $\|u\|_{L^r(\partial D)} \le C\|f\|_{L^p_{(0,1)}(\partial D)}$.*

(ii) *For $p > (n-1)M + 2$ we have $\|u\|_{\Lambda_\alpha(\partial D)} \le C\|f\|_{L^p_{(0,1)}(\partial D)}$ for $\alpha = 1/M - ((n-1)M+2)/(Mp)$. Here $\|u\|_{\Lambda_\alpha(\partial D)}$ is the Hölder norm of order α on ∂D.*

The constants in (i) *and* (ii) *depend only on p, M, and D.*

For the proof of the theorem we define a nonisotropic polydisc suited to the geometry of boundaries of convex domains of finite type and estimate the support function S, components of Q, $\bar{\partial}Q$ and $d_z Q$ in this polydisc. In this case (3.1) is the key inequality for the estimates of these factors in the integral solution operators for $\bar{\partial}_b$. We can see the proof of Theorem 4.2 in [AC].

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