DEFECT RELATION FOR RATIONAL FUNCTIONS AS TARGETS

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ABSTRACT. The second main theorem in Nevanlinna theory is proved when targets are rational functions. We use Ahlfors' theory of covering surfaces for a proof.

1. INTRODUCTION

The purpose of this paper is to prove the following.

Theorem . *Let f be a transcendental meromorphic function on the complex plane $\mathbb{C}$. Let $a_1, \cdots, a_q$ be distinct rational functions on $\mathbb{C}$. Then there is a set $E \subset \mathbb{R}_{>0}$ of finite linear measure such that*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r,a_i,f) + o(T(r,f)) \quad \textit{for} \quad r \to \infty, \quad r \notin E.$$

The notations $T(r,f)$ and $\overline{N}(r,a_i,f)$ are standard in Nevanlinna theory (cf.[H],[N2]) and also given in the following section.

Our theorem gives a special case of so-called *second main theorem for small functions*, which was suggested by R. Nevanlinna ([N1]) and improved by Ch. Osgood and N. Steinmetz ([O],[St]). In the forthcoming paper [Y], we shall prove the general case of this problem.

We briefly mention our method of the proof.

We consider the complex projective line $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ equipped with the Riemannian metric coming from the length element $\frac{1}{\sqrt{\pi}} \frac{|dw|}{1+|w|^2}$, which is normalized such that the total area of $\mathbb{P}^1$ is equal to 1. Since a_i is rational function, the value $a_i(\infty) \in \mathbb{P}^1$ is well defined.

As already pointed out by A. Sauer [Sa], when $a_i(\infty) \neq a_j(\infty)$ for $i \neq j$, our theorem follows from the following argument.

For $i = 1, \dots, q$, take a small spherical disc E_i in $\mathbb{P}^1$ centred at $a_i(\infty)$ such that $\overline{E_i} \cap \overline{E_j} = \emptyset$ for $i \neq j$. Let $R(r_0, r)$ be the ring domain $\{z \in \mathbb{C};\ r_0 < |z| < r\}$. Apply Ahlfors' theory of covering surfaces to the subcovering $f : R(r_0, r) \to \mathbb{P}^1$ to get

$$\sum_{i=1}^{q} \mathrm{card}(\mathcal{H}_i) \geq (q-2)S - hL. \tag{1.1}$$

Here $S = S(r)$ is the mean sheet number, L is the length of the relative boundary and $\mathcal{H}_i$ is the set of islands over E_i of the covering $f : R(r_0, r) \to \mathbb{P}^1$. By Rouché's Theorem (cf. Lemma 3), we have

$$\mathrm{card}(\mathcal{H}_i) \leq \overline{n}(a_i, f, R(r_0, r)),$$

when $r_0 \gg 0$. Hence using (1.1), we get non-integrated version

$$\sum_{i=1}^{q} \overline{n}(a_i, f, R(r_0, r)) \geq (q-2)S - hL$$

of our theorem. Taking the integral $\int_1^r \frac{dt}{t}$ of the both hand side of this inequality, we get our theorem (cf. J. Miles [M]).

The next simple case is that $a_1(\infty) = a_2(\infty)$ and $a_i(\infty) \neq a_j(\infty)$ for $2 \leq i \neq j \leq q$. For $i = 2, \dots, q$, take a small spherical disc E_i centred at $a_i(\infty)$ such that $\overline{E_i} \cap \overline{E_j} = \emptyset$ for $i \neq j$. Apply Ahlfors' theory of covering surfaces to the subcovering $f : R(r_0, r) \to \mathbb{P}^1$ to get

$$-\sum_{H \in \mathcal{H}_2} \rho(H) - \sum_{P \in \mathcal{P}_2} \rho^+(P) + \sum_{i=3}^{q} \mathrm{card}(\mathcal{H}_i) \geq (q-3)S - hL. \tag{1.2}$$

Here S, L and $\mathcal{H}_i$ are the same as above and $\mathcal{P}_2$ is the set of peninsulas over E_2 of the covering $f : R(r_0, r) \to \mathbb{P}^1$.

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Next, what we have to do is to *separate* the functions a_1 and a_2. To do this, we consider the function $\lambda(w) = \frac{w-a_1}{a_2-a_1}$. Let E_1' and E_2' be small spherical discs centred at 0 and 1, respectively. Assume that $\infty \notin \overline{E_1'} \cup \overline{E_2'}$ and $\overline{E_1'} \cap \overline{E_2'} = \emptyset$. Then we apply Ahlfors' theory to the covering

$$\lambda(f) = \frac{f-a_1}{a_2-a_1} : f^{-1}(E_2) \to \mathbb{P}^1$$

to get the inequality

$$\sum_{H\in\mathcal{H}_2} \rho(H) + \sum_{P\in\mathcal{P}_2} \rho^+(P) + \sum_{i=1}^{2} \mathrm{card}(\mathcal{H}_i') \geq S' - hL'. \tag{1.3}$$

Here $S' = S'(r)$ is the mean sheet number, L' is the length of the relative boundary and $\mathcal{H}_i'$ is the set of islands over E_i' of the covering $\lambda(f) : R(r_0, r) \to \mathbb{P}^1$. Combining (1.2) and (1.3), we get

$$\sum_{i=1}^{2} \mathrm{card}(\mathcal{H}_i') + \sum_{i=3}^{q} \mathrm{card}(\mathcal{H}_i) \geq (q-3)S + S' - h(L+L'). \tag{1.4}$$

Here the cancellation of the term

$$\sum_{H\in\mathcal{H}_2} \rho(H) + \sum_{P\in\mathcal{P}_2} \rho^+(P) \tag{1.5}$$

is very important in this paper. In Lemma 2, we study inequality of type (1.2) and (1.3) in general form.

By Rouché's Theorem, we get

$$\mathrm{card}(\mathcal{H}_1') \leq \overline{n}\left(0, \frac{f-a_1}{a_2-a_1}, R(r_0, r)\right) = \overline{n}(a_1, f, R(r_0, r))$$

and

$$\mathrm{card}(\mathcal{H}_2') \leq \overline{n}\left(1, \frac{f-a_1}{a_2-a_1}, R(r_0, r)\right) = \overline{n}(a_2, f, R(r_0, r))$$

for $r_0 \gg 0$. Hence using (1.4), we get non-integrated version

$$\sum_{i=1}^{q} \overline{n}(a_i, f, R(r_0, r)) \geq (q-3)S + S' - h(L+L')$$

of our theorem. Taking the integral $\int_1^r \frac{dt}{t}$, we obtain our theorem as before. Here, we also use the fact

$$\int_1^r \frac{S'(t)}{t} dt = \int_1^r \frac{S(t)}{t} dt + O(\log r),$$

which follows from the fact $T(r, a_i) \leq O(\log r)$.

To prove the general case of our theorem, we first construct a tree Γ which has information to separate the functions $a_1, \dots, a_q$. Then we apply Lemma 2 for adjacent vertices of Γ to get analogous inequalities for (1.2) and (1.3), and take summation for every edges of Γ to get analogous inequality for (1.4). Here, as above, the cancellation of the terms such as (1.5) is very important. Using Rouché's Theorem, we obtain non-integrated version of our theorem (Lemma 5). Taking the integral $\int_1^r \frac{dt}{t}$, we obtain our theorem. Here we also need a combinatorial lemma to estimate the right hand side of the integration of Lemma 5 (cf. Lemma 4).

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2. Notations

In this paper, all domains of a Riemann surface are taken such that whose boundaries, if exist, are piecewise analytic. We also assume that all curves on a Riemann surface are piecewise analytic.

Let $\mathscr{F}$ be a Riemann surface. We say that F is a *finite domain* of $\mathscr{F}$ when F is a compactly contained, connected domain of $\mathscr{F}$ and F is bordered by a finite disjoint union of Jordan curves. Then F is compact if and only if $\mathscr{F}$ is compact and $F = \mathscr{F}$. We denote by $\overline{F}$ the closure of F and by ∂F the boundary of F.

Take a triangulation of $\overline{F}$ by a finite number of triangles, where $\overline{F}$ may be a bordered surface. We define the *characteristic* $\rho(F)$ of F by

$$-[\text{number of interior vertices}] + [\text{number of interior edges}] - [\text{number of triangles}].$$

Then it is well known that this definition is independent of the choice of the triangulation. This characteristic is normalized such that $\rho(\text{disc}) = -1$ as usual in Ahlfors' theory. We also put $\rho^+(F) = \max\{0, \rho(F)\}$.

Let Ω be an open subset of $\mathscr{F}$. Let f and a be meromorphic functions on $\mathscr{F}$. Assume that $f \not\equiv a$. Put

$$\overline{n}(a, f, \Omega) = \text{card}\left(\{z \in \Omega;\ f(z) = a(z)\}\right).$$

We denote by $\omega_{\mathbb{P}^1}$ the Fubini-Study form on the projective line $\mathbb{P}^1$; i.e.,

$$\omega_{\mathbb{P}^1} = \frac{1}{(1+|w|^2)^2}\frac{\sqrt{-1}}{2\pi}dw \wedge d\overline{w}.$$

Put

$$A(f, \Omega) = \int_\Omega f^*\omega_{\mathbb{P}^1}.$$

Let γ be a Jordan arc on $\mathscr{F}$. We denote by

$$l(f, \gamma)$$

the length of the curve $f|_\gamma : \gamma \to \mathbb{P}^1$ with respect to the associated Kähler metric for $\omega_{\mathbb{P}^1}$, whose length element is

$$\frac{1}{\sqrt{\pi}}\frac{|dw|}{1+|w|^2}. \tag{2.1}$$

Let f be a meromorphic function on $\mathbb{C}$. We use the following notations, which are standard in Nevanlinna theory. For a meromorphic function a on $\mathbb{C}$ such that $f \not\equiv a$, define the truncated counting function by

$$\overline{N}(r, a, f) = \int_1^r \frac{\overline{n}(a, f, \mathbb{C}(t))}{t}dt,$$

where $\mathbb{C}(t) = \{z \in \mathbb{C};\ |z| < t\}$. We define the spherical characteristic function by

$$T(r, f) = \int_1^r \frac{A(f, \mathbb{C}(t))}{t}dt.$$

Then by Shimizu-Ahlfors theorem, this function $T(r, f)$ is equal to the usual characteristic function up to bounded term in r.

3. Review of Ahlfors' Theory

Let F_0 be a finite domain of $\mathbb{P}^1$. Let $\mathscr{F}$ be a Riemann surface, $F \subset \mathscr{F}$ be a finite domain and ζ be a non-constant meromorphic function on $\overline{F}$. Assume that $\zeta(\overline{F}) \subset \overline{F_0}$. Then we may consider $\zeta : F \to F_0$ as a covering surface in the sense of [N2, p.323].

We call $\zeta^{-1}(F_0) \cap \partial F$ as relative boundary and

$$l(\zeta, \zeta^{-1}(F_0) \cap \partial F)$$

as length of the relative boundary, which is often denoted by L.

Let $\Omega \subset F_0$ be an open subset which is bounded by a finite number of disjoint Jordan curves. We call

$$S_\Omega = \frac{A(\zeta, \zeta^{-1}(\Omega))}{\int_\Omega \omega_{\mathbb{P}^1}}$$

for the mean sheet number of ζ over Ω. We often write S_{F_0} as S and call for the mean sheet number of ζ.

We apply Ahlfors' theory to the above situation $\zeta : F \to F_0$.

Covering Theorem 1. ([N2, p.328]) There exists a positive constant $h = h(F_0, \Omega) > 0$ which is independent of F and ζ such that

$$|S - S_\Omega| \leq hL. \tag{3.1}$$

Main Theorem. ([N2, p.332]) There exists a positive constant $h = h(F_0) > 0$ which is independent of F and ζ such that

$$\rho^+(F) \geq \rho(F_0)S - hL. \tag{3.2}$$

4. Topological Lemma

Let $\mathscr{F}$ be a Riemann surface. Let Ω and G be two open subsets in $\mathscr{F}$. We define two subsets $\mathcal{I}(G,\Omega)$, $\mathcal{P}(G,\Omega)$ of the set of connected components of $G\cap\Omega$ by the following manner. Let G' be a connected component of $G\cap\Omega$, then G' is contained in $\mathcal{I}(G,\Omega)$ if and only if G' is compactly contained in Ω, otherwise G' is contained in $\mathcal{P}(G,\Omega)$. Then a connected component G' in $\mathcal{I}(G,\Omega)$ is also a connected component of G.

Let $F\subset\mathscr{F}$ be a finite domain and ζ be a non-constant meromorphic function on $\overline{F}$. Let E be a domain in $\mathbb{P}^1$. We consider the following condition for ζ and E;

(4.1) $$\text{Let } a\in\overline{F} \text{ be a branch point of } \zeta. \text{ Then } \zeta(a)\notin\partial E.$$

We will use this condition just for simplicity (see argument in [N2, p.342]).

The following lemma will be used in a proof of Lemma 2.

Lemma 1. *Assume that a finite number of disjoint simple closed curves γ_i $(i=1,\cdots p)$ divide $\mathbb{P}^1$ into connected domains $D_1,\cdots,D_{p+1}$. Let ζ be a non-constant meromorphic function on $\overline{F}$, where F is a finite domain of a Riemann surface $\mathscr{F}$. Assume that the condition (4.1) is satisfied for ζ and D_i $(1\le i\le p+1)$. Put $\mathcal{A}=\bigcup_{i=1}^{p+1}\mathcal{I}\left(\zeta^{-1}(D_i),F\right)$, $\mathcal{B}=\bigcup_{i=1}^{p+1}\mathcal{P}\left(\zeta^{-1}(D_i),F\right)$. Then we have*

$$\rho^+(F)\ge\sum_{A\in\mathcal{A}}\rho(A)+\sum_{B\in\mathcal{B}}\rho^+(B).$$

Proof. Let $\sigma_1,\cdots,\sigma_s$ be the curves on F which lie over the curves γ_i $(i=1,\cdots p)$. Here note that the curves $\sigma_1,\cdots,\sigma_s$ are simple by the condition (4.1). Let t be the number of curves σ_j $(1\le j\le s)$ which are not closed. Assume that $\sigma_1,\ldots,\sigma_t$ are not closed, and $\sigma_{t+1},\ldots,\sigma_s$ are closed. First, the cross-cuts $\{\sigma_i\}_{i=1}^t$ divide F into domains $F_1',\ldots,F_u'$, where $u\le t+1$. Next, the loop-cuts $\{\sigma_i\}_{i=t+1}^s$ divide each domain F_j' $(1\le j\le u)$ into domains $F_{j,1}'',\ldots,F_{j,w_j}''$. Then we have $\mathcal{A}\cup\mathcal{B}=\{F_{j,w}''\}_{1\le j\le u,1\le w\le w_j}$. Note that if one of $F_{j,1}'',\ldots,F_{j,w_j}''$ is simply connected and also contained in $\mathcal{B}$, then there are no loop-cuts $\{\sigma_i\}_{i=t+1}^s$ on F_j', i.e., $w_j=1$ and $F_j'=F_{j,1}''$. In this case, F_j' is also non-compact.

Let r be the number of connected components in $\mathcal{B}$ which are simply connected. Then by the above observation, we have $r\le u$ and

(4.2) $$r\le t+1.$$

If $r=t+1$, then $r=u=t+1$, and every F_j' are simply connected and non-compact. Using

$$\rho(F)=\sum_{1\le j\le u}\rho(F_j')+t \qquad \text{(cf. [N2, p.323 (1.1)])}$$

and $\rho(F_j')=-1$, we get $\rho(F)=-1$.

Since we have equality

(4.3) $$\rho(F)=\sum_{B\in\mathcal{A}\cup\mathcal{B}}\rho(B)+t,$$

we have the following inequality

$$\rho(F)\ge\sum_{A\in\mathcal{A}}\rho(A)+\sum_{B\in\mathcal{B}}\rho^+(B)+\eta$$

where $\eta=0$ if $r\le t$ and $\eta=-1$ if $r=t+1$. But in the case $\eta=-1$, we have $\rho(F)=-1$. Hence we have

$$\rho^+(F)\ge\sum_{A\in\mathcal{A}}\rho(A)+\sum_{B\in\mathcal{B}}\rho^+(B).$$

This proves our lemma. □

5. Application of Ahlfors' Theory

The following lemma is a modification of Ahlfors' second main theorem (see the remark below).

Lemma 2. *Let $E^\dagger$ be a Jordan domain in $\mathbb{P}^1$ or $\mathbb{P}^1$ itself. Let $E_1,\cdots,E_p,E_\infty$ be Jordan domains in $\mathbb{P}^1$. Assume that the closures $\overline{E_j}$ of E_j's $(j=1,\cdots,p,\infty)$ are mutually disjoint. Then there is a positive constant $h>0$ which only depends on $E_1,\cdots,E_p,E_\infty$ with the following property: Let F be a finite domain of a Riemann surface $\mathscr{F}$ and υ, ζ be two non-constant meromorphic functions on $\overline{F}$. Assume that*

(5.1) $$\zeta\left(\upsilon^{-1}(\mathbb{P}^1\backslash E^\dagger)\cap\overline{F}\right)\subset E_\infty$$

and that ζ and E_j satisfy the condition (4.1) for $j=1,\cdots,p,\infty$.
Put

$$\mathcal{H}^I=\mathcal{I}\left(\upsilon^{-1}(E^\dagger),F\right),\ \mathcal{H}^P=\mathcal{P}\left(\upsilon^{-1}(E^\dagger),F\right),$$
$$\mathcal{G}_j^I=\mathcal{I}\left(\zeta^{-1}(E_j),F\right),\ \mathcal{G}_j^P=\mathcal{P}\left(\zeta^{-1}(E_j),F\right)\quad \textit{for } j=1,\cdots,p,$$

and

$$\mathcal{G}^I_\infty = \mathcal{I}\left(\zeta^{-1}(E_\infty), F \cap \upsilon^{-1}(E^\dagger)\right).$$

Let S be the mean sheet number and L be the length of the relative boundary with respect to the covering $\zeta : F \to \mathbb{P}^1$. Then we have the following inequality.

$$(5.2)\quad \vartheta(\zeta,\upsilon) + \sum_{H^I\in\mathcal{H}^I}\rho(H^I) + \sum_{H^P\in\mathcal{H}^P}\rho^+(H^P) - \sum_{j=1}^{p}\sum_{G^I_j\in\mathcal{G}^I_j}\rho(G^I_j) - \sum_{j=1}^{p}\sum_{G^P_j\in\mathcal{G}^P_j}\rho^+(G^P_j) - \sum_{G^I_\infty\in\mathcal{G}^I_\infty}\rho(G^I_\infty) \geq (p-1)S - hL,$$

where $\vartheta(\zeta,\upsilon)$ is the number of connected components H^I in $\mathcal{H}^I$ such that $\zeta(H^I) \subset E_\infty$.

Remark. (1) In the case $E^\dagger = \mathbb{P}^1$, the condition (5.1) is satisfied. Moreover if F is non-compact, then we have $\mathcal{H}^P = \{F\}$, $\mathcal{H}^I = \emptyset$ and $\vartheta(\zeta,\upsilon) = 0$.

(2) Consider the case that $E^\dagger = \mathbb{P}^1$ and F is a simply connected, non-compact domain. Using the facts that $\rho^+(F) = 0$ and $\rho^+(G^P_j) \geq 0$, we have

$$-\sum_{j=1}^{p}\sum_{G^I_j\in\mathcal{G}^I_j}\rho(G^I_j) - \sum_{G^I_\infty\in\mathcal{G}^I_\infty}\rho(G^I_\infty) \geq (p-1)S - hL.$$

In this case, for $j = 1,\dots,p,\infty$, the sets $\mathcal{G}^I_j$ are the sets of islands over E_j with respect to the covering $\zeta : F \to \mathbb{P}^1$. Using $\rho(G^I_j) \geq -1$, we get

$$\sum_{j=1,\dots,p,\infty}\left(\text{number of islands over } E_j \text{ w.r.t. } \zeta\right) \geq (p-1)S - hL,$$

which is famous Ahlfors' second main theorem.

(3) To get the inequality (1.3), we apply our lemma to $\upsilon = f$, $\zeta = \lambda(f)$, $E^\dagger = E_2$, $E_1 = E'_1$ and $E_2 = E'_2$. See a proof of Lemma 5.

Proof of Lemma 2. Let γ_i $(i = 1,\cdots,p,\infty)$ be the boundary of E_i, which is a simple closed curve on $\mathbb{P}^1$.

We first consider the subcovering $\zeta_I : H^I \to \mathbb{P}^1$ $(H^I \in \mathcal{H}^I)$ of the covering $\zeta : F \to \mathbb{P}^1$. Since H^I is compactly contained in F, the boundary ∂H^I of H^I does not meet the boundary of F. Hence, by the assumption (5.1), we have

$$(5.3)\qquad \zeta(\partial H^I) \subset E_\infty.$$

By this and the condition (4.1), we conclude that the curves $\sigma^I_1,\cdots,\sigma^I_s$ lying over the curves γ_j $(i = 1,\cdots,p,\infty)$ are simple closed curves on H^I. By this system of loop cuts (σ^I_j), H^I is divided into four classes of connected domains $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$. $\mathcal{A}_1$ is the set of connected components of $\zeta_I^{-1}(E_1 \cup \cdots \cup E_p)$. Put $\mathcal{A}_2 = \mathcal{I}\left(\zeta_I^{-1}(E_\infty), H^I\right)$ and $\mathcal{A}_3 = \mathcal{P}\left(\zeta_I^{-1}(E_\infty), H^I\right)$. Let Ω be the domain $\mathbb{P}^1 \backslash \left(\bigcup_{i=1}^p \overline{E_i} \cup \overline{E_\infty}\right)$ and let $\mathcal{A}_4$ be the set of connected components of $\zeta_I^{-1}(\Omega)$. Since the curves σ^I_j are closed, we have

$$(5.4)\qquad \rho(H^I) = \sum_{A_1\in\mathcal{A}_1}\rho(A_1) + \sum_{A_2\in\mathcal{A}_2}\rho(A_2) + \sum_{A_3\in\mathcal{A}_3}\rho(A_3) + \sum_{A_4\in\mathcal{A}_4}\rho(A_4).$$

The components in $\mathcal{A}_4$ are covering surfaces of Ω and by (5.3), these covering surfaces do not have relative boundaries. Hence by the Hurwitz formula, using $\rho(\Omega) = p - 1$, we have

$$\sum_{A_4\in\mathcal{A}_4}\rho(A_4) \geq s_I\rho(\Omega) = s_I(p-1).$$

Here s^I denotes the mean sheet number over the domain Ω of the covering surface $\zeta_I : H^I \to \mathbb{P}^1$. Using the equality (5.4), we have

$$\rho(H^I) - \sum_{A_1\in\mathcal{A}_1}\rho(A_1) - \sum_{A_2\in\mathcal{A}_2}\rho(A_2) - \sum_{A_3\in\mathcal{A}_3}\rho(A_3) = \sum_{A_4\in\mathcal{A}_4}\rho(A_4) \geq s_I(p-1).$$

If $\zeta_I(H^I) \not\subset E_\infty$, then components in $\mathcal{A}_3$ is not simply connected, so $\rho(A_3) \geq 0$. Hence

$$(5.5)\qquad \rho(H^I) - \sum_{A_1\in\mathcal{A}_1}\rho(A_1) - \sum_{A_2\in\mathcal{A}_2}\rho(A_2) \geq s_I(p-1).$$

On the other hand, if $\zeta_I(H^I) \subset E_\infty$, then we have

$$(5.6)\qquad 1 + \rho(H^I) - \sum_{A_1\in\mathcal{A}_1}\rho(A_1) - \sum_{A_2\in\mathcal{A}_2}\rho(A_2) \geq s_I(p-1).$$

This is because $\mathcal{A}_1 = \mathcal{A}_2 = \emptyset$, $\rho(H^I) \geq -1$ and $s_I = 0$. Here note that H^I is non-compact in this case, because ζ is non-constant. Using (5.5) and (5.6), we have

$$\vartheta(\zeta, v) + \sum_{H^I \in \mathcal{H}^I} \rho(H^I) - \sum_{H^I \in \mathcal{H}^I} \sum_{A_1 \in \mathcal{A}_1} \rho(A_1) - \sum_{H^I \in \mathcal{H}^I} \sum_{A_2 \in \mathcal{A}_2} \rho(A_2) \geq (p-1) \sum_{H^I \in \mathcal{H}^I} s_I. \tag{5.7}$$

Next we consider the subcovering $\zeta_P : H^P \to \mathbb{P}^1$ of $\zeta : F \to \mathbb{P}^1$ for a component $H^P \in \mathcal{H}^P$. Let $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_6$ be as follows.

$$\mathcal{B}_1 = \mathcal{I}\left(\zeta_P^{-1}\left(\cup_{i=1}^p E_i\right), H^P\right), \mathcal{B}_2 = \mathcal{P}\left(\zeta_P^{-1}\left(\cup_{i=1}^p E_i\right), H^P\right), \mathcal{B}_3 = \mathcal{I}\left(\zeta_P^{-1}(E_\infty), H^P\right),$$
$$\mathcal{B}_4 = \mathcal{P}\left(\zeta_P^{-1}(E_\infty), H^P\right), \mathcal{B}_5 = \mathcal{I}\left(\zeta_P^{-1}(\Omega), H^P\right), \mathcal{B}_6 = \mathcal{P}\left(\zeta_P^{-1}(\Omega), H^P\right).$$

By Lemma 1, we have

$$\rho^+(H^P) \geq \sum_{B \in \mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5} \rho(B) + \sum_{B \in \mathcal{B}_2 \cup \mathcal{B}_4 \cup \mathcal{B}_6} \rho^+(B). \tag{5.8}$$

Since components in $\mathcal{B}_5 \cup \mathcal{B}_6$ are covering surfaces of Ω, using the Hurwitz formula and the main theorem (3.2), there is a positive constant h' which depend only on E_i's such that

$$\sum_{B_5 \in \mathcal{B}_5} \rho(B_5) + \sum_{B_6 \in \mathcal{B}_6} \rho^+(B_6) \geq s_P(p-1) - h' L_P$$

where s_P is the mean sheet number over Ω for the covering surface $\zeta_P : H^P \to \mathbb{P}^1$ and L_P is the sum of the lengths of the relative boundaries of $B_6 \to \Omega$ over $B_6 \in \mathcal{B}_6$. By summing up for $H^P \in \mathcal{H}^P$ and from (5.8), we have

$$\sum_{H^P \in \mathcal{H}^P} \rho^+(H^P) - \sum_{H^P \in \mathcal{H}^P} \sum_{B \in \mathcal{B}_1 \cup \mathcal{B}_3} \rho(B) - \sum_{H^P \in \mathcal{H}^P} \sum_{B' \in \mathcal{B}_2} \rho^+(B')$$
$$\geq (p-1) \sum_{H^P \in \mathcal{H}^P} s_P - h' \sum_{H^P \in \mathcal{H}^P} L_P. \tag{5.9}$$

Here we use the fact $\rho^+(B_4) \geq 0$ for $B_4 \in \mathcal{B}_4$.

Now consider the covering $\zeta : F \to \mathbb{P}^1$. By the assumption (5.1), we have

$$\zeta^{-1}(\Omega) \subset \bigcup_{H^I \in \mathcal{H}^I} H^I \cup \bigcup_{H^P \in \mathcal{H}^P} H^P \quad \text{on } F.$$

Hence we have

$$\sum_{H^P \in \mathcal{H}^P} L_P \leq L. \tag{5.10}$$

(Recall that L is the length of the relative boundary for $\zeta : F \to \mathbb{P}^1$.) Also using the covering theorem 1 (3.1), we have

$$S_\Omega = \sum_{H^I \in \mathcal{H}^I} s_I + \sum_{H^P \in \mathcal{H}^P} s_P \geq S - h'' L \tag{5.11}$$

for some positive constant h'' which only depend on E_i's. Here S_Ω is the mean sheet number of the covering $\zeta : F \to \mathbb{P}^1$ over Ω.

Again by the assumption (5.1), we have

$$\zeta^{-1}(\overline{E_i}) \subset \bigcup_{H^I \in \mathcal{H}^I} H^I \cup \bigcup_{H^P \in \mathcal{H}^P} H^P \quad \text{for } 1 \leq i \leq p \quad \text{on } F.$$

Hence we have $\bigcup_{1 \leq j \leq p} \mathcal{G}_j^I = \bigcup_{H^I \in \mathcal{H}^I} \mathcal{A}_1 \cup \bigcup_{H^P \in \mathcal{H}^P} \mathcal{B}_1$ and $\bigcup_{1 \leq j \leq p} \mathcal{G}_j^P = \bigcup_{H^P \in \mathcal{H}^P} \mathcal{B}_2$. These imply that

$$\sum_{j=1}^p \sum_{G_j^I \in \mathcal{G}_j^I} \rho(G_j^I) = \sum_{H^I \in \mathcal{H}^I} \sum_{A_1 \in \mathcal{A}_1} \rho(A_1) + \sum_{H^P \in \mathcal{H}^P} \sum_{B_1 \in \mathcal{B}_1} \rho(B_1) \tag{5.12}$$

and

$$\sum_{j=1}^p \sum_{G_j^P \in \mathcal{G}_j^P} \rho^+(G_j^P) = \sum_{H^P \in \mathcal{H}^P} \sum_{B_2 \in \mathcal{B}_2} \rho^+(B_2). \tag{5.13}$$

We also have $\mathcal{G}_\infty^I = \bigcup_{H^I \in \mathcal{H}^I} \mathcal{A}_2 \cup \bigcup_{H^P \in \mathcal{H}^P} \mathcal{B}_3$, so we have

$$\sum_{G_\infty^I \in \mathcal{G}_\infty^I} \rho(G_\infty^I) = \sum_{H^I \in \mathcal{H}^I} \sum_{A_2 \in \mathcal{A}_2} \rho(A_2) + \sum_{H^P \in \mathcal{H}^P} \sum_{B_3 \in \mathcal{B}_3} \rho(B_3). \tag{5.14}$$

Summing (5.7),(5.9) and using (5.10), (5.11), (5.12), (5.13), (5.14), we obtain (5.2). □

6. Rouché's Theorem

We denote by $\mathrm{dist}(x, y)$ the distance of $x, y \in \mathbb{P}^1$ with respect to the Fubini-Study metric on $\mathbb{P}^1$ (cf. (2.1)).

Lemma 3. *Let $E \subset \mathbb{P}^1$ be a Jordan domain such that $b \in E$. Then there is a positive constant $C = C(E, b)$ with the following property: Let F be a finite domain in a Riemann surface $\mathscr{F}$ and let ζ be a meromorphic function on $\mathscr{F}$ such that $\zeta(F) = E$ and $\zeta(\partial F) = \partial E$. Then for a meromorphic function α on $\mathscr{F}$ such that $\mathrm{dist}(\alpha(z), b) < C$ for $z \in \overline{F}$, there is a point $z \in F$ with $\zeta(z) = \alpha(z)$.*

Proof. Using an isomorphism of $\mathbb{P}^1$ which preserves the Fubini-Study metric, we may assume that $\infty \notin \overline{E}$. Put $d = \min_{w \in \partial E} |w - b|$. Let $C = C(E, b)$ be a constant such that

$$\{z \in \mathbb{C}; \mathrm{dist}(z, b) < C\} \subset \{z \in \mathbb{C}; |z - b| < d\}.$$

By Cauchy's residue theorem, we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial F} \frac{\zeta'(z)}{\zeta(z) - b} dz = n(b, \zeta, F) - n(\infty, \zeta, F) = n(b, \zeta, F) > 0,$$

and

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial F} \frac{\zeta'(z) - \alpha'(z)}{\zeta(z) - \alpha(z)} dz = n(0, \zeta - \alpha, F) - n(\infty, \zeta - \alpha, F).$$

Here $n(b, \zeta, F)$ is the number of solutions of $\zeta(z) = b$ on F with counting multiplicities and similar for other terms.

Hence it suffices to show that

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial F} \frac{\eta'(z)}{\eta(z)} dz = 0,$$

where $\eta(z) = \frac{\zeta(z) - \alpha(z)}{\zeta(z) - b}$. Since we have

$$|\eta(z) - 1| = \left| \frac{b - \alpha(z)}{\zeta(z) - b} \right| < 1 \quad \text{for} \quad z \in \partial F,$$

we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial F} \frac{\eta'(z)}{\eta(z)} dz = \frac{1}{2\pi} \int_{\partial F} d \arg \eta(z) = 0,$$

which proves our lemma. □

7. Construction of Tree

We start the proof of our theorem. To prove our theorem, using an automorphism of $\mathbb{P}^1$, we may assume without loss of generality that $a_i(\infty) \neq 0, \infty$ for $i = 1, \dots, q$. Add the new function $a_{q+1}(z) \equiv 0$ to our rational functions. In the following, we prove

$$(q - 1) T(r, f) \leq \sum_{i=1}^{q+1} \overline{N}(r, a_i, f) + o(T(r, f)) \quad \text{when} \quad r \to \infty, \quad r \notin E$$

for some set $E \subset \mathbb{R}_{>0}$ of finite linear measure. This immediately implies our theorem because of the inequality $\overline{N}(r, a_{q+1}, f) \leq T(r, f) + O(1)$.

Put $(q + 1) = \{1, \cdots, q + 1\}$. For a subset $\Phi \subset (q + 1)$, put $\mathcal{C}_\Phi = \{(i, j);\ i, j \in \Phi,\ i \neq j\}$. In the case $\mathrm{card}\,\Phi \geq 2$, we can take $(s, t) \in \mathcal{C}_\Phi$ such that

$$\frac{a_i - a_j}{a_s - a_t}(\infty) \neq \infty \quad \text{for all } (i, j) \in \mathcal{C}_\Phi. \tag{7.1}$$

To see this, for $(k, l) \in \mathcal{C}_\Phi$, we define $\mathcal{C}_\Phi(k, l) \subset \mathcal{C}_\Phi$ by

$$\mathcal{C}_\Phi(k, l) = \left\{ (i, j) \in \mathcal{C}_\Phi;\ \frac{a_i - a_j}{a_k - a_l}(\infty) = \infty \right\}.$$

Then it is not difficult to see that $\mathcal{C}_\Phi(k', l') \subset \mathcal{C}_\Phi(k, l)$ for $(k', l') \in \mathcal{C}_\Phi(k, l)$ and $(k', l') \notin \mathcal{C}_\Phi(k', l')$. Hence $\mathcal{C}_\Phi(k', l') \subsetneq \mathcal{C}_\Phi(k, l)$ for $(k', l') \in \mathcal{C}_\Phi(k, l)$. Since $\mathcal{C}_\Phi$ is a finite set, there exists $(s, t) \in \mathcal{C}_\Phi$ such that $\mathcal{C}_\Phi(s, t) = \emptyset$, hence (7.1) holds.

Now we define the equivalence relation $\sim_\Phi$ on the set Φ by

$$i \sim_\Phi j \quad (i, j \in \Phi) \Longleftrightarrow \frac{a_i - a_j}{a_s - a_t}(\infty) = 0,$$

and the function $\lambda_\Phi(w)$ by

$$\lambda_\Phi(w) = \frac{w - a_t}{a_s - a_t}. \tag{7.2}$$

Remark. (1) If there is $(k, l) \in \mathcal{C}_\Phi$ such that $a_k(\infty) \neq a_l(\infty)$, then we have $a_s(\infty) \neq a_t(\infty)$. Hence $i \sim_\Phi j$ for $i, j \in \Phi$ if and only if $a_i(\infty) = a_j(\infty)$.

(2) For a meromorphic (resp. rational) function g on $\mathbb{C}$, the function

$$\lambda_\Phi(g)(z) = \frac{g(z) - a_t(z)}{a_s(z) - a_t(z)}$$

is meromorphic (resp. rational) on $\mathbb{C}$.

Let $\mathcal{S}$ be the set of all subsets of $(q+1)$. Let $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$ be the classification of Φ by the equivalence relation $\sim_\Phi$. Put $\omega(\Phi) = \{\Phi_1, \cdots, \Phi_r\}$ which is a subset of $\mathcal{S}$. Since $a_s \not\sim_\Phi a_t$, we have $r \geq 2$. We define the sequence $V_0, V_1, \ldots$ of subsets of $\mathcal{S}$ by the following inductive rule. Put $V_0 = \{(q+1)\}$. Define V_{i+1} from V_i by

$$V_{i+1} = \bigcup_{\Phi \in V_i, \mathrm{card}\,\Phi \geq 2} \omega(\Phi).$$

Then this sequence $V_0, V_1, \ldots, V_k$ is finite, i.e., $\mathrm{card}\,\Phi = 1$ for all $\Phi \in V_k$ for some $k \geq 0$. Put $V = V_0 \cup \cdots \cup V_k$, which is a disjoint union.

Next we define the graph Γ by the following rule. The set of all vertices of Γ, denoted by $\mathrm{vert}(\Gamma)$, is equal to V. Two vertices v and v' in $\mathrm{vert}(\Gamma)$ are adjacent if and only if $v' \in \omega(v)$ or $v \in \omega(v')$. Then Γ is a tree, i.e., a connected graph without cycles. We denote the vertex $(q+1)$ by v_o and call the initial vertex. We call a vertex v with $\mathrm{card}\, v = 1$ a terminal vertex. Then Γ has $q+1$ terminal vertices $\{1\}, \ldots, \{q+1\}$.

For a non-terminal vertex v, put $\Pi(v) = \omega(v) \subset \mathrm{vert}(\Gamma)$. For a non-initial vertex v, let $v^\flat$ be the vertex such that $v \in \Pi(v^\flat)$. Then $v^\flat$ is uniquely determined by v. Let d_v be the number of vertices v' which are adjacent to v. Put

$$\mathrm{vert}(\Gamma)_{\mathrm{n.t.}} = \{v \in \mathrm{vert}(\Gamma);\ v \text{ is not terminal}\}.$$

Lemma 4. $\sum_{v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} (d_v - 2) = q - 1.$

Proof. We have

$$(\text{number of terminal vertices of } \Gamma) + \sum_{v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} d_v = 2 \times (\text{number of edges of } \Gamma)$$

and

$$(\text{number of edges of } \Gamma) = \mathrm{card}(\mathrm{vert}(\Gamma)) - 1.$$

Hence we get

$$q + 1 + \sum_{v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} d_v = -2 + \sum_{v \in \mathrm{vert}(\Gamma)} 2 = -2 + 2(q+1) + \sum_{v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} 2,$$

which proves our lemma. □

8. Non-Integrated Version of Theorem

For $v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}$, put

$$\zeta_v = \lambda_v(f)$$

which is a meromorphic function on $\mathbb{C}$. Here λ_v is defined by (7.2), putting $\Phi = v$.

We prove the following non-integrated version of Theorem.

Lemma 5. *There are positive constants $r_0 > 0$ and $h > 0$ such that*

$$\sum_{1 \leq i \leq q+1} \overline{n}(a_i, f, R(r_0, r)) \geq \sum_{v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} \big((d_v - 2) A(\zeta_v, R(r_0, r)) - hl(\zeta_v, \partial R(r_0, r))\big)$$

for $r > r_0$.

Proof. For $v \in \mathrm{vert}(\Gamma)$, we define $\iota(v) \in v \subset (q+1)$ by the following rule. If $v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}$, then put $\iota(v) = t$ where t is defined by (7.1) putting $\Phi = v$. If v is terminal, then take $\iota(v)$ such that $v = \{\iota(v)\}$.

Put

$$\mathrm{edge}(\Gamma) = \{(v, v');\ v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}, v' \in \Pi(v)\}.$$

For $(v, v') \in \mathrm{edge}(\Gamma)$, we define the following objects. Put

$$\alpha_{v,v'} = \lambda_v(a_{\iota(v')}),$$

which is a rational function on $\mathbb{C}$. Then by the definition of Γ, we have

$$\alpha_{v,v'}(\infty) \neq \infty \quad (\text{cf. } (7.1)).$$

Let E_∞ be a small spherical disc in $\mathbb{P}^1$ centred at ∞ such that

$$0 \notin E_\infty \quad \text{and} \quad \alpha_{v,v'}(\infty) \notin \overline{E_\infty} \quad \text{for all} \quad (v, v') \in \mathrm{edge}(\Gamma).$$

We also assume that E_∞ and ζ_v satisfy the condition (4.1) for all $v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}$. Let $E^v_{v'}$ be a small spherical disc in $\mathbb{P}^1$ centred at $\alpha_{v,v'}(\infty)$ such that

- $\overline{E^v_{v'}} \cap \overline{E^v_{v''}} = \emptyset$ for $v' \neq v'' \in \Pi(v)$ (note that $\alpha_{v,v'}(\infty) \neq \alpha_{v,v''}(\infty)$ by $\iota(v') \not\sim_v \iota(v'')$),
- $\overline{E_\infty} \cap \overline{E^v_{v'}} = \emptyset$ for all $(v, v') \in \mathrm{edge}(\Gamma)$,
- $E^v_{v'}$ and ζ_v satisfy the condition (4.1) for all $(v, v') \in \mathrm{edge}(\Gamma)$.

Put

$$\mathrm{edge}(\Gamma)_{\mathrm{n.t.}} = \{(v, v') \in \mathrm{edge}(\Gamma);\ v' \text{ is not terminal}\}.$$

For $(v, v') \in \mathrm{edge}(\Gamma)_{\mathrm{n.t.}}$, recall that we defined λ_v and $\lambda_{v'}$ by

$$\lambda_v = \frac{w - a_t}{a_s - a_t} \quad \text{and } \lambda_{v'} = \frac{w - a_{t'}}{a_{s'} - a_{t'}} \quad \text{(cf.(7.2))}.$$

Put

$$\delta_{v,v'} = \frac{a_s - a_t}{a_{s'} - a_{t'}},$$

which is a rational function. Then we have

$$\delta_{v,v'}(\infty) = \infty \quad (\text{since } s' \sim_v t'), \tag{8.1}$$

and

$$\lambda_{v'} = \delta_{v,v'}(\lambda_v - \alpha_{v,v'}). \tag{8.2}$$

In the following, we put $D(r) = \{z \in \mathbb{C};\ |z| > r\}$.

Claim 1: There is a positive constant $r_1 > 0$ such that

$$\zeta_{v'}\left(\zeta_v^{-1}(\mathbb{P}^1 \backslash E^v_{v'}) \cap D(r_1)\right) \subset E_\infty$$

for all $(v, v') \in \mathrm{edge}(\Gamma)_{\mathrm{n.t.}}$.

Proof. By (8.2), we have

$$\zeta_{v'} = \delta_{v,v'}(\zeta_v - \alpha_{v,v'}). \tag{8.3}$$

Since $E^v_{v'}$ is a neighborhood of $\alpha_{v,v'}(\infty)$, there are positive constant $C > 0$ and $r_2 > 0$ such that

$$|\zeta_v(y) - \alpha_{v,v'}(y)| > C \quad \text{on} \quad y \in \zeta_v^{-1}(\mathbb{P}^1 \backslash E^v_{v'}) \cap D(r_2)$$

for all $(v, v') \in \mathrm{edge}(\Gamma)_{\mathrm{n.t.}}$. Hence by (8.1) and (8.3), the image

$$\zeta_{v'}\left(\zeta_v^{-1}(\mathbb{P}^1 \backslash E^v_{v'}) \cap D(r)\right)$$

is contained in arbitrary small neighborhood of $\infty \in \mathbb{P}^1$ when $r \to \infty$. This proves our claim. $\square$

We take a positive constant r_0 such that

- $r_0 > r_1$,
- $\mathrm{dist}(\alpha_{v,v'}(z), \alpha_{v,v'}(\infty)) < C(E^v_{v'}, \alpha_{v,v'}(\infty))$ on $z \in D(r_0)$ for all $(v, v') \in \mathrm{edge}(\Gamma)$,
- the rational functions $a_1, \cdots, a_q$ have no pole on $z \in D(r_0)$,
- $a_i(z) \neq a_j(z)$ on $z \in D(r_0)$ for all $1 \leq i \neq j \leq q + 1$.

Now for $(v, v') \in \mathrm{edge}(\Gamma)$ and $r > r_0$, we define the integer $\tau_{v,v'}(r)$ by

$$\tau_{v,v'}(r) = -\sum_{G \in \mathcal{P}\left(\zeta_v^{-1}(E^v_{v'}), R(r_0, r)\right)} \rho^+(G) - \sum_{G \in \mathcal{I}\left(\zeta_v^{-1}(E^v_{v'}), R(r_0, r)\right)} \rho(G)$$

when $v' \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}$, and by

$$\tau_{v,v'}(r) = -\sum_{G \in \mathcal{I}\left(\zeta_v^{-1}(E^v_{v'}), R(r_0, r)\right)} \rho(G)$$

when v' is terminal.

In the following claim, we formally put $\tau_{v_o^\flat, v_o}(r) = 0$. (Note that $v_o^\flat$ is not defined.)

Claim 2: There is a positive constant $h > 0$ such that

$$\mathrm{IE}(v): \quad -\tau_{v^\flat, v}(r) + \sum_{v' \in \Pi(v)} \tau_{v,v'}(r) \geq (d_v - 2)\, A\left(\zeta_v, R(r_0, r)\right) - hl\left(\zeta_v, \partial R(r_0, r)\right)$$

for all $r > r_0$ and $v \in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}$.

Proof. We first consider the case $v = v_o$. Then we have $\mathrm{card}\, \Pi(v_o) = d_{v_o}$. By the assumption made in the beginning of Section 7, for $i = 1, \ldots, q$, we have $a_i(\infty) \neq a_{q+1}(\infty)$, so $i \not\sim_{v_o} q + 1$. Hence the vertex $\{q + 1\}$ in Γ, denoted by $\tilde{v}$, is contained in $\Pi(v_o)$. Apply Lemma 2 to the case

$$\mathscr{F} = \mathbb{C},\ F = R(r_0, r),\ \zeta = \upsilon = \zeta_{v_o},$$
$$E^\dagger = \mathbb{P}^1,\ \{E_i\}_{i=1}^p = \{E^{v_o}_{v'}\}_{v' \in \Pi(v_o) \backslash \{\tilde{v}\}},\ E_\infty = E^{v_o}_{\tilde{v}}.$$

Then we obtain

$$\rho^+(R(r_0,r)) + \sum_{v'\in\Pi(v_o)} \tau_{v_o,v'}(r) - \sum_{\substack{v'\in\Pi(v_o)\setminus\{\tilde{v}\}\\ v':\text{terminal}}} \left(\sum_{G\in\mathcal{P}\left(\zeta_{v_o}^{-1}(E_{v'}^{v_o}),R(r_0,r)\right)} \rho^+(G) \right)$$
$$\geq (d_{v_o}-2)\, A\left(\zeta_{v_o}, R(r_0,r)\right) - h_{v_o} l\left(\zeta_{v_o}, \partial R(r_0,r)\right)$$

for some positive constant h_{v_o} independent of r. Here we note that by the fact $\int_{\mathbb{P}^1}\omega_{\mathbb{P}^1}=1$, the mean sheet number of the covering $\zeta_{v_o}: R(r_0,r)\to\mathbb{P}^1$ is equal to $A\left(\zeta_{v_o}, R(r_0,r)\right)$. Using the facts that $\rho^+(R(r_0,r))=0$ and $\rho^+(G)\geq 0$, we obtain our claim in this case.

Next we consider the case $v\neq v_o$. Then we have $\operatorname{card}\Pi(v)=d_v-1$. By Claim 1, we may apply Lemma 2 to the case that

$$\mathscr{F}=\mathbb{C},\ F=R(r_0,r),\ \zeta=\zeta_v,\ \upsilon=\zeta_{v^\flat},$$
$$E^\dagger=E_v^{v^\flat},\ \{E_i\}_{i=1}^p=\{E_{v'}^v\}_{v'\in\Pi(v)},\ E_\infty=E_\infty.$$

Then we obtain

$$(8.4)\quad \phi_v(r)-\tau_{v^\flat,v}(r)+\sum_{v'\in\Pi(v)}\tau_{v,v'}(r) - \sum_{\substack{v'\in\Pi(v)\\ v':\text{terminal}}} \left(\sum_{G\in\mathcal{P}\left(\zeta_v^{-1}(E_{v'}^v),R(r_0,r)\right)} \rho^+(G) \right)$$
$$\geq (d_v-2)\, A\left(\zeta_v, R(r_0,r)\right) - h_v l\left(\zeta_v, \partial R(r_0,r)\right)$$

for some positive constant h_v independent of r. Here $\phi_v(r)$ is defined by

$$\phi_v(r)=\vartheta(\zeta_v,\zeta_{v^\flat}) - \sum_{G\in\mathcal{I}(\zeta_v^{-1}(E_\infty),R(r_0,r)\cap\zeta_{v^\flat}^{-1}(E_v^{v^\flat}))} \rho(G).$$

Subcaim: (1) $\vartheta(\zeta_v,\zeta_{v^\flat})=0$.
(2) $-\sum_{G\in\mathcal{I}(\zeta_v^{-1}(E_\infty),R(r_0,r)\cap\zeta_{v^\flat}^{-1}(E_v^{v^\flat}))}\rho(G)=0$.

Proof of Subclaim. We first prove (1). Take $G\in\mathcal{I}(\zeta_{v^\flat}^{-1}(E_v^{v^\flat}),R(r_0,r))$. Then by the definition of r_0 and Lemma 3, there is a point $z\in G$ such that

$$\zeta_{v^\flat}(z)=\alpha_{v^\flat,v}(z)\ (\neq\infty).$$

Note that $\delta_{v^\flat,v}$ has no pole on $R(r_0,r)$. Hence by (8.3), we have $\zeta_v(z)=0\notin E_\infty$. Hence $\zeta_v(G)\not\subset E_\infty$. This proves (1).

Next we prove (2). More precisely, we prove $I=\mathcal{I}(\zeta_v^{-1}(E_\infty),R(r_0,r)\cap\zeta_{v^\flat}^{-1}(E_v^{v^\flat}))=\emptyset$.

Suppose there exists $G\in I$. Then there is a point $z\in G$ such that $\zeta_v(z)=\infty$ (note that $\infty\in E_\infty$). On the other hand, we have $\zeta_{v^\flat}(z)\neq\infty$ because $\infty\notin E_v^{v^\flat}$. But these contradict to (8.3), because we take r_0 such that $\delta_{v^\flat,v}$ and $\alpha_{v^\flat,v}$ have no pole in $R(r_0,r)$. Hence, $I=\emptyset$, which proves our subclaim. □

Using this subclaim and the fact $\rho^+(G)\geq 0$ in (8.4), we also obatin our claim in the case v is not initial.

Putting $h=\max_{v\in\text{vert}(\Gamma)_{\text{n.t.}}} h_v$, we conclude our proof of the claim. □

Now, summing up the inequalities IE(v) for all $v\in\text{vert}(\Gamma)_{\text{n.t.}}$, we get

$$(8.5)\qquad \sum_{v:\text{terminal}}\tau_{v^\flat,v}(r)\geq\sum_{v\in\text{vert}(\Gamma)_{\text{n.t.}}}\left((d_v-2)\,A\left(\zeta_v,R(r_0,r)\right)-hl\left(\zeta_v,\partial R(r_0,r)\right)\right).$$

Here we note that terms $\tau_{v,v'}(r)$ for $(v,v')\in\text{edge}(\Gamma)_{\text{n.t.}}$ appear in the inequality IE(v') with the coefficient -1, and in the inequality IE(v) with the coefficient $+1$. Hence these terms are canceled, and we have

$$\sum_{v\in\text{vert}(\Gamma)_{\text{n.t.}}}(\text{left hand side of IE}(v)) = -\tau_{v_o^\flat,v_o}(r)+\sum_{v:\text{terminal}}\tau_{v^\flat,v}(r)=\sum_{v:\text{terminal}}\tau_{v^\flat,v}(r).$$

By Lemma 3 and the definition of r_0, for a terminal vertex v, we have

$$\begin{aligned}\tau_{v^\flat,v}(r)&\leq\operatorname{card}\left(\mathcal{I}\left(\zeta_{v^\flat}^{-1}(E_v^{v^\flat}),R(r_0,r)\right)\right)\\ &\leq\overline{n}(\alpha_{v^\flat,v},\zeta_{v^\flat},R(r_0,r))\\ &=\overline{n}(\lambda_{v^\flat}(a_{\iota(v)}),\lambda_{v^\flat}(f),R(r_0,r))\\ &=\overline{n}(a_{\iota(v)},f,R(r_0,r)).\end{aligned}$$

Hence by (8.5) and the equality

$$\sum_{v:\text{terminal}}\overline{n}(a_{\iota(v)},f,R(r_0,r))=\sum_{i=1}^{q+1}\overline{n}(a_i,f,R(r_0,r)),$$

we get our lemma. □

9. Conclusion of Proof of Theorem

Integrating the inequality of Lemma 5, we get

$$\sum_{1\leq i\leq q+1} \overline{N}(r,a_i,f) + O(\log r) \geq \sum_{v\in \mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} \left((d_v-2)T(r,\zeta_v) - hL(r,\zeta_v)\right).$$

Here, we put

$$L(r,\zeta_v) = \int_1^r \frac{l(\zeta_v, \partial\mathbb{C}(t))}{t} dt.$$

By the proof of [M, Theorem], we have

$$L(r,\zeta_v) < o\left(T(r,\zeta_v)\right) \quad \text{when} \quad r\to\infty, \quad r\notin E_v$$

for some set $E_v \subset \mathbb{R}_{>0}$ of finite linear measure. Since we have $T(r,a_i) < O(\log r)$ for $i=1,\dots,q+1$, we have

$$T(r,f) \leq T(r,\zeta_v) + O(\log r) \quad \text{when} \quad r\to\infty.$$

Hence putting $E = \bigcup_{v\in\mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} E_v$, we have

$$\left(\sum_{v\in\mathrm{vert}(\Gamma)_{\mathrm{n.t.}}} (d_v-2)\right) T(r,f) \leq \sum_{1\leq i\leq q+1} \overline{N}(r,a_i,f) + o\left(T(r,f)\right) \quad \text{when} \quad r\to\infty,\ r\notin E.$$

Here note that since f is transcendental, we have $\lim_{r\to\infty}\frac{\log r}{T(r,f))} = 0$. Using Lemma 4, we obtain

$$(q-1)T(r,f) \leq \sum_{1\leq i\leq q+1} \overline{N}(r,a_i,f) + o\left(T(r,f)\right) \quad \text{when} \quad r\to\infty, \quad r\notin E.$$

Since we have

$$\overline{N}(r,a_{q+1},f) = \overline{N}(r,\infty,f) \leq T(r,f) + O(1) \quad \text{when } r\to\infty,$$

we get our theorem.

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