Poincaré's remark on Neumann's algorithm

Norman Levenberg (Auckland University)
Hiroshi Yamaguchi (Nara Women's University)

Introduction.

Let $D \Subset \mathbf{R}^3$ be a domain with a connected C^∞ smooth boundary Σ. For $\varphi \in C^\infty(\Sigma)$ we consider the double-layer potential

$$\mathcal{W}\varphi(\boldsymbol{x}) \quad = \quad \frac{1}{4\pi} \iint_\Sigma \frac{\partial}{\partial \boldsymbol{n}_y} \left(\frac{1}{\|\boldsymbol{y} - \boldsymbol{x}\|} \right) \varphi(\boldsymbol{y}) dS_y, \qquad \boldsymbol{x} \in \mathbb{R}^3,$$

where $\boldsymbol{n}_y$ is the unit outer normal vector to Σ at $\mathbf{y}$. Throughout this note we set $D^+ = D$ and $D^- = \mathbf{R}^3 \setminus \overline{D}$, and given any object F defined $\mathbb{R}^3 \setminus \Sigma$, we set $F(\boldsymbol{x}) = F^\pm(\boldsymbol{x})$ for $\boldsymbol{x} \in D^\pm$. Then $\mathcal{W}^\pm\varphi$ is of class C^∞ on $\overline{D}^\pm$, harmonic on $D^\pm$, and the discontinuity along Σ is of the following form:

$$\mathcal{W}^\pm\varphi = \mathcal{W}\varphi \ \mp \ \frac{1}{2}\varphi \qquad \text{on } \Sigma.$$

Given $f \in C^\infty(\Sigma)$, if there exists $\varphi \in C^\infty(\Sigma)$ satisfying the integral equation

$$\varphi = -2f + 2\mathcal{W}\varphi \qquad \text{on } \Sigma, \tag{1}$$

then $\mathcal{W}^+\varphi$ is the solution of the Dirichlet problem for f on Σ. Given initial data $\varphi_0 = f$ we recursively define

$$\varphi_n = -2f + 2\mathcal{W}\varphi_{n-1} \ \ (n = 1, 2, \ldots) \qquad \text{on } \ \Sigma.$$

This yields the formal solution

$$\varphi = -2f - 2^2\mathcal{W}f - 2^3\mathcal{W}^{(2)}f - 2^4\mathcal{W}^{(3)}f - \cdots \qquad \text{on } \ \Sigma, \tag{2}$$

where $\mathcal{W}^{(n)}f = \mathcal{W}(\mathcal{W}^{(n-1)}f)$ $(n = 2, 3, \ldots)$.

In case the domain D is convex (C. Neumann (1887)), or in case D is diffeomorphic to the ball (H. Poincaré [3](1896)), the series (2) is convergent on Σ. To be precise, there exists a unique constant C such that

$$\widetilde{\varphi} = -2(f - C) - 2^2\mathcal{W}(f - C) - 2^3\mathcal{W}^{(2)}(f - C) - 2^4\mathcal{W}^{(3)}(f - C) - \cdots$$

is uniformly convergent on Σ. Thus, $\mathcal{W}^+\widetilde{\varphi} + C$ is the solution for f on D.

Poincaré remarked in [3] that the same result should be true for any domain D. Then E.R. Neumann, in his Jablonowski prize-winning paper (1905), showed that, indeed, Poincaré's conjecture is true for any domain D.

In the first Hayama symposium in 1995, U. Cegrell and the second author introduced the notion of the equilibrium magnetic field (as a generalization of the solenoid) and developed a natural algorithm for constructing it. In a similar manner, in the C^ω category, we will develop a modification of C. Neumann's algorithm from the viewpoint of static electromagnetism. Then we could clearly understand why the algorithm are varid and what the algorithm means in the electromagnetism.

1 Magnetic field induced by surface current.

We shortly give the preliminaries of the static electromagnetism (see [5]). For $\rho \in C_0^\infty(\mathbb{R}^3)$ we consider the single-layer potential

$$\begin{aligned} U(\boldsymbol{x}) &= N\rho(\boldsymbol{x}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{1}{\|\boldsymbol{y}-\boldsymbol{x}\|}\rho(\boldsymbol{y})dv_y, \quad \boldsymbol{x}\in\mathbb{R}^3, \\ E(\boldsymbol{x}) &= -\nabla U(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3. \end{aligned}$$

We call ρdv_x, U and E an (electric) volume charge , the scalar potential induced by ρdv_x, and the electric field induced by ρdv_x.

For $J = (f_1, f_2, f_3) \in V_0^\infty(\mathbb{R}^3)$ with $\mathrm{div}J = 0$ in $\mathbb{R}^3$ we consider the single-layer potentials

$$\begin{aligned} A(\boldsymbol{x}) &= NJ(\boldsymbol{x}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{1}{\|\boldsymbol{y}-\boldsymbol{x}\|}J(\boldsymbol{y})dv_y, \quad \boldsymbol{x}\in\mathbb{R}^3, \\ B(\boldsymbol{x}) &= \mathrm{rot}\,A(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3. \end{aligned}$$

We call $J\,dv_x$, A and B an (electric) volume current, the vector potential induced by $J\,dv_x$, and the magnetic field induced by $J\,dv_x$.

Causality theorem: $\mathrm{div}E = \rho$ and $\mathrm{rot}B = J$ hold (Maxwell).

We call $d\nu$ a signed measure in $\mathbb{R}^3$ a generalized (electric) charge, and

$$\begin{aligned} U(\boldsymbol{x}) &= N_{d\nu}(\boldsymbol{x}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{1}{\|\boldsymbol{y}-\boldsymbol{x}\|}d\nu(\boldsymbol{y}), \quad \boldsymbol{x}\in\mathbb{R}^3\setminus K, \\ E(\boldsymbol{x}) &= -\nabla U(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3\setminus\widetilde{K}, \end{aligned}$$

where K and $\widetilde{K}$ are some subsets in $\mathbb{R}^3$ in which $U(\boldsymbol{x})$ and $E(\boldsymbol{x})$ are defined, the scalar potential and the electric field induced by $d\nu$.

Let $d\nu = (d\nu_1, d\nu_2, d\nu_3)$ a triple of signed measures in $\mathbb{R}^3$. If there exists a sequence of volume current densities $\{J_n\,dv_x\}_n$ such that $J_n\,dv_x \to d\nu$ $(n\to\infty)$ in the sense of distribution (componentwise), we call $d\nu$ a generalized (electric) current, and

$$\begin{aligned} A(\boldsymbol{x}) &= N_{d\mu}(\boldsymbol{x}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{1}{\|\boldsymbol{y}-\boldsymbol{x}\|}d\mu(\boldsymbol{y}), \quad \boldsymbol{x}\in\mathbb{R}^3\setminus K, \\ B(\boldsymbol{x}) &= \mathrm{rot}\,A(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3\setminus\widetilde{K}, \end{aligned}$$

the vector potential and the magnetic firld induced by $d\mu$.

Causality theorem: $\mathrm{div}E = d\nu$ and $\mathrm{rot}B = d\mu$ hold in the sense of distribution.

We consider the following special generalized current. Let $D \Subset \mathbb{R}^3$ be a domain bounded by a connected C^∞ smooth boundary Σ. Let $J = (f_1, f_2, f_3)$ be a vector-valued function of class C^∞ on Σ. We write it by $J \in V^\infty(\Sigma)$. If $J\,dS_x$ on Σ (where dS_x is a surface element of Σ at $\boldsymbol{x}$) is a generalized current, then we call $J\,dS_x$ a surface current on Σ. We write the vector potential and the magnetic field induced by $J\,dS_x$ by A_J and B_J. Then $B_J^\pm$ is extended of class C^∞ on $\overline{D}^\pm$ and have the following discontinuity form:

$$B_J^+(\zeta) - B_J^-(\zeta) = \boldsymbol{n}_\zeta \times J(\zeta) \quad \text{for } \zeta \in \Sigma.$$

Proposition 1.1. (see [1]) *Let* $\boldsymbol{f} = (f_1, f_2, f_3) \in V^\infty(\Sigma)$ *and put*

$$\begin{aligned} \boldsymbol{g} &= \boldsymbol{f} \times \boldsymbol{n}_x = (g_1, g_2, g_3) \quad on \quad \Sigma, \\ \sigma &= g_1 dx + g_2 dy + g_3 dz \ (= \boldsymbol{g} \bullet d\boldsymbol{x}) \quad on \quad \Sigma. \end{aligned}$$

Then $\boldsymbol{f} dS_x$ *on* Σ *is surface current on* Σ*, if and only if* $\boldsymbol{f}$ *is tangential on* Σ *and* σ *is closed* 1*-form on* Σ.

If a surface current JdS_x on Σ induces a magnetic field B_J which is identically 0 in D^-, then we say in [5] that JdS_x is an equilibrium surface current on Σ and B_J is the equilibrium magnetic field for JdS_x. In this case B_J^+ becomes tangential on Σ. We then had the following theorem.

Theorem 1.1. (cf. Main Theorem in [5]) *Let* $\{\gamma\}_{j=1,\ldots,q}$ *be a* 1*-dimensional homology base of* D*. Then, for any fixed* $i\,(i = 1, \ldots, q)$*, there exists a unique equilibrium surface current* $\mathcal{J}_i dS_x$ *on* Σ *which induces a magnetic field* $\mathcal{B}_i$ *such that* $\int_{\gamma_j} \mathcal{B}_i \bullet d\boldsymbol{x} = \delta_{ij}$ $(j = 1, \ldots, q)$*, where* δ_{ij} *is Kronecker's delta.*

We put $\chi_D = 1$ on D^+ and $= 0$ on D^-. The following lemma is a generalization of the classical Helmholtz theorem.

Lemma 1.1. (Electromagnetic orthogonal decompsition)(cf. U. Cegrell & H. Yamaguchi [1] and P. Sagré & V. P. Khavin [4])

$$\chi_D \boldsymbol{f} = E \dotplus B \qquad on \ \ \mathbb{R}^3 \setminus \Sigma,$$

where E *is the electric field induced by charge* $d\nu$ *and* B *is the magnetic field induced by current* $d\mu$ *such that*

$$d\nu = \begin{cases} (\operatorname{div} \boldsymbol{f}) dv_x, & x \in D, \\ (\boldsymbol{f} \bullet \boldsymbol{n}_x) dS_x, & x \in \Sigma, \end{cases} \qquad d\nu = \begin{cases} (\operatorname{rot} \boldsymbol{f}) dv_x, & x \in D, \\ (\boldsymbol{f} \times \boldsymbol{n}_x) dS_x, & x \in \Sigma_x. \end{cases}$$

The following lemma shows the electromagnetic meaning of the double-layer potential $\mathcal{W}f$ mentionned in Introduction.

Lemma 1.2. (see [1]) *Let* $f \in C^\infty(\Sigma)$ *and define* $J = (\nabla f) \times \boldsymbol{n}_x$ *on* Σ*. Then* $J\,dS_x$ *is a surface current on* Σ *whose magnetic field* B_J *is identical with* $\nabla \mathcal{W}f$ *in* $\mathbb{R}^3 \setminus \Sigma$*, that is,*

$$\nabla \mathcal{W}f(\mathbf{x}) = \operatorname{rot} \frac{1}{4\pi} \int_\Sigma \frac{J(\mathbf{y})}{\|\mathbf{y} - \mathbf{x}\|}\, dS_y \qquad on \ \ \mathbb{R}^3 \setminus \Sigma.$$

From now on we assume $\Sigma(= \partial D)$ is a connected C^ω smooth closed surface and denote by $H(\overline{D})$ the class of all harmonic functions on $\overline{D}$. For $u \in H(\overline{D})$ we simply write p_u by the single-layer potentila for $\partial u / \boldsymbol{n}_x$ on Σ and q_u by the double-layer potential for u on Σ. It is well-known that

$$\chi_D u(\boldsymbol{x}) = p_u(\boldsymbol{x}) \ - \ q_u(\boldsymbol{x}) \quad \text{on} \quad \mathbb{R}^3 \setminus \Sigma.$$

We consider the following seven norms: $\boldsymbol{e}_0(u) = \|\nabla u\|_D$ and

$$\boldsymbol{e}_1(u) \sim \boldsymbol{e}_6(u) = \{\|\nabla p_u\|_{\mathbb{R}^3\setminus\Sigma},\ \|\nabla q_u\|_{\mathbb{R}^3\setminus\Sigma},\ \|\nabla p_u\|_{D^\pm},\ \|\nabla q_u\|_{D^\pm}\}.$$

We then showed the following equivalence condition.

Lemma 1.3. (see [1]) *There exists a constant $M > 1$ and $1 > k > 0$ such that*

$$\boldsymbol{e}_i(u)/M \leq \boldsymbol{e}_j(u) \leq M\boldsymbol{e}_i(u) \ (i,j = 0,1,\ldots,6),$$
$$\boldsymbol{e}_i(u) \leq k\boldsymbol{e}_0(u) \ (i = 1,2,\ldots,6)$$

Combining these three lemmas we have

Proposition 1.2. *Let $u \in H(\overline{D})$ and consider the electromagnetic orthogonal decomposition of $\chi_D \nabla u$:*

$$\chi_D \nabla u = E_u \dot{+} B_u \quad on \ \ \mathbb{R}^3 \setminus \Sigma.$$

Then

$$E_u = -\nabla p_u, \qquad B_u = \nabla q_u,$$

and the corresponding inequalities among seven norms: $\{\|\nabla u\|_D, \ \|E_u\|_{\mathbb{R}^3\setminus\Sigma} \sim \|B_u\|_{D^\pm}\}$ hold as in Lemma 1.3.

From this proposition we have the following important remark.

Remark 1.1. *We use the same notations in Proposition 1.2. Then*

1. *The tangential component of E_u^+ on Σ is determined by the tangential component of the initial data ∇u on Σ (since so is B_u^+).*
2. *The normal component of B_1^+ on Σ is determined by the normal component of ∇u on Σ (since so is E_1^+).*

Let $u \in H(\overline{D})$ and consider the electromagnetic orthogonal decomposition of $\chi_D u$:

$$\chi_D \nabla u \quad = \quad E_1 \dot{+} B_1 \quad \text{on} \ \ \mathbb{R}^3 \setminus \Sigma.$$

We define recursively

$$\begin{aligned} \chi_D E_1^+ &= \widetilde{E}_2 \dot{+} \widetilde{B}_2 \quad \text{on} \ \ \mathbb{R}^3 \setminus \Sigma, \\ \chi_D B_1^+ &= \widehat{E}_2 \dot{+} \widehat{B}_2 \quad \text{on} \ \ \mathbb{R}^3 \setminus \Sigma, \end{aligned}$$

and, for $n = 2, 3, \ldots$

$$\begin{aligned} \chi_D \widetilde{E}_n^+ &= \widetilde{E}_{n+1} \dot{+} \widetilde{B}_{n+1} \quad \text{on} \ \ \mathbb{R}^3 \setminus \Sigma, \\ \chi_D \widehat{B}_n^+ &= \widehat{E}_{n+1} \dot{+} \widehat{B}_{n+1} \quad \text{on} \ \ \mathbb{R}^3 \setminus \Sigma. \end{aligned}$$

So, we have the following formal summations in which the first is the summation of electric charges induced by surface charges on Σ and the second is that of magnetic fields induced by surface currents on Σ:

$$\chi_D \nabla u = E_1 + \widehat{E}_2 + \widehat{E}_3 + \cdots \quad \text{on} \ \ D^+, \tag{3}$$
$$\chi_D \nabla u = B_1 + \widetilde{B}_2 + \widetilde{B}_3 + \cdots \quad \text{on} \ \ D^+. \tag{4}$$

Theorem 1.2. *Both formal summations (3) and (4) are strongly convergent in $D^+ (= D)$ (i.e., convergent in $L^2(D)$).*

[Proof] Proposition 1.2 and Lemma 1.3 imply that

$$\|\chi_D \nabla u - (E_1 + \widehat{E}_2 + \widehat{E}_3 + \cdots + \widehat{E}_n)\|_{D^+} = \|\widetilde{B}_n\|_{D^+} \leq k^n \|\nabla u\|_D,$$

and

$$\|\chi_D \nabla u - (B_1 + \widetilde{B}_2 + \widetilde{B}_3 + \cdots + \widetilde{B}_n)\|_{D^+} = \|\widetilde{E}_n\|_{D^+} \leq k^n \|\nabla u\|_D.$$

It follows from $0 < k < 1$ that both sumations (3) and (4) are strongly convergent in D. □

2 Algorithm of surface current.

Let Σ be C^ω smooth and let $\boldsymbol{f}$ be a C^ω tangential vector field on Σ. Our problem is to verify the existence and describe the construction of an algorithm for the surface current $\mathcal{J}\, dS_x$ on Σ which induces the magnetic field $\mathcal{B}$ such that the tangential component of $\mathcal{B}^+$ on Σ (precisely, the continuously extension of $\mathcal{B}^+$ to Σ) is equal to $\boldsymbol{f}$.

Theorem 2.1. *The necessary and suffcient conditions for $\boldsymbol{f}$ of the existence of such surface current $\mathcal{J} dS_x$ on Σ are*

1. *$f \bullet dx$ is closed on Σ, and*
2. *$\int_\gamma \boldsymbol{f} \bullet d\boldsymbol{x} = 0$ for any 1-cycle γ with $\gamma \sim 0$ on $\overline{D}$.*

[Proof] Assume that for a given $\boldsymbol{f} \in V^\omega(\Sigma)$ there exists such surface current $\mathcal{J} dS_x$ on Σ. Then it induces the magnetic field $\mathcal{B}$ such that $\mathcal{B}^+ = \boldsymbol{f}$ on Σ. Since $\mathcal{B}^+$ is closed 1-form on $\overline{D^+}$, it follows that $\boldsymbol{f}$ satisfies 1 and 2.

Conversely, assume that $\boldsymbol{f} \in V^\omega(\Sigma)$ satisfies 1 and 2. By Theorem 1.1 there exists an equilibrium surface current $\mathfrak{J} dS_x$ which induces the magnetic field $\mathfrak{B}$ on $\mathbb{R}^3 \setminus \Sigma$ such that $\mathfrak{B}^+ \bullet d\boldsymbol{x}$ on Σ has the same period as $\boldsymbol{f} \bullet d\boldsymbol{x}$ for any 1-cycle γ on Σ with $\gamma \not\sim 0$ on $\overline{D}$. Note that $\mathfrak{B}^+$ is tangential on Σ. We can thus find a function $h \in C^\omega(\Sigma)$ such that $dh = (\boldsymbol{f} - \mathfrak{B}^+) \bullet d\boldsymbol{x}$ on Σ. We extend h on Σ to a C^ω function $\widetilde{h}$ in a neighborhood of Σ in $\mathbb{R}^3$ so that the tangential component of $\nabla\widetilde{h}$ on Σ is $\boldsymbol{f} - \mathfrak{B}^+$ on Σ. We then construct a double-layer potntial $\mathcal{W}\varphi$ on $\mathbb{R}^3 \setminus \Sigma$ such that the boundary value of $\mathcal{W}\varphi$ on Σ from D^+ is identical with h on Σ. By Lemma 1.2, $\mathcal{H} := \nabla \mathcal{W}\varphi$ is the magnetic field induced by surface current $J_\varphi dS_x := (\nabla\varphi \times \boldsymbol{n}_x) dS_x$ on Σ. Since $\widetilde{h} = h = \mathcal{W}\varphi$ on Σ, it follows that the tangential component of $\mathcal{H}^+$ on Σ is equal to $\boldsymbol{f} - \mathfrak{B}^+$. Hence, $\mathcal{J} dS_x \Sigma := \mathfrak{J} dS_x + J_\varphi dS_x$ is a desired surface current on Σ. □

We shall show the algorithm for constructing $\mathcal{J} dS_x$ and $\mathcal{B}$ for a given tangential vector $\boldsymbol{f}$ on Σ which satisfies 1 and 2.

1^{st} step. Let $J(\boldsymbol{x}) dS_x = (\boldsymbol{f} \times \boldsymbol{n}_x)\, dS_x$ on Σ; this is a surface current on Σ by Proposition 1.1. It thus induces a magnetic field B_0 on $\mathbb{R}^3 \setminus \Sigma$.

2^{nd} step. Let $J_1(\boldsymbol{x}) dS_x = (J(\boldsymbol{x}) - B_0^+(\boldsymbol{x}) \times \boldsymbol{n}_x) dS_x$ on Σ; this is a surface current on Σ by Proposition 1.1. It thus induces a magnetic field B_1 on $\mathbb{R}^3 \setminus \Sigma$.

3^{rd} step. If a surface current $J_\nu(\boldsymbol{x}) dS_x$ on Σ and its induced magnetic field B_ν in $\mathbb{R}^3 \setminus \Sigma$ are defined for $\nu \geq 1$, then we set

$$J_{\nu+1}(\boldsymbol{x}) dS_x = (J_\nu(\boldsymbol{x}) - B_\nu^+(\mathrm{x}) \times \boldsymbol{n}_x)\, dS_x, \qquad \boldsymbol{x} \in \Sigma.$$

This is a surface current density on Σ which induces a magnetic field $B_{\nu+1}$ in $\mathbf{R}^3 \setminus \Sigma$.

4^{th} step. We have inductively defined a sequence $\{J_\nu \, dS_x\}_\nu$ of surface current densities on Σ and a sequence $\{B_\nu\}_\nu$ of magnetic fields on $\mathbf{R}^3 \setminus \Sigma$.

Then

$$\lim_{\nu\to\infty} (J + J_1 + \ldots + J_\nu) \, dS_x = \mathcal{J} \, dS_x \text{ (as distributions)},$$
$$\lim_{\nu\to\infty} \|(B_0 + B_1 + \ldots + B_\nu) - \mathcal{B}\|_{\mathbf{R}^3\setminus\Sigma} = 0.$$

[Proof] We have the following sequence of electromagnetic orthogonal decompositions:

$$\begin{array}{rcll} \chi_D \mathcal{B} & = & E_1 \dot{+} B_1 & \text{on } D^+ \cup D^-, \\ \chi_D E_1 & = & E_2 \dot{+} B_2 & \text{on } D^+ \cup D^-, \\ \chi_D E_2 & = & E_3 \dot{+} B_3 & \text{on } D^+ \cup D^-, \\ \vdots & & \vdots & \vdots \end{array}$$

We thus have

$$B_m + B_{m+1} \cdots + B_{m+p} = \chi_D E_m - E_{m+p} \quad \text{on } D^+,$$

so that Proposition 1.2 and Lemma 1.3 imply

$$\|B_m + B_{m+1} \cdots + B_{m+p}\|_{D^+} \le \|\chi_D E_m\|_{D^+} + \|E_{m+p}\|_{D^+} \le 2k^m \|\mathcal{B}\|_{D^+}.$$

It follows that $\mathcal{B} = \sum_{\nu=1}^\infty B_\nu$ is convergent in $L^2(D^+)$, and hence, in $L^2(\mathbb{R}^3)$ by Proposition 1.2 and Lemma 1.3.

By causality theorem we have $J_\nu dS_x = \operatorname{rot} B_\nu$ on Σ in the sense of distribution. It follows that $\mathcal{J} dS_x = (J + \sum_{\nu=1}^\infty J_\nu) dS_x$ converges in the sense of distribution. □

Remark 2.1. *Let $F \in C^\omega(\Sigma)$ and let u be the solution oF the Dirichlet problem for f on D. If we let $\boldsymbol{f}$ denote the tangential component of ∇F on Σ, then $\mathcal{B}^+ = \nabla u$ in D and $\mathcal{J} = \nabla\varphi$ on Σ where φ is the solution of equation (1).*

References

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