Ansov Flows and 3D Contact Structures

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§0 Introduction : Foliations and Contact structures

In the present article, two features concerning the title are treated. The first half is to review how foliation, or more specifically, Anosov foliations appeared in 3-dimensional contact topology and 4-dimensional symplectic geometry. Globally convex symplectic 4-manifolds are constructed from Anosov flows on 3-manifolds which exhibits some differece between strict pseudo convexity in complex geometry and symplectic convexity.

The second is to propose further possibilities of applying the relations between Anosov foliations and contact structures to 3-dimensional contact topology.

Foliations and contact structures are very far away as geometric structures, and have been playing totally different roles in dynamical systems. In higher dimensions, as hyperplane distributions there exist steps of differences in integrability between them. However, in the 3-dimensional case, they are next to each other and from purely geometric points of view they have so many similarities which came to be recognized recently.

In mid 90's, Eliashberg-Thurston and the author realized that a pair of contact structures is naturaly associated with an Anosov flow on a 3-manifold. This phenomenon was generarized to the notion of projectively Anosov flows. In these cases contact structures approximate foliations.

Eliashberg-Thurston extends this idea ultimately and proved that every foliation but one is approximated both by positive and negative contact structures.

In this paper do not go into this general situation. Rather we still stick to a very special case given by (algebraic) Anosov flows.

§1 Anosov flows and non-Stein symplectic 4-manifolds

The story goes back to the following question by Calabi;

Question 1.1 Does there exist a (globally) convex symplectic manifold with disconnected ends?

Let us begin with explaining this question. In the case of complex manifolds with pseudo convex boundary (*i.e.*, a Grauert domain) the boundary is connected. In the world of symplectic geometry, we have corresponding notions. Let us think of Stein manifolds. A Stein manifold is a complex manifold which admits a proper strictly pruli-subharmonic function ϕ bounded below. Then, thanks to the strict pruli-subharmonicity, $g(\cdot,\cdot) = -dJ^*d\phi(\cdot, J(\cdot))$ gives a Riemannian metric and $\omega(\cdot,\cdot) = -dJ^*d\phi(\cdot,\cdot)$ defines a symplectic strucutre on the manifold. Let Z be the gradient vector field of ϕ *w.r.t.* this Riemannian metric.

1.2 Then we have the following properties.

1) $\mathcal{L}_Z\omega = \omega$.

2) Putting $\lambda = \iota_Z\omega$, then $d\lambda = \omega$.
3) On the preimage of a regular value of ϕ, Z is outward normal.

Definition 1.3 1) An open symplectic manifold (Ω, ω) which admits a vector field Z satisfying 1) is called a globally convex symplectic manifold (Eliashberg-Gromov [EG]).
2) If a compact symplectic manifold (W, ω) with boundary admits such a vector field Z only around the boundary and Z is outward normal on the boundary, then W is said to have a contact type (symplectically convex) boundary (Weinstein [W]). Then, $\lambda|_{\partial W}$ defines a positive contact structure on ∂W.

Conversely, if a contact manifold (with a contact 1-form) bounds such a symplectic manifold, the contact manifold is said to be strongly fillable and the symplectic manifold is called a strong filling of the contact manifold. The notion of fillability is, in dimension 3, weakened much more ([G]) and now plays very impotant rolles in symplectic and contact topology.

From the Morse theory of ϕ and L, it is easy to see that a Stein manifold has a homotopy type of at most half of its dimension. Especially, the boundary is connected and it is still true for general Grauert domains. Therefor it is natural to ask if there exists some difference between complex geometry and symplectic geometry in the topology of such convex manifolds. In the case of compact manifolds, we also knew that there exists a lot of examples of non-Kähler symplectic manifolds.

A counter example was first constructed by D. McDuff [Mc]. Her construction looked unnecessarily complicated. So that her construction was modified, simplified in terms of the Lie algebra $sl(2;\mathbb{R})$ by E. Ghys and by the author, and related to Anosov flows, *e.g.*, the geodesic flow of hyperbolic surfaces [Mi1]. Constructions are made on simple 4-manifolds, *i.e.*, $W = [-1, 1] \times M^3$. For this porpose, we should know two contact structures on a single manifold M with opposite orientations, because as the boundary of W, two boundary components are in different orientaions.

Example 1.4 Let us review two standard families of contact structures, one is related to the classical mechanics and the other is to the quantum mechanics. Take a Riemannian surface Σ and take the geodesic flow ϕ_t on the unit tangent bundle $S^1T\Sigma$. Let $\xi_0 = \dot{\phi_t}^{\perp}$ be the perpendicular distribution to the flow. ξ_0 is tangent to the fibres. This is the **Liouville contact structure**. For surfaces of constant curvature, the Liouville contact structure is described in terms of the 3-dimensional Lie groups. For $\Sigma = S^2$, T^2, or hyperbolic surfaces $\Sigma_{g\geq 2}$, we take $SO(3)$, $Euc(R^2)$, or $PSL(2;\mathbb{R})$ respectively. In each case, the manifold $S^1T\Sigma$ is taken as the quotient of these Lie groups by a co-compact lattice, and the contact structure xi_0 is defined as a left invariant plane field.

In non-flat case, on the same 3-manifolds, we can associate contact structures of different nature. If an S^1-bundle admits an S^1-connection with nowhere vanishing curvature, then, the connection form defines a conntact structure ξ_1 *i.e.*, the horizontal distribution is a contact structure. ξ_1 is transverse to the fibres. These structure are offten called **prequantum bundles**. Again, these structures are also described as a left invariant plane field in the case of unit tangent bundle $S^1T\Sigma$ of $\Sigma = S^2$ or hyperbolic surfaces $\Sigma_{g\geq 2}$.

For S^2, two examples are more or less the same, *i.e.*, they coinsides up to isotopy. However, for $\Sigma_{g\geq 2}$ the classical mechanical ones and the quantum ones have different orientations. These are the contact structures which were used in McDuff's construction.

McDuff's construction was simplified in the following way. Take two contact forms $\alpha_\pm$ on $S^1T\Sigma = \Gamma \setminus PSL(2;\mathbb{R})$ as left invariant 1-forms $\in psl(2;\mathbb{R})^*$, which define the classical mechanical contact structure (*i.e.*, the Liouville structure) and the quantum mechanical one as described above. Then connect them in $psl(2;\mathbb{R})^*$ by the segment $\alpha_t = \frac{1}{2}\{(1-t)\alpha_- + (1+t)\alpha_+\}$. Then regard $\lambda = \{\alpha_t\}$ as a 1-form on $W = [-1,+1] \times M$ or on $\Omega = \mathbb{R} \times M$, and let $\omega = d\lambda$. Then the vector field Z which is determined by $\iota_Z\omega = \lambda$ gives rise to a global convex structure on (Ω, ω). Evidently, Ω has the same homotopy type as that of M.

This construction is possible in $psl(2;\mathbb{R})^*$, because the quadratic form ν on $psl(2;\mathbb{R})^*$ defined by $\nu(\alpha) = \alpha \wedge d\alpha \in \bigwedge^3 psl(2;\mathbb{R})^* \cong \mathbb{R}$ is not semi-definite, *i.e.*, ν takes both positive and negative values. In other words, there exist both positive and negative left invariant contact structures. This happens, among unimodular 3-dimensional Lie algebra, also for *solv* and never for others. Here *Solv* is the 3-dimensional Lie group which is associated to the suspension of a hyperbolic automorhism of T^2 and *solv* is its Lie algebra. Aaccordingly, the above construction holds for *Solv*.

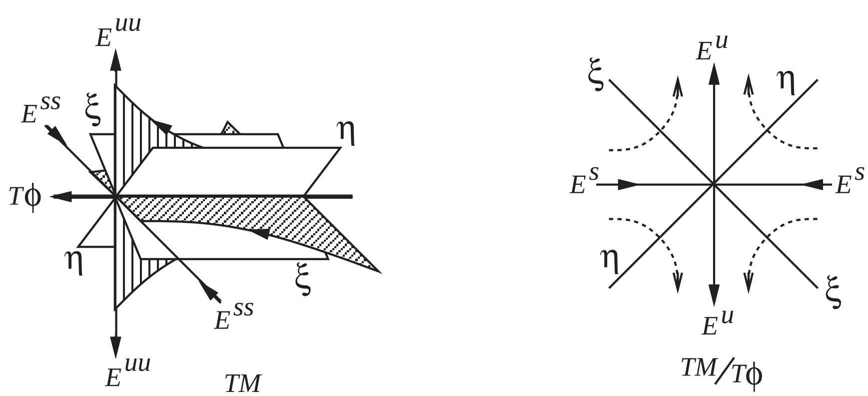
These are also only Lie algebras which admits algebraic Anosov flows (*i.e.*, left invariant Anosov flows) on their quotient by cocompact lattice. Once this fact is recognized, it is natural to realize the simple geometric idea in the Proposition below which is one of the key observations.

Definition 1.5 A non singular smooth flow ϕ_t on M^3 is an **Anosov flow** if there exists a ϕ_t-invariant C^0-decomposition of the tangent bundle $TM = T\phi \oplus E^{uu} \oplus E^{ss}$ into three line bundles which satisfies the following for some (and eventually any) Riemannian metric: For some $c > 0$,

$$\forall v \in E^{uu}, \quad \forall t > 0, \qquad \|(\phi_t)_* v\| \geq \exp(ct)\|v\|,$$
$$\forall v \in E^{ss}, \quad \forall t < 0, \qquad \|(\phi_t)_* v\| \geq \exp(ct)\|v\|.$$

$E^s = T\phi \oplus E^{ss}$ と $E^u = T\phi \oplus E^{uu}$ defines (un)stable Ansov foliations $\mathcal{F}^u$ and $\mathcal{F}^s$.

Figure 1.6



Anosov Flow $\Longrightarrow$ Bi-Contact Structure

Proposition 1.7 ([Mi1], [ET]) Let ϕ_t be an Anosov flow on a 3-manifold M, and $\mathcal{F}^u$ and $\mathcal{F}^s$ be its (un)stable foliations. As in the Figure 1.6, we take two plane field ξ and η which are tangent to the flow and make an angle of $\pi/4$ between $\mathcal{F}^u$ and $\mathcal{F}^s$. Then ξ and η are positive and negative contact structures respectively.

We call a taransverse pair of contact structures with opposite orientations a **bi-contact structure**.

In general $\mathcal{F}^u$ and $\mathcal{F}^s$ have only the smoothness of class C^1. Therefore, if ξ and η exactly keep an angle of $\pi/4$ between $\mathcal{F}^u$ and $\mathcal{F}^s$, then they have the same smoothness. To obtain smooth ones, it is enough to take approximations by smooth plane fields tangent to ϕ in the C^1-topology. Then the construction of the convex symplectic structures out of algebraic Anosov flows works for any Anosov flows.

Theorem 1.8 ([Mi1]) Let ϕ be an Anosov flow on a closed 3-manifold M. Then $\mathbb{R} \times M$ admits a globally convex symplectic structure, while it has a homotopy type of dimension 3.

§2 Asymptotic Linking Pairing

In the rest of this paper we propose an approach to 3-dimensional contact topology from foliated cohomology through asymptotic linking pairing.

§2.1 Asymptotic Linking and Foliations · Contact Structures

Definition 2.9 Let B^2 be the space of exact 2-forms on a closed oriented 3-mailfold M. Then,

$$lk(d\alpha, d\beta) = \int_M \alpha \wedge d\beta \,, \qquad d\alpha, d\beta \in B^2$$

defines a symmetric non-degenerate bilinear pairing on B^2, which is called the **asymptotic linking pairing**.

Trying to define the signature ($\infty - \infty$ even if possible) of the linking pairing lk is one of the motivation for what follows. Thus it seduces us to look for large positive (or negative) definite subspaces or null subspaces.

Proposition 2.10 Let $\xi = \ker \alpha$ and $\eta = \ker \beta$ be positive and negative contact structures, and $\mathcal{F} = \ker \omega$ be a foliation on M. Then, as a subspace of (B^2, lk)

1) $P_\xi = \{d(f\alpha); f \in C^\infty(M)\} = \{d\alpha'; \alpha'|_\xi = 0\}$ is positive definite.
2) $Q_\eta = \{d(f\beta); f \in C^\infty(M)\} = \{d\beta'; \beta'|_\eta = 0\}$ is negative definite.
3) $N_{\mathcal{F}} = \{d(f\omega); f \in C^\infty(M)\} = \{d\omega'; \omega'|_{T\mathcal{F}} = 0\}$ is a null subspace.

The follwoing remark suggests to investigate $N_{\mathcal{F}}^\perp / N_{\mathcal{F}}$.

Remark 2.11 Let λ be a symmetric non-degenerate bilinear pairing on a finite dimensional vector space V and N be a null subspace. Then λ is naturally induced on $N^\perp/N$ wchich is denoted by λ_N and has the same signature as λ, *i.e.*,

$$\mathrm{sgn}\lambda_N = \mathrm{sgn}\lambda \,.$$

§2.2 Contact Invariant via Linking Pairing

Definition 2.12 For a contact structure ξ, a contact embedding from $(T^2 \times [0, n\pi], \xi_0 = \ker[\cos z dx - \sin z dy])$ is called a **torsion embedding** of length n and the **torsion invariant** of (M, ξ) is defined to be $Tor(M, \xi) = \sup\{n; (M, \xi)$ admits a torsion embedding of length n $\}$.

Remark 2.13 Eliashberg's classifying theorem [E] implies that if ξ is over twisted then $Tor(M, |xi) = \infty$.

Proposition 2.14 If a positive contact structure ξ admits a torsion embedding ϕ of length n, then there exists a subspace P_ϕ of B^2 with the following properties.

1) $\dim P_\phi = n$, 2) P_ϕ is positive definite, 3) $P_\phi \perp P_\xi$ *w.r.t. lk.*

Definition 2.15 Hence, we define the **analytic torsion invariant** $ATor(\xi)$ to be $ATor(\xi) = \sup\{\dim P; P$ is positive definite, $P \perp P_\xi\}$.

Under this definition, the above Proposition is nothing but " $Tor(\xi) \leq ATor(\xi)$ " . If we can show the finiteness of the torsion or the analytic torsion invariant for some contact structure, then it implies its tightness.

§3 Foliatted Cohomology and Linking Pairing

§3.1 Foliated Cohomology

Let us review what is the foliated cohomology. Let $\mathcal{F}$ be a smooth codimension 1 foliation on a manifold M which is transversely oriented and is defined by a 1-form ω, *i.e.*, $T\mathcal{F} = \ker \omega$.

Definition 3.16 First Definition of Foliated de Rham Complex $\Omega^*(\mathcal{F})$

Foliated(leafwise) de Rham complex is a de Rham theory along leaves with transverse smoothness. Thus it is defined as follows.

$\Omega^k(\mathcal{F}) = C^\infty(M; \bigwedge^k T^*\mathcal{F}) = \{$ smooth family of k-forms on each leaves $\}$
$d_\mathcal{F} : \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F})$: the exterior differential along the leaves .

Then we obtain the foliated (leafwise) de Rham complex $(\Omega^*(\mathcal{F}), d_\mathcal{F})$ and its cohomology theory $H^*(\mathcal{F})$ which is called the **foliated** (or leafwise cohomology).

Second Definition of Foliated de Rham Complex $\Omega^*(\mathcal{F})$ (equivalent to the first)

$\mathcal{I}^k(\mathcal{F}) = \{$ k-forms on M which vanish on each leaves $\}$
$\mathcal{I}^*(\mathcal{F}) = \bigoplus_k \mathcal{I}^k(\mathcal{F})$(= the differential ideal generated by ω in $\Omega^*(M)$)

In other words, let $r = r_\mathcal{F}$ be the dual of the natural inclusion $\iota = \iota_\mathcal{F} : T\mathcal{F} \rightarrowtail TM$. Then, we get naturally the following short exact sequence.

$$0 \longrightarrow \mathcal{I}^*(\mathcal{F}) \xrightarrow{i} \Omega^*(M) \xrightarrow{r} \Omega^*(\mathcal{F}) \longrightarrow 0$$

i.e., $\Omega^*(\mathcal{F})$ is defioned as the quotient $\Omega^*(M)/\mathcal{I}^*(\mathcal{F})$.

From the second definition, we get the following long exact sequence:

$$\cdots \longrightarrow H^1(\mathcal{I}^*(\mathcal{F})) \xrightarrow{i} H^1(M) \xrightarrow{r} H^1(\mathcal{F}) \xrightarrow{d} H^2(\mathcal{I}^*(\mathcal{F})) \longrightarrow \cdots$$

This gives rise to the following construction of the characteristic pairing CJ, which generalizes a framework of defining exotic characteristic classes ([AD]).

Definition 3.17 The exterior product $\Omega^1(\mathcal{F}) \otimes \mathcal{I}^2(\mathcal{F}) \xrightarrow{\wedge} \Omega^3(M)$ which is well-defined and the connecting homomorphism in the long exact sequence are composed to define the characteristic pairing $CJ : H^1(\mathcal{F}) \otimes H^1(\mathcal{F}) \longrightarrow \mathbb{R}$ as

$$H^1(\mathcal{F}) \otimes H^1(\mathcal{F}) \xrightarrow{Id \otimes d} H^1(\mathcal{F}) \otimes H^2(\mathcal{I}^*(\mathcal{F})) \xrightarrow{\wedge} H^3(M) \cong \mathbb{R},$$

which is a symmetric and bilinear.

Example 3.18 (**Reeb class, Godbillon-Vey class**) The Frobenius integrability implies that there exists a 1-form η satisfying $d\omega = \omega \wedge \eta$. η restricts to leaves to define a foliated cohomology class $[\eta](= [r(\eta)]) \in H^1(\mathcal{F})$ which is called the **Reeb class**, which measures the distorsion of the transverse distance by holonomy transformation. $gv(\mathcal{F}) = [\eta \wedge d\eta] \in H^3(M)$ is the **Godbillon-Vey invariant**, which is also formulated in our context as $CJ([\eta], [\eta])$.

§3.2 Foliated Cohomology and Asymptotic Linking

Now let us formulate the relation between the foliated cohomology and the asymptotic linking pairing.

Theorem 3.19 There exists a natural **surjective** homomorphism

$$\Phi : H^1(\mathcal{F}) \twoheadrightarrow N(\mathcal{F})^{\perp_{lk}} / N(\mathcal{F})$$

which is given by the composition of a lift $Z^1(\mathcal{F}) \to \Omega^1(M)$ and the exterior differential on the manifold and satisfies the following.

$$CJ = \Phi^* lk \quad i.e., \quad [\alpha], [\beta] \in H^1(\mathcal{F}),\ CJ([\alpha], [\beta]) = lk(\Phi([\alpha]), \Phi([\beta])) = lk(d\alpha, d\beta)\,.$$

§3.3 1st Foliated Cohomology of Algebraic Anosov Foliations

A priori, we can not expect foliated cohomology to be finite dimensional nor to be separable (*i.e.*, Hausdorff), because leafwise exterior differential $d_{\mathcal{F}}$ has only a partial ellipticity. However, sometimes it happens and the 1st cohomology of Algebraic Anosov foliations is the case.

Theorem 3.20 ([MM]) The 1st cohomology of algebraic Anosov foliation is spanned by $H^1(M)$ and the Reeb class $[\eta]$. More precisely,

1) The case of $\widetilde{PSL(2;\mathbb{R})}$: $H^1(\mathcal{F}^u) \cong H^1(\mathcal{F}^s) \cong H^1(M) \oplus \mathbb{R}[\eta]$,

2) The case of $Solv$: $H^1(\mathcal{F}^u) \cong H^1(\mathcal{F}^s) \cong H^1(M) = \mathbb{R}[\eta]$.

Let us explain very briefly how it is computed in the case of $\widetilde{Solv,}$ while we still do not know the real reason why it is thus computed in the case of $\widetilde{PSL}(2;\mathbb{R})$.

It is easy to see that the Mayer-Vietoris arguments also works i n the foliated cohomology. In the $Solv$-case, the foliation is a suspension of an irrational linear foliation on T^2 over S^1. Therefore by the Wang sequence, the problem reduces to compute $H^1(\mathcal{F}_a)$ for the irrational linear foliation on T^2 with slope a. This is again, a foliated bundle over S^1 so that everything reduces to the following well-known problem.

Q : For a given smooth function on $S^1 = \mathbb{R}/\mathbb{Z}$, does there exist a smooth function $g(x) \in C^\infty(S^1)$ which satisfies the coboundary equation $f(x) = g(x+a) - g(x)$?

The answer depends on whether the slope a is a Liouville number or no. Anyway the integral $\int_{S^1} f(x)dx$ is an obvious obstruction to the solution, but if a is not of Liouville, there is no further obstruction (*i.e.*, the problem of 'small denominator') in finding a smooth solution g by Fourier expansion. In our case, the slope a is the tangent of an eigen direction of a hyperbolic automorphism of T^2 (*i.e.*, an element of $SL(2;\mathbb{Z})$), that means it is an algebraic number so that it is not of Liouville. Hence we obtain $H^1(\mathcal{F}^u) \cong H^1(\mathcal{F}_a) \cong \mathbb{R}$.

§4 Concluding Remarks

The following is a direct corollary to Theorem 3.19 and Theorem 3.20.

Theorem 4.21 $N_\mathcal{F}$ satisfies the following properties both for $\mathcal{F} = \mathcal{F}^u$ and $\mathcal{F}^s$ of Algebraic Anosov flows.

1) The case of $\widetilde{PSL}(2;\mathbb{R})$: $N_\mathcal{F}^\perp / N_\mathcal{F} = \mathbb{R}[d\eta]$, $lk_{N_\mathcal{F}}$ is not trivial(Godbillon-Vey).

2) The case of $Solv$: $N_\mathcal{F}^\perp / N_\mathcal{F} = 0$.

In these cases, $N_\mathcal{F}$ is a maximal nullsubspace, and in 1) we have $N_\mathcal{F}^\perp = N_\mathcal{F} \oplus \mathbb{R}[d\eta]$, and in 2) $N_\mathcal{F}^\perp = N_\mathcal{F}$.

Now let us consider the $Solv$ case, for simplicity, and thin about what the following computation implies for the associated bi-contact structure (ξ, η). (Here, η denotes the negative contact structure. Please forgive the author for bad use of η. Traditions did not take care of ... .)

$$(P_\xi \oplus Q_\eta)^\perp = (N_{\mathcal{F}^u} \oplus N_{\mathcal{F}^s})^\perp = N_{\mathcal{F}^u}^\perp \cap N_{\mathcal{F}^s}^\perp = N_{\mathcal{F}^u} \cap N_{\mathcal{F}^s} = 0\,.$$

If we ook at this result without remembering topology, infinite dimensionality and non-definiteness, it seems to suggested that

$$ATor(\xi) = ATor(\eta) = 0$$

because it seems very dificult to find a room to put P_ϕ into B^2. IF the argument could be true, then the tightness of ξ and η follows immediately, as well as, $Tor = 0$ for the standard contact structure on S^3.

Problem Find an analytic framework which justifies this arugument.

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