Several topics from the value distribution theory

Junjiro Noguchi
School of Mathematical Sciences
The University of Tokyo

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Introduction

We will discuss some topics and open problems, which are old and new of the value distribution theory in several complex variables. In the last section we will obtain a version of Nevanlinna's Lemma on logarithmic derivative for continuous forms and *pseudo-holomorphic* curves into compact *almost* complex manifolds.

1 Truncation of counting functions in the second main theorem.

We will discuss the importance of the truncation that appears in the estimate of the second main theorem type. We will discuss the truncation of counting functions for entire holomorphic curves in Abelian and semi-Abelian varieties. In this context we will also revisit Nochka's proof of Cartan-Nochka's second main theorem for entire holomorphic curves in complex projective spaces.

Let $f : \mathbf{C} \to A$ be an entire holomorphic curve into a semi-Abelian variety A:

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \to 0,$$

where A_0 is an Abelian variety. Let $\bar{A} = A \times_{(\mathbf{C}^*)^t} (\mathbf{P}^1(\mathbf{C}))^t$ be a compactification of A. Let $\bar{D}$ be an effective reduced divisor on $\bar{A}$ without component contained in $\partial A = \bar{A} \setminus A$. Set $D = \bar{D} \cap A$ and the stabilizer $\mathrm{St}(D) = \{a \in A; a + D = D\}^0$, where $\{\cdot\}^0$ stands for the idenitity component. Let $J_k(A)$ be the k-th jet space and let $J_k(f) : \mathbf{C} \to J_k(A)$ denote the k-th jet lifting. We denote $X_k(f)$ the (algebraic) Zariski-closure of $J_k(f)(\mathbf{C})$ in the logarithmic k-jet space $J_k(\bar{A}, \log \partial A)$.

We set

$$
\begin{aligned}
T_f(r, L(\bar{D})) &= \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* c_1(L(\bar{D})), \\
N_k(r, f^*\bar{D}) &= \int_1^r \frac{n_k(t, f^*\bar{D})}{t} dt, \\
N(r, f^*\bar{D}) &= N_\infty(r, f^*\bar{D}), \\
m_f(r, \bar{D}) &= \int_{|z|=r} \log \frac{1}{\|\sigma_D(f(z))\|} \frac{d\theta}{2\pi}.
\end{aligned}
$$

Theorem 1.1. (First Main Theorem; cf. [10])

$$T_f(r, L(\bar{D})) = N(r, f^*\bar{D}) + m_f(r, \bar{D}) + O(1).$$

The order of f is defined by

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r, L(\bar{D}))}{\log r} \le \infty.$$

Let $\partial A = \cup_{i=1}^t B_i$ be the Whiteney stratification such that $\operatorname{codim} B_i = i$. Consider the following boundary condition:

(B) $\bar{D}$ contains no stratum of B_t.

Theorem 1.2. (Noguchi-Winkelmann-Yamanoi [12]) *Assume* (B) *and* $\mathrm{St}(D) = \{0\}$. *Let $f : \mathbf{C} \to A$ be an arbitrary entire holomorphic curve. Then there is a number $k_0 = k_0(f, D)$, or $= k_0(\rho_f, D)$ if $\rho_f < \infty$, such that*

$$T_f(r, L(\bar{D})) \le N_{k_0}(r, f^*D) + S_f(r, L(\bar{D})),$$

where

$$S_f(r, L(\bar{D})) \le \delta \log r + O(\log T_f(r, L(\bar{D})))||_{E(\delta)}.$$

The key of the proof of Theorem 1.2 was

Lemma 1.3. ([12] Lemma 5.1) *Let*

$$I_k : J_k(\bar{A}, \log \partial A) \cong \bar{A} \times \mathbf{C}^{nk} \to \mathbf{C}^{nk} \quad (\dim A = n)$$

be the jet projection. Then there is such a k_0 as above that

$$I_{k_0}(J_{k_0}(\bar{D}, \log \partial A)) \not\supset W_{k_0} = I_{k_0}(X_{k_0}(f)).$$

Remark **1.4.** In the case of Abelian varieties, besides Lemma 1.3, the other part of the proof had been established already (cf. [10]). Our appraoch is to distinguish the image of f from $\bar{D}$ by making use of jets, and to get lower bound for the avarage of a sort of distance from $f(z)$ to $\bar{D}$. This approach is different to that of a paper of Siu and Yeung[1], where they tried to construct a jet differential vanishing on an ample divisor; unfortunately, the statement of the claimed lemma at p. 1147 of their paper was incorrect (cf. [12] *Remark* 5.34). In a preprint of Siu and Yeung[2], they analyse the intersection of $X_k(f)$ and $J_k(D)$ in terms of Chern numbers in the case of Abelian variety A, and showed the above Lemma 1.3 with k_0 having a better dependence (cf. *Claim 1*, p. 3 of the preprint[2]):

$$k_0 = k_0(c_1(D)^n).$$

As far as D is allowed to have singularities, k_0 must depend on D.

Example. Let $A = (\mathbf{C}/(\mathbf{Z} + i\mathbf{Z}))^2$ and set

$$f : z \in \mathbf{C} \to (z, z^2) \in A.$$

Let D be a curve with a cusp of order p at $0 \in A$. Then $f^*D \geq p(\mathbf{Z} + i\mathbf{Z})$. Thus, for an arbitrarily given k_0, we take $p > k_0$ so that

$$T_f(r, L(D)) \sim r^4,$$
$$T_f(r, L(D)) - N_{k_0}(r, f^*D) \geq (p - k_0)r^2(1 + o(1)).$$

On the other hand, Yamanoi lately proved

Theorem 1.5. *Let A be an Abelian variety with an ample line bundle L, and let $f : \mathbf{C} \to A$ be an algebraically nondegenerate entire holomorphic curve. Let $Y \subset J_k(A)$ be a subvariety of* $\operatorname{codim} Y \geq 2$. *Then for an arbitrary number $\epsilon > 0$*

$$N(r, f^*Y) \leq \epsilon T_f(r, L)||_{E(\epsilon)}.$$

In particular, setting $Y = J_1(D) \subset J_1(A)$, one gets for $L = L(D)$

$$T_f(r, L) \leq N_1(r, f^*D) + \epsilon T_f(r, L)||_{E(\epsilon)}.$$

Therefore we see that there is a delicate but big difference in the two types of estimates:

$$\text{“}\epsilon T_f(r, L)||_{E(\epsilon)}\text{”},$$
$$\text{“}\delta \log r + O(\log T_f(r, L(\bar{D})))||_{E(\delta)}\text{”}.$$

We go back to the classical case of the projective space and hyperplanes.

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[1]Siu, Y.-T. and Yeung, S.-K., Defects for ample divisors of Abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees, Amer. J. Math. **119** (1997), 1139-1172.

[2]Siu, Y.-T. and Yeung, S.-K., Addendum to “Defects for ample divisors of Abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees, American Journal of Mathematics **119** (1997), 1139-1172”, ArXiv:math.CV/0208059v1 7 Aug 2002.

Theorem 1.6. (Cartan '33; Nochka '83; cf., e.g., [8]) *Let $H \to \mathbf{P}^N(\mathbf{C})$ be the hyperplane bundle. Let $f : \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$ be an entire holomorphic curve such that the smallest linear subspace containing $f(\mathbf{C})$ is $\mathbf{P}^n(\mathbf{C})$. Let $H_j, 1 \leqq j \leqq q$ be hyperplanes of $\mathbf{P}^N(\mathbf{C})$ in general position. Then we have*

$$(q - 2N + n - 1)T_f(r, L) \leqq \sum_{j=1}^{q} N_n(r, f^*H_j) + S_f(r, H). \tag{1.7}$$

Conjecture 1. Let the notation be as above. Assume that $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is algebraically nondegenerate. Then

$$(q - 2N + n - 1)T_f(r, L) \leqq \sum_{j=1}^{q} N_1(r, f^*H_j) + \epsilon T_f(r, H)||_{E(\epsilon)}.$$

2 Jet image of entire holomorphic curves in Abelian and semi-Abelian varieties.

This is a work joint with J. Winkelmann.

Let $f : \mathbf{C} \to A$ be an entire holomorphic into a semi-Abelian A. Let $X_k(f)$ be the (algebraic) Zariski image of $J_k(f) : \mathbf{C} \to J_k(A)$. Let $I_k : J_k(A) \to \mathbf{C}^{nk}$ be the jet projection.

First, note that $X_0(f)$ is a translation of a semi-Abelian subgroup (log-Bloch-Ochiai's Theorem ([4], [5]).

We have the following structure theorem.

Theorem 2.1. ([11]) *Assume that f is algebraically nondegenerate. If $\rho_f < \infty$, or if A is a simple Abelian variety, then there is a subvariety $W_k \subset \mathbf{C}^{nk}$ such that*

$$X_k(f) = A \times W_k. \tag{2.2}$$

In general, the product structure (2.2) does not hold:

Proposition 2.3. *There exist a 3-dimensional Abelian variety A and a holomorphic map $f : \mathbf{C} \to A$ with the following properties:*

(i) *f is algebraically nondegenerate.*

(ii) *$X_1(f)$ is* not *a direct product inside $J_1(A)$; i.e., there exists no subvariety $W_1 \subset \mathbf{C}^3$ such that $X = A \times W_1$.*

Remark. (i) In [7] Proposition 1.8 (ii) (for the semi-Abelian case), and in [15] (for the Abelian case) it was claimed that product structure (2.2) holds. But it was not correct.

(ii) Note that the proofs of Lang's conjecture in [15] and [7], which claims the algebraic degeneracy of holomorphic curves into an Abelian or semi-Abelian variety omitting a divisor, is O.K.!

3 Example of Kobayashi hyperbolic projective hypersurface with arithmetic finiteness property.

Lang's conjecture ('74) claims that an algebraic variety V defined over a number field k carries only finitely many k-rational points if $V_{\mathbf{C}}$ with some $k \hookrightarrow \mathbf{C}$ is Kobayashi hyperbolic.

Kobayashi's conjecture ('70) says that a generic projective hypersurface of $\mathbf{P}^n(\mathbf{C})$ of high degree ($\geq 2n+1$) should be Kobayashi hyperbolic.

In view of these two conjectures, it is interesting to know if there exists such a hyperbolic projective hypersurface X defined over rationals that the set $X(k)$ of k-rational points of X is always finite for an arbitrarily given number field k; we refer this property as the ***arithmetic finiteness property***.

By examples of Masuda-Noguchi [2] there are Kobayashi hyperbolic projective hypersurface with finitely many S-unit points over an arbitrary number field (cf. [6]).

In deed we will show that Shirosaki's construction of Kobayashi hyperbolic projective hypersurfaces ([13]) provides such an example.

We take co-prime positive integers $d, e \in \mathbf{N}$ such that

$$d > 2e + 8, \tag{3.1}$$

and set

$$P(w_0, w_1) = w_0^d + w_1^d + w_0^e w_1^{d-e}. \tag{3.2}$$

Following to Shirosaki [13], we define inductively

$$\begin{aligned} P_0(w_0, w_1) &= P(w_0, w_1), \\ P_n(w_0, w_1, \ldots, w_n) &= P_{n-1}(P(w_0, w_1), \ldots, P(w_{n-1}, w_n)), \quad n \geq 2. \end{aligned}$$

Then, P_n is homogeneous and of degree d^n. Set

$$X = \{P_n(w_0, w_1, \ldots, w_n) = 0\} \subset \mathbf{P}^n_{\mathbf{Q}}. \tag{3.3}$$

By Shirosaki $X_{\mathbf{C}}$ is hyperbolic for $e \geqq 2$.

Theorem 3.4. ([8]) *Assume that $e \geqq 2$. Then X defined by (3.3) satisfies the arithmetic finiteness property.*

Remark **3.5.** Note that P_n decomposes to d components of homogeneous polynomials $P_{nj}, 1 \leq j \leq d$ of degree d^{n-1}. Thus the varieties $X_{nj} = \{P_{nj} = 0\}$ of degree d^{n-1} is Kobayahsi hyperbolic and carries the arithmetic finiteness property.

4 Entire mappings of order < 2.

Entire holomorphic or meromorphic mappings of order $\rho_f < 2$ into a compact complex manifold M seem to have a special role.

Let $n = \dim M$. We denote by $S^l\Omega^r(M)$ the symmetric l-power of the sheaf of holomorphic r-forms on M; in particular, $K^l_M = S^l\Omega^n_M$.

We say that a meromorphic mapping $f : \mathbf{C}^n \to M$ is *differentially nondegenerate* if the rank of the differential df of f has the maximal rank n at some point of M where f is holomorphic.

Theorem 4.1. (Kodaira '68) *Let* $\dim M = 2$. *Assume* $b_1(M) = 0$ *and* $h^0(M, K^l_M) = 0$ *for all* $l > 0$. *Then* M *is rationl.*

Theorem 4.2. (Kodaira '71) *Let* M *be a compactification of* $\mathbf{C}^2$. *Then* M *is rationl.*

He deduced $h^0(M, K^l_M) = 0$ from the imbedding $\mathbf{C}^2 \hookrightarrow M$.

Theorem 4.3. ([9]) *Assume that there is a differentially nondegenerate meromorphic mapping* $f : \mathbf{C}^n \to M (n = \dim M)$ *with* $\rho_f < 2$. *Then for arbitrary* $l_k \geqq 0, \sum_{k=1}^n l_k > 0$,

$$h^0(M, S^{l_1}\Omega^1(M) \otimes \cdots \otimes S^{l_n}\Omega^n(M)) = 0.$$

Theorem 4.4. ([9]) *Let* M *be a Kähler surface. Assume that there is a differentially nondegenerate meromorphic mapping* $f : \mathbf{C}^2 \to M$ *with* $\rho_f < 2$. *Then* M *is rational.*

Note that $\rho_f \geq 2$ for tori.

Conjecture 2. If M is compact and Kähler, and if M admits a differentially nondegenerate meromorphic mapping $f : \mathbf{C}^n \to M$ with $\rho_f < 2$. Then M is rational.

5 Lemma on differential for pseudo-holomorphic curves.

Let w be the affine coordinate of $\mathbf{C}$. Then dw/w is a logarithmic 1-froms on $\mathbf{P}^1(\mathbf{C})$ with poles at 0 and ∞.

Let $f(z)$ be a meromorphic function. Then

$$f^* \frac{dw}{w} = \frac{f'}{f} dz.$$

Nevanlinna's lemma on logarithmic derivative says

Lemma 5.1. (Nevanlinna '25)

$$m\left(r, \frac{f'}{f}\right) = \int_{|z|=r} \log^+ \left| \frac{f'(z)}{f(z)} \right| \frac{d\theta}{2\pi} = S_f(r).$$

Let M be a projective algebraic variety with an ample line bundle $L \to M$ and let ω be a *holomorphic* 1-form on M. Let $f : \mathbf{C} \to M$ be an entire holomorphic curve. Set

$$f^*\omega = \xi(z)dz.$$

Lemma 5.2. (Ochiai [14])

$$m(r, \xi) = S_f(r, L).$$

This had had to be assumed in Bloch [1] and was proved by Ochiai [14] for the analytic part of the proof of Bloch-Ochiai's Theorem: At p. 22 of [1] A. Bloch wrote

> "La démonstration du troisième de ces lemmes, relatif à la croissance d'une système de fonctions abéliennes dont les arguments sont fonctions d'une variable, présente encore une lacune qui n'est qu'à moitié comblée".

At p. 36 of [1] he also wrote

> "Toutefois, pour ne pas introduire dans ce qui suit une restriction ne tenant pas à la nature des choses, mais seulment au procédé de démonstration, nous l'admettrons comme vraie dans tous les cas".

It was extended for Deligne's logarithmic 1-forms by Noguchi [4].

We prove a similar lemma for entire *pseudo-holomorphic* curves into a compact *almost* complex manifold.

Let M be a compact almost complex manifold with almost complex structure J and hermitian metric $h = (h_{i\bar{j}})$. Let $f : \mathbf{C} \to M$ be a pseudo-holomorphic curve (J-holomorphic mapping). We define the order function of f with respect to h by

$$f^*h = s(z)\frac{1}{\pi}dz \cdot d\bar{z},$$
$$T_f(r, h) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} s(z)\frac{i}{2\pi}dz \wedge d\bar{z}.$$

Lemma 5.3. *Let the notation be as above. Let η be an arbitrary continuous 1-form on M. Set $f^*\eta = \zeta_1(z)dz + \zeta_2(z)d\bar{z}$. Then*

$$m(r, \zeta_i) \leq \delta \log r + O(\log T_f(r, h))||_{E(\delta)}, \qquad i = 1, 2.$$

I hope that there may be some application of Lemma 5.3 to the theory of pseudo-holomorphic curves.

The proof is as follows. The next is called Borel's lemma.

Lemma 5.4. (cf. [10]) *Let $\phi(r)$ be a continuous, increasing function on $\mathbf{R}^+$ such that $\phi(r_0) > 0$ for some $r_0 \in \mathbf{R}^+$. Then for $\delta > 0$ we have*

$$\frac{d}{dr}\phi(r) < \phi(r)^{1+\delta}||_{E(\delta)}.$$

Proof. Without loss of generality, we may assume that η is a $(1,0)$-form. Set $f^*\eta = \zeta(z)dz$.

Since the norm $\|\eta\|_h$ of η with respect to h is bounded, there is a constant $C > 0$ such that

$$|\zeta(z)|^2 \leq Cs(z).$$

For simplicity, let $C = 1$, and let $0 < \delta < 1$. Then, making use of Lemma 5.4, we have

$$\begin{aligned}
\int_{|z|=r} \log^+ |\zeta| \frac{d\theta}{2\pi} &\leqq \frac{1}{2} \int_{|z|=r} \log(1+|\zeta|^2) \frac{d\theta}{2\pi} \leqq \frac{1}{2} \log\left(1 + \int_{|z|=r} s(z) \frac{d\theta}{2\pi}\right) \\
&\leqq \frac{1}{2} \log\left(1 + \frac{1}{2r}\frac{d}{dr} \int_{\Delta(r)} s(z) \frac{1}{\pi} r dr \wedge d\theta\right) \\
&\leqq \frac{1}{2} \log\left(1 + \frac{1}{2r}\left(\int_{\Delta(r)} s(z) \frac{i}{2\pi} dz \wedge d\bar{z}\right)^{1+\delta}\right) ||_{E_1(\delta)} \\
&\leqq \frac{1}{2} \log\left(1 + \frac{r^\delta}{2}\left(\frac{d}{dr}\int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*h\right)^{1+\delta}\right) ||_{E_1(\delta)} \\
&\leqq \frac{1}{2} \log\left(1 + \frac{r^\delta}{2}\left(\int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*h\right)^{(1+\delta)^2}\right) ||_{E_2(\delta)} \\
&\leqq \frac{1}{2} \log\left(1 + \frac{r^\delta}{2} \left(T_f(r,h)\right)^{(1+\delta)^2}\right) ||_{E_2(\delta)} \\
&\leqq 2\log^+ T_f(r,h) + \delta \log r + O(1) ||_{E_2(\delta)} \\
&= S_f(r,h).
\end{aligned}$$

Q.E.D.

6 An extension problem

Let Δ^n be the n-dimensional unit polydisk. Let M be a compact complex manifold with the univerrsal covering which is a bounded polynomially convex domain $\tilde{M}$ of $\mathbf{C}^m$.

Theorem 6.1. (T. Nishino [3] for $n = m = 1$; Masakazu Suzuki [16], [17] for $n, m \geq 1$) *Let the notation be as above. Assume that an arbitrary decktransformation of $\tilde{M} \to M$ can be holomorphically extendable to a neighborhood of the closure $\overline{\tilde{M}}$ in $\mathbf{C}^m$. Let $E \subset \Delta^n$ be a closed polar subset with respect to a plurisubharmonic function. Then every holomorphic mapping $f : \Delta \setminus E \to M$ can be holomorphically extendable to $\hat{f} : \Delta^n \to M$.*

Problem. Does the extension of f over polar E hold for a compact Kobayashi hyperbolic manifold M?

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