REALIZING REAL LIE GROUPS AS AUTOMORPHISM GROUPS OF COMPLEX MANIFOLDS

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In general, the automorphism group of a complex manifold can be quite large. For example, $(x, y) \mapsto (x, y + f(x))$ defines an automorphism of $\mathbb{C}^2$ for every holomorphic function $f \in \mathcal{O}(\mathbb{C})$. Thus the automorphism group of $\mathbb{C}^2$ is too large to be a (finite-dimensional) Lie group.

In contrast, if X is a complex manifold which is hyperbolic in the sense of Kobayashi ([2]), then the automorphism group of X, equipped with the compact-open-topology, admits the structure of a real Lie group (finite-dimensional, with countably many connected components.)

It is natural to ask which real Lie groups can be realized in this way. More precisely we ask:

Question. *Is it true that for every real Lie group G there exists a complex manifold X, hyperbolic in the sense of Kobayashi, such that G is isomorphic to the group $\mathrm{Aut}(X)$ of all automorphisms of X?*

For compact Lie groups a positive answer has been provided by Bedford and Dadok [1], and independently by Saerens and Zame [3].

Recently, we have been able to give an affirmative answer in the following two cases:

- G is connected ([4]) or
- G is discrete ([5]).

In all theses cases one can construct X in such a way that X is furthermore Stein and *complete* hyperbolic.

The first step is to relate G to some standard group. In the case of a connected Lie group we utilize the theorem of Ado which states that the Lie algebra of G can be embedded into the Lie algebra of some $GL_n(\mathbb{R})$. In the discrete case we use the fact that every countable group is a quotient of the non-commutative free group with countably infinitely many generators.

Using this relation of G with a standard group one can construct a complex manifold X_0 on which G acts. Then the problem is to ensure

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that the automorphism group of X_0 is not larger than G. This is achieved by passing to a suitable "generically choosen" modification X of X_0.

Here the idea is to construct X in such a way that

- there is a boundary bX in some sense,
- every automorphism of X extends to bX almost everywhere,
- the G-orbits in bX are free and totally real,
- the boundary bX is "locally isomorphic" at two points $p, q \in bX$ only if p and q are in the same G-orbit.

Then it is possible to deduce that every automorphism of X must already be given by an element of G.

Naturally the details of this strategy are quite different for the two opposite cases where G is connected resp. discrete.

For these details we refer to our papers [4],[5].

References

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