RESIDUES OF CHERN CLASSES ON SINGULAR VARIETIES

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This is a summary of [S4], which will be published elsewhere.

Let E be a complex vector bundle of rank r over some reasonable space X of real dimension m. Then, for $i = 1, \dots, r$, we have the i-th Chern class $c^i(E)$ in $H^{2i}(X)$. If we use the obstruction theory, it is the primary obstruction to constructing $r - i + 1$ sections linearly independent everywhere. The Chern-Weil theory provides us with a canonical way of constructiong a closed $2i$-form representing the class $c^i(E)$ in the de Rham cohomology. For an ℓ-tuple of sections $\mathbf{s} = (s_1, \dots, s_\ell)$ of E, we denote by $S = S(\mathbf{s})$ its singular set, i.e., the set of points where the s_i's fail to be linealy independent. For $i \geq r - \ell + 1$, there is a natural lifting $c^i_S(E, \mathbf{s})$ in $H^{2i}(X, X \setminus S)$ of $c^i(E)$, which we call the "localization of $c^i(E)$ at S with respect to $\mathbf{s}$". Suppose S is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then, by the Alexander homomorphism $H^{2i}(X, X \setminus S) \to H_{m-2i}(S) = \oplus_\lambda H_{m-2i}(S_\lambda)$, the class $c^i_S(E, \mathbf{s})$ determines, for each λ, the "residue" $\mathrm{Res}_{c^i}(\mathbf{s}, E; S_\lambda)$ in $H_{m-2i}(S_\lambda)$. If X is compact, we have the "residue formula"

$$\sum_\lambda (\iota_\lambda)_* \mathrm{Res}_{c^i}(\mathbf{s}, E; S_\lambda) = c^i(E) \frown [X],$$

where $\iota_\lambda \hookrightarrow X$ denotes the inclusion and $[X]$ the fundamental class of X. The formula itself is of rather trivial nature. However, everytime we have an explicit expression for the residues, it becomes really an interesting one.

Here in this note, we consider the case where X is a (locally) complete intersection variety of dimension n. We also assume that $r - \ell + 1 = n$ and look at $c^n(E)$ so that the residue $\mathrm{Res}_{c^n}(\mathbf{s}, E; S_\lambda)$ under consideration is a number. In the case S_λ consists of an isolated point p, we give analytic, algebraic and topological expressions for $\mathrm{Res}_{c^n}(\mathbf{s}, E; p)$. As a consequence we have the fact that these three expressions are the same, which is rather well-known in some cases, in particular in the case X is non-singular, $r = n$ and $\ell = 1$ (see, e.g., [GH], [O]). We also give some examples and applications.

We work in the framework of Chern-Weil theory adapted to the Čech-de Rham cohomology and we represent the residue by a certain integral of the "Bott difference form" (see (2.1) below). The analytic expression is given in terms

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of Grothendieck residues on varieties and is obtained by computing the residues directly (Theorem 3.2). The algebraic one is given as the dimension of a certain $\mathbb{C}$-algebra (Theorem 3.5) and the topological one as the degree of a certain map (Theorem 3.7). The last two equalities are proved by the following steps, noting that the residue, the dimension and the mapping degree under consideration all satisfy the conservation law under perturbations of sections :
(1) reducing to the case of non-singular base space,
(2) reducing the number of sections, and going to the case of $\ell = 1$,
(3) reducing to the case of non-degenerate singularities, where everything is 1.

1. Residues of Chern classes

We refer to [S2, Ch.IV, 2, Ch.VI, 4] and [S3] for details of this section.

Let V be an analytic variety of pure dimension n in a complex manifold W of dimension $n+k$. We denote by $\mathrm{Sing}(V)$ the singular set of V and let $V' = V \backslash \mathrm{Sing}(V)$ be the non-singular part.

Let S be a compact set in V (V may not be compact). We assume that S has a finite number of connected components, $S \supset \mathrm{Sing}(V)$ and that S admits a regular neighborhood in W. Let $\tilde{U}_1$ be a regular neighborhood of S in W and $\tilde{U}_0$ a tubular neighborhood of $U_0 = V \setminus S$ in W. We consider the covering $\mathcal{U} = \{\tilde{U}_0, \tilde{U}_1\}$ of the union $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$, which may be assumed to have the same homotopy type as V.

For a complex vector bundle E over $\tilde{U}$ of rank r ($\geq n$), the n-th Chern class $c^n(E)$ is in $H^{2n}(\tilde{U}) \simeq H^{2n}(V)$. The corresponding class in $H^{2n}(V)$ is denoted by $c^n(E|_V)$. The class $c^n(E)$ is represented by a Čech-de Rham cocycle $c^n(\nabla_\star)$ on $\mathcal{U}$ given as

$$c^n(\nabla_\star) = (c^n(\nabla_0), c^n(\nabla_1), c^n(\nabla_0, \nabla_1)),$$

where $c^n(\nabla_i)$ denotes the n-th Chern form of a connection ∇_i for E on $\tilde{U}_i$, $i = 0,\ 1$, and $c^n(\nabla_0, \nabla_1)$ the Bott difference form. Note that it is sufficient if ∇_0 is defined only on U_0, since there is a C^∞ retraction of $\tilde{U}_0$ onto U_0. Suppose we have an ℓ-tuple $\mathbf{s} = (s_1, \ldots, s_\ell)$ of C^∞ sections linearly independent everywhere on U_0, $\ell = r - n + 1$, and let ∇_0 be $\mathbf{s}$-trivial, i.e., $\nabla_0(s_i) = 0$, $i = 1, \ldots, \ell$. Then we have the vanishing $c^n(\nabla_0) = 0$ and the above cocycle $c^n(\nabla_\star)$ defines a class $c^n_S(E|_V, \mathbf{s})$ in $H^{2n}(V, V \setminus S; \mathbb{C})$. It is sent to $c^n(E|_V)$ by the canonical homomorphism $j^* : H^{2n}(V, V \setminus S; \mathbb{C}) \to H^{2n}(V, \mathbb{C})$.

Let $(S_\lambda)_\lambda$ be the connected components of S. Then, for each λ, $c^n_S(E|_V, \mathbf{s})$ defines the residue $\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; S_\lambda)$. For each λ, we choose a neighborhood $\tilde{U}_\lambda$ of S_λ in $\tilde{U}_1$, so that the $\tilde{U}_\lambda$'s are mutually disjoint. Let $\tilde{R}_\lambda$ be a real $2(n+k)$-dimensional manifold with C^∞ boundary $\partial\tilde{R}_\lambda$ in $\tilde{U}_\lambda$ containing S_λ in its interior such that $\partial\tilde{R}_\lambda$ is transverse to V. We set $R_\lambda = \tilde{R}_\lambda \cap V$. Then the residue is a number given by

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; S_\lambda) = \int_{R_\lambda} c^n(\nabla_1) - \int_{\partial R_\lambda} c^n(\nabla_0, \nabla_1). \tag{1.1}$$

We also have the residue formula :

Proposition 1.2. *If V is compact,*

$$\sum_{\lambda} \mathrm{Res}_{c^n}(\mathbf{s}, E|_V; S_\lambda) = \int_V c^n(E).$$

Comparing with the obstruction theoretic definition of Chern classes, we see that the residue $\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; S_\lambda)$ is in fact an integer. However, we may prove this fact more directly, in the cases under consideration, from the arguments below.

2. Residues at an isolated singularity

Let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$, as before. We do not exclude the case $k=0$, where $V=W$ is a complex manifold of dimension n.

Suppose now that V has at most an isolated singularity at p and let E be a holomorphic vector bundle of rank r ($\geq n$) on a small coordinate neighborhood $\tilde{U}$ of p in W. Sometimes we identify $\tilde{U}$ with a neighborhood of 0 in $\mathbb{C}^{n+k}$ and p with 0. We may assume that E is trivial and let $\mathbf{e}=(e_1,\ldots,e_n)$ be a holomorphic frame of E on $\tilde{U}$. Let $\ell = r-n+1$ and suppose we have an ℓ-tuple of holomorphic sections $\tilde{\mathbf{s}}$ of E on $\tilde{U}$. Suppose that $S(\tilde{\mathbf{s}}) \cap V = \{p\}$. Then we have $\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p)$ with $\mathbf{s} = \tilde{\mathbf{s}}|_V$. Let $\tilde{R}$ be a compact real $2(n+k)$-dimensional manifold with C^∞ boundary in $\tilde{U}$ containing p in its interior such that $\partial\tilde{R}$ is transverse to V and set $R = \tilde{R} \cap V$. We also set $U = \tilde{U} \cap V$ and let ∇_0 be an $\mathbf{s}$-trivial connection for E on $U \setminus \{p\}$. We choose ∇_1 to be $\mathbf{e}$-trivial. Then we have $c^n(\nabla_1) = 0$ and from (1.1),

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p) = -\int_{\partial R} c^n(\nabla_0, \nabla_1). \tag{2.1}$$

In the following, we give fundamental properties of residues, in particular, the fact that they are positive integers and satisfy the "conservation law" under perturbations of sections.

First, suppose $V=W$ is a complex manifold of dimension n and write $\tilde{U}$ and $\tilde{\mathbf{s}}$ by U and $\mathbf{s}$, respectively. Thus our assumption is $S(\mathbf{s}) = \{p\}$.

Let us also assume that $r=n$. Thus $\ell = 1$ and we have only one section s. We write $s = \sum_{i=1}^n f_i\, e_i$ with f_i holomorphic functions on U.

Lemma 2.2. *If $r=n$ and $\ell=1$, we have*

$$\mathrm{Res}_{c^n}(s, E; p) = \int_{\partial R} f^* \beta_n,$$

where β_n denotes the Bochner-Martinelli kernel on $\mathbb{C}^n$ and $f=(f_1,\ldots,f_n)$.

We may think of s (or f) as a map from ∂R to $\mathbb{C}^n \setminus \{0\}$, which has the homotopy type of S^{2n-1}.

Corollary 2.3. *If $r = n$ and $\ell = 1$,*

$$\mathrm{Res}_{c^n}(s, E; p) = \deg s|_{\partial R},$$

the mapping degree of $s|_{\partial R}$. Thus $\mathrm{Res}_{c^n}(s, E; p)$ is a positive integer.

We say that p is a non-degenerate zero of s if $\det \frac{\partial(f_1,\ldots,f_n)}{\partial(z_1,\ldots,z_n)}(p) \neq 0$. In this case, $(f_1, \ldots, f_n)$ form a coordinate system around 0. Hence we have

Corollary 2.4. *If p is a non-degenerate zero of s,*

$$\mathrm{Res}_{c^n}(s, E; p) = 1.$$

Now we go back to the general case of vector bundle E of rank $r \geq n$ with an ℓ-tuple $\mathbf{s} = (s_1, \ldots, s_\ell)$ of sections, $\ell = r - n + 1$, still assuming $V = W$. We consider the bundle $E^* = E \times T$ over $U^* = U \times T$, where T is a small neighborhood of 0 in $\mathbb{C} = \{t\}$. Suppose we have an ℓ-tuple of holomorphic sections $\mathbf{s}^*$ of E^* on U^* such that $\mathbf{s}^*(z, 0) = \mathbf{s}(z)$. For t in T, we set $E_t = E^*|_{U\times\{t\}}$ and $\mathbf{s}_t(z) = \mathbf{s}^*(z, t)$. We call such an $\mathbf{s}^*$ (or $\mathbf{s}_t$) a perturbation of $\mathbf{s}$. Sometimes we identify $U \times \{t\}$ with U and $E|_t$ with E. Since we assumed that $S(\mathbf{s}) = \{p\}$, by the upper semi-continuity of $\dim S(\mathbf{s}_t)$, $S(\mathbf{s}_t)$ consists at most of a finite number of points.

Lemma 2.5. *The sum $\sum_{q\in S(s_t)} \mathrm{Res}_{c^n}(\mathbf{s}_t, E_t; q)$ is constant. In particular,*

$$\mathrm{Res}_{c^n}(\mathbf{s}, E; p) = \sum_{q\in S(\mathbf{s}_t)} \mathrm{Res}_{c^n}(\mathbf{s}_t, E_t; q),$$

which is a non-negative integer.

Remark 2.6. If $\ell = 1$, there exists always a "good perturbation" of s, i.e., a holomorphic sections s^* of E^* near 0 such that $s^*(z, 0) = s(z)$ and that s_t has only non-degenerate zeros, for $t \neq 0$ ([GH, Ch.5]).

Next we consider the case $k > 0$. Let p be an isolated singular point in V and suppose that V is a complete intersection defined by $h = (h_1, \ldots, h_k) : (\tilde{U}, p) \to (\mathbb{C}^k, 0)$. Let T be a small neighborhood of 0 in $\mathbb{C}^k$. For a point t in T, we set $V_t = h^{-1}(t)$. Let $C(h)$ denote the critical set of h and $D(h) = h(C(h))$ the discriminant, which is a hypersurface in T (see, e.g., [Lo]). We have $\mathrm{Sing}(V_t) = C(h) \cap V_t$, which consists of at most a finite number of points. We set $\mathbf{s}_t = \tilde{\mathbf{s}}|_{V_t}$ and $S(\mathbf{s}_t) = S(\tilde{\mathbf{s}}) \cap V_t$. By the assumption $S(\tilde{\mathbf{s}}) \cap V = \{p\}$, we have $\dim S(\tilde{\mathbf{s}}) \leq k$. Hence $S(\mathbf{s}_t)$ also consists of at most a finite number of points. Note that even if q is in $\mathrm{Sing}(V_t)$, if $q \notin S(\mathbf{s}_t)$, then $\mathrm{Res}_{c^n}(\mathbf{s}_t, E|_{V_t}; q) = 0$.

Lemma 2.7. *The sum $\sum_{q\in S(\mathbf{s}_t)} \mathrm{Res}_{c^n}(\mathbf{s}_t, E|_{V_t}; q)$ is constant. In particular,*

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p) = \sum_{q\in S(\mathbf{s}_t)} \mathrm{Res}_{c^n}(\mathbf{s}_t, E|_{V_t}; q),$$

which is a non-negative integer.

Note that, by Lemma 3.3 below, $\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p)$ is in fact a positive integer.

3. Various expressions

(I) Analytic

First we recall Grothendieck residues relative to a subvariety. Let $\tilde{U}$ be a neighborhood of 0 in $\mathbb{C}^{n+k}$ and V a subvariety of dimension n in $\tilde{U}$ which contains 0 as at most an isolated singular point. Also, let $f_1, \ldots, f_n$ be holomorphic functions on $\tilde{U}$ and $V(f_1, \ldots, f_n)$ the variety defined by them. We assume that $V(f_1, \ldots, f_n) \cap V = \{0\}$. For a holomorphic n-from ω on $\tilde{U}$, the Grothendieck residue relative to V is defined by (e.g., [S2, Ch.IV, 8])

$$\mathrm{Res}_0 \left[\begin{matrix} \omega \\ f_1, \ldots, f_n \end{matrix} \right]_V = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where Γ is the n-cycle in V given by

$$\Gamma = \{\, q \in \tilde{U} \cap V \mid |f_i(q)| = \varepsilon_i, \quad i = 1, \ldots, n \,\}$$

for small positive numbers ε_i. It is oriented so that $d\arg(f_1) \wedge \cdots \wedge d\arg(f_n) \geq 0$.

If $k = 0$, it reduces to the usual Grothendieck residue (e.g., [GH, Ch.5]), in which case we omit the suffix V.

If V is a complete intersection defined by $h_1 = \cdots = h_k = 0$ in $\tilde{U}$, we have

$$\mathrm{Res}_0 \left[\begin{matrix} \omega \\ f_1, \ldots, f_n \end{matrix} \right]_V = \mathrm{Res}_0 \left[\begin{matrix} \omega \wedge dh_1 \wedge \cdots \wedge dh_k \\ f_1, \ldots, f_n, h_1, \ldots, h_k \end{matrix} \right].$$

Now we go back to the situation of Section 2. We write $\tilde{s}_i = \sum_{j=1}^r f_{ij}\, e_j$, $i = 1, \ldots, \ell$, with f_{ij} holomorphic functions on $\tilde{U}$. Let F be the $\ell \times r$ matrix whose (i,j)-entry is f_{ij}. We set

$$\mathcal{I} = \{\, (i_1, \ldots, i_\ell) \mid 1 \leq i_1 < \cdots < i_\ell \leq r \,\}.$$

For an element $I = (i_1, \ldots, i_\ell)$ in $\mathcal{I}$, let F_I denote the $\ell \times \ell$ matrix consisting of the columns of F corresponding to I and set $\varphi_I = \det F_I$. If we write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_\ell}$, we have

$$\tilde{s}_1 \wedge \cdots \wedge \tilde{s}_\ell = \sum_{I \in \mathcal{I}} \varphi_I\, e_I.$$

Note that $S(\tilde{\mathbf{s}})$ is the set of common zeros of the φ_I's. From the assumption $S(\tilde{\mathbf{s}}) \cap V = \{p\}$, we have ([S3, Lemma 5.6]) :

Lemma 3.1. *We may choose a holomorphic frame* $\mathbf{e} = (e_1, \ldots, e_r)$ *of* E *so that there exist* n *elements* $I^{(1)}, \ldots, I^{(n)}$ *in* $\mathcal{I}$ *with* $V(\varphi_{I^{(1)}}, \ldots, \varphi_{I^{(n)}}) \cap V = \{p\}$.

Letting $\mathbf{e}$ and $\varphi_{I^{(1)}}, \ldots, \varphi_{I^{(n)}}$ be as above, we have the following ([S3, Theorem 5.7]) :

Theorem 3.2. *We have*

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p) = \mathrm{Res}_p \begin{bmatrix} \sigma_n(\Theta) \\ \varphi_{I^{(1)}}, \ldots, \varphi_{I^{(n)}} \end{bmatrix}_V,$$

where $\sigma_n(\Theta)$ is a holomorphic n-form given in terms of the entries of F and is computable.

Special cases

1. The case $\ell = 1$ and $r = n$. Let $\mathbf{e} = (e_1, \ldots, e_n)$ be an arbitrary frame of E and write $s = \sum_{i=1}^n f_i e_i$. Then we may set $\varphi_{I^{(i)}} = f_i$, $i = 1, \ldots, n$, and we have $\sigma_n(\Theta) = df_1 \wedge \cdots \wedge df_n$.

2. The case $n = 1$ and $\ell = r$. Let $\mathbf{e} = (e_1, \ldots, e_r)$ be an arbitrary frame of E and write $s_i = \sum_{j=1}^r f_{ij}\, e_j$, $i = 1, \ldots, r$. Let $F = (f_{ij})$ and set $\varphi = \det F$. Then we may set $\varphi_{I^{(1)}} = \varphi$ and we have $\sigma_n(\Theta) = d\varphi$.

See [S3] for more cases where the form $\sigma_n(\Theta)$ is computed explicitly.

(II) Algebraic

The proofs in this subsection are done using commutative algebra (see, e.g., [E], [Ma]) and are inspired by [EG] and [Lo, Ch.4].

We consider the situation of Section 2. We suppose that V is a complete intersection defined by $h : (\tilde{U}, p) \to (\mathbb{C}^k, 0)$. Let T be a small neighborhood of 0 in $\mathbb{C}^k$ and, for a point t in T, we set $V_t = h^{-1}(t)$. Also let $S(\mathbf{s}_t) = S(\tilde{\mathbf{s}}) \cap V_t$, as before. From the assumption $S(\tilde{\mathbf{s}}) \cap V = \{p\}$ we have

Lemma 3.3. $\dim S(\tilde{\mathbf{s}}) = k$ *and* $S(\mathbf{s}_t)$ *is a non-empty finite set.*

Let F and φ_I be defined as in (I) above. We denote by $\mathcal{O}_{\tilde{U}}$ the sheaf of germs of holomorphic functions on $\tilde{U}$, by $\mathcal{F}$ the ideal sheaf in $\mathcal{O}_{\tilde{U}}$ generated by the φ_I's, by $\mathcal{I}(V) = (h_1, \ldots, h_k)$ the ideal sheaf of V in $\mathcal{O}_{\tilde{U}}$ and by $\mathcal{F}(V)$ the ideal sheaf generated by $\mathcal{F}$ and $\mathcal{I}(V)$. Also, for $t = (t_1, \ldots, t_k) \in T$, we denote by $\mathcal{F}(V_t)$ the ideal sheaf generated by $\mathcal{F}$ and $\mathcal{I}(V_t) = (h_1 - t_1, \ldots, h_k - t_k)$.

Lemma 3.4. *In the above situation,*

$$\dim_{\mathbb{C}} \mathcal{O}_{\tilde{U},p}/\mathcal{F}(V)_p = \sum_{q \in S(\mathbf{s}_t)} \dim_{\mathbb{C}} \mathcal{O}_{\tilde{U},q}/\mathcal{F}(V_t)_q.$$

Theorem 3.5. *We have*

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p) = \dim_{\mathbb{C}} \mathcal{O}_{\tilde{U},p}/\mathcal{F}(V)_p.$$

Remark 3.6. The assumption that V is a complete intersection is necessary only to ensure that V admits a "smoothing" in $\tilde{U}$.

(III) Topological

Let V, $\tilde{U}$, E and $\mathbf{s}$ be as in Section 2. We assume that V is a complete intersection in $\tilde{U}$ with at most an isolated singularity at p. Let $W_\ell(\mathbb{C}^r)$ denote the Stiefel manifold of ℓ-frames in $\mathbb{C}^r$. It is known that the space $W_\ell(\mathbb{C}^r)$ is $2(r-\ell)$-connected and $\pi_{2n-1}(W_\ell(\mathbb{C}^r)) \simeq \mathbb{Z}$ (recall $2r-2\ell+1 = 2n-1$). Let L denote the link of (V,p). Note that both of $W_\ell(\mathbb{C}^r)$ and L have a natural generator for the $(2n-1)$-st homology. Thus the degree of the map

$$\mathbf{s}|_L : L \to W_\ell(\mathbb{C}^r)$$

is well-defined.

Theorem 3.7. *We have*

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p) = \deg \mathbf{s}|_L.$$

4. Examples and applications

(4a) Index of a holomorphic 1-form of Ebeling and Guzein-Zade

Let V be a complete intersection in $\tilde{U}$ with an isolated singularity at p and defined by $(h_1, \ldots, h_k)$, as before. Also, let L be the link of (V,p). For a holomorphic 1-form θ on $\tilde{U}$, we consider the $(k+1)$-tuple $\tilde{\mathbf{s}} = (\theta, dh_1, \ldots, dh_k)$ of sections of $T^*\tilde{U}$, which is of rank $n+k$. Thus $r-\ell+1 = n+k-(k+1)+1 = n$. We assume that $S(\tilde{\mathbf{s}}) \cap V = \{p\}$. Let $\mathbf{s} = \tilde{\mathbf{s}}|_V$, which defines a map of $V \setminus \{p\}$ to $W_\ell(\mathbb{C}^r)$. It should be emphasized that here we take the restrictions of components of $\tilde{\mathbf{s}}$ as sections and not as differential forms.

Following [EG], with different naming and notation, we define the *V-index* $\mathrm{Ind}_V(\theta, p)$ of θ at p by

$$\mathrm{Ind}_V(\theta, p) = \deg \mathbf{s}|_L.$$

Then by Theorem 3.7, it coincides with $\mathrm{Res}_{c^n}(\mathbf{s}, T^*\tilde{U}|_V; p)$ and by Theorems 3.2 and 3.5, it has analytic and algebraic expressions. In fact the algebraic one is already given in [EG].

(4b) Multiplicity of a function on a local complete intersection

We refer to [IS] for details of this subsection. Let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$. We assume that there exist a holomorphic vector bundle N of rank k and a holomorphic section σ of N, generically transverse to the zero section, with $V = \sigma^{-1}(0)$. Thus V is a local complete intersection defined by the local components of σ. Note that the restriction of N to the non-singular part V' coincides with the normal bundle of V' in W. We denote the virtual bundle $(T^*W - N^*)|_V$ by τ_V^* and call it the virtual cotangent bundle of V. Let g be a C^∞ function on W and let f and f' be its restrictions to V

and V', respectively. We define the singular set $S(f)$ of f by $S(f) = \mathrm{Sing}(V) \cup C(f')$. As in the case of vector bundles, we may define the localization of the n-th Chern class of τ_V^* by df, which in turn defines the residue $\mathrm{Res}_{c^n}(df, \tau_V^*; S)$ at each compact connected component S of $S(f)$. We define the *virtual multiplicity* $\widetilde{m}(f, S)$ of f at S by

$$\widetilde{m}(f, S) = \mathrm{Res}_{c^n}(df, \tau_V^*; S). \tag{4.1}$$

The *multiplicity* of f at S is then defined by

$$m(f, S) = \widetilde{m}(f, S) - \mu(V, S), \tag{4.2}$$

where, $\mu(V, S)$ denotes the (generalized) Milnor number of V at S as defined in [BLSS] (cf. [P], [PP] in the case $k = 1$). Note that if S consists of a point p, it is the usual Milnor number $\mu(V, p)$ of the isolated complete intersection singularity (V, p) ([Mi], [H], see also [Lo]).

Note that, if S is in V', we have $\mathrm{Res}_{c^n}(df, \tau_V^*; S) = \mathrm{Res}_{c^n}(df, T^*V'; S)$. On the other hand, in this case we have $\mu(V, S) = 0$. Thus if f is holomorphic and if S consists of an isolated point p, $m(f, S)$ coincides with the usual multiplicity of f at p (cf. Section 3 (I), Special case 1).

Let $g : W \to C$ be a holomorphic map onto a complex curve C and set $f = g|_V$, $f' = g|_{V'}$ and $S(f) = \mathrm{Sing}(V) \cup C(f')$. We assume that $S(f)$ is compact. We further set $V_0 = V \setminus S(f)$ and $f_0 = g|_{V_0}$. Thus df_0 is a non-vanishing section of the bundle $T^*V_0 \otimes f_0^*TC$, which is of rank n. If we look at $c^n(\varepsilon)$, $\varepsilon = \tau_V^* \otimes f^*TC$ and we see that there is a canonical localization $c_S^n(\varepsilon, df)$ in $H^{2n}(V, V \setminus S; \mathbb{C})$ of $c^n(\varepsilon)$.

Let $(S_\lambda)_\lambda$ be the connected components of S and let $(R_\lambda)_\lambda$ be as in Section 1. Then $c_S^n(\varepsilon, df)$ defines, for each λ, the residue $\mathrm{Res}_{c^n}(df, \tau_V^* \otimes f^*TC; S_\lambda)$, which is given by a formula similar to (1.1). If V is compact, by Proposition 1.2, we have

$$\sum_\lambda \mathrm{Res}_{c^n}(df, \tau_V^* \otimes f^*TC; S_\lambda) = \int_V c^n(\tau_V^* \otimes f^*TC).$$

In particular, if $S(f)$ consists only of isolated points, the above formula reduces to

$$\sum_{p \in S(f)} m(f, p) = (-1)^n \left(\chi(V) - \chi(F)\,\chi(C)\right), \tag{4.3}$$

where F is a general fiber of f ([IS, Corollary 5.6]). This generalizes a formula in [I] (see also [F, Example 14.1.5] and [HL, VI 3]) for a singular variety V.

If S_λ consists of a single point p, the residue $\mathrm{Res}_{c^n}(df, \tau_V^*; p)$ is given as follows. Let $\tilde{U}$ be a small neighborhood of p in W so that the bundle N admits a

frame $(\nu_1, \ldots, \nu_k)$ on $\tilde{U}$. We write $\sigma = \sum_{i=1}^{k} h_i \nu_i$ with h_i holomorphic functions on $\tilde{U}$. Then V is defined by $(h_1, \ldots, h_k)$ in $\tilde{U}$. Consider the $(k+1)$-tuple of sections

$$\tilde{\mathbf{s}} = (dg, dh_1, \ldots, dh_k)$$

of $T^*\tilde{U}$. By the assumption, we have $S(\tilde{\mathbf{s}}) \cap V = \{p\}$. Since the rank of $T^*\tilde{U}$ is $n+k$, we have the residue $\mathrm{Res}_{c^n}(\mathbf{s}, T^*\tilde{U}|_V; p)$, $\mathbf{s} = \tilde{\mathbf{s}}|_V$. Then we have ([IS, Theorem 4.6])

$$\widetilde{m}(f, p) = \mathrm{Res}_{c^n}(\mathbf{s}, T^*\tilde{U}|_V; p). \tag{4.4}$$

The virtual multiplicity $\widetilde{m}(f,p)$ was defined as the residue of df on the virtual bundle τ_V^* (cf. (4.1)) and this definition led us to a global formula as (4.3). The identity (4.4) shows that it coincides with the residue of $\mathbf{s} = (dg|_V, dh_1|_V, \ldots, dh_k|_V)$ on the vector bundle $T^*\tilde{U}|_V$. Thus we have various expressions for $\widetilde{m}(f,p)$ as given in Section 3 ; by Theorem 3.2 we have a way to compute $\widetilde{m}(f,p)$ explicitly, by Theorem 3.5 we may express

$$\widetilde{m}(f, p) = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(g, h_1, \ldots, h_k), h_1, \ldots, h_k), \tag{4.5}$$

where $J(g, h_1, \ldots, h_k)$ denotes the Jacobian ideal of the map $(g, h_1, \ldots, h_k)$, i.e., the ideal generated by the $(k+1) \times (k+1)$ minors of the Jacobian matrix $\frac{\partial(g, h_1, \ldots, h_k)}{\partial(z_1, \ldots, z_{n+k})}$, and by Theorem 3.7,

$$\widetilde{m}(f, p) = \mathrm{Ind}_V(dg, p). \tag{4.6}$$

From (4.2), (4.5) and a formula in [Gr], [Lê], we get

$$m(f, p) = \mu(V_g, p), \tag{4.7}$$

where V_g denotes the complete intersection defined by $(g, h_1, \ldots, h_k)$, assuming $g(p) = 0$.

(4c) Some others

Let V be a complete intersection defined by $(h_1, \ldots, h_k)$ in $\tilde{U}$ and p an isolated singularity of V, as before.

The n-the polar multiplicity $m_n(V,p)$ of Gaffney ([Ga]) is defined by

$$m_n(V, p) = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(\ell, h_1, \ldots, h_k), h_1, \ldots, h_k),$$

where ℓ is a general linear function. By (4.5) and (4.6), we may write

$$m_n(V, p) = \mathrm{Ind}_V(d\ell, p) = \tilde{m}(\ell|_V, p).$$

Also, in the expression

$$\mathrm{Eu}(V, p) = 1 + (-1)^{n+1} \mu(V_\ell, p)$$

for the Euler obstruction $\mathrm{Eu}(V,p)$ of V at p (cf. [D], [K], see also [BLS]), we have by (4.7),

$$\mu(V_\ell, p) = m(\ell|_V, p).$$

Note that these local invariants appear in the comparison of the Schwartz-MacPherson, Mather and Fulton-Johnson classes of a local complete intersection with isolated singularities (cf. [OSY], [S1]).

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