# LEMMA ON LOGARITHMIC DERIVATIVE AND TRUNCATED COUNTING FUNCTION

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## §0. Introduction.

It is well-known that Roth's theorem and Nevanlinna's second main theorem are strikingly similar. It is therefore very important to construct the geometry unifying Diophantos/Nevanlinna theories, which fully explains this similarity. The difficulty in the attempt of constructing the unifying geometry is that Nevanlinna theory is based on the calculus for functions on the complex plane while Diophantine approximation is the theory on the ring of integers and therefore the concept of the derivative is intrinsic in Nevanlinna theory while it is not defined in Diophantine approximation. Related to this difference between these theories is the absence of the ramification counting function (counting zeros of the "derivative") in Roth's theorem. It is then natural for us to expect that the geometric framework which somehow "recovers" the Diophantine analogue of the ramification counting function (which is "lost" in Roth's theorem) occupies an essential part of the unifying geometry under question. The purpose of my lecture is to introduce a new Diophantos/Nevanlinna analogue which emerges from the attempt of constructing such geometric framework. For this purpose we begin with the basic nature which is common to both Diophantos/Nevanlinna theories. Namely both theories consists of the inequalities which measure the difference of the two states, i.e., the "exact states" and the "approximation states":

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## §1. Approximation States v.s. Exact States.

The Nevanlinna-Cartan second main theorem is the following:

**Theorem 1.** *Let $D = \{D_1, \ldots, D_q\}$ be a collection of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then there exists a finite number of hyperplanes $Z_D$ depending only on $D$ such that the following statement holds: Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve such that $f(\mathbb{C}) \not\subset D$. Assume that the image of $f$ is not contained in $Z_D$. Let $W(f) = f^{(1)} \wedge f^{(2)} \wedge \cdots \wedge f^{(n)}$ be the Wronskian of $f$ defined in terms of affine coordinates of $\mathbb{P}^n(\mathbb{C})$. Then*

$$(*) \qquad \sum_{i=1}^{q} m_{f,D_i}(r) + T_{f,K_{\mathbb{P}^n(\mathbb{C})}}(r) \leq S_f(r)/\!/ \,.$$

*Under the stronger assumption that $f$ is linearly non-degenerate, then we have a stronger inequality*

$$(1) \qquad \sum_{i=1}^{q} m_{f,D_i}(r) + T_{f,K_{\mathbb{P}^n(\mathbb{C})}}(r) + N_{W(f),0}(r) \leq S_f(r)/\!/ \,.$$

$(*)$ should be improved into a form involving some ramification counting function. However, such a stronger version (i.e., the second main theorem with a ramification counting function such that the "exceptional subspaces" $Z_D$ is a proper algebraic subset) presently unavailable.

We interpret Theorem 1 by comparing two states: the "exact" and the "approximation" states. Given holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we say that $f$ is in the exact state (resp. the approximation state) if $f(\mathbb{C}) \subset D$ (resp. $f(\mathbb{C}) \not\subset D$) holds.

Cartan's proof of Theorem 1 consists of three steps.

The first step is the characterization of the exact state. Given holomorphic curve $f$ being in the exact state w.r.to any linear divisor if and only if

$$W(f) \equiv 0 \ . \tag{2}$$

In this stage we cannot distinguish the linear divisor $D$ under under question.

The second step is to measure how given $f$ in the approximation state deviates from being in the exact state w.r.to the particular linear divisor $D$. This is done by Nevanlinna's lemma on logarithmic derivative:

$$m_{f,D}(r) \leq m_{W(f),0}(r) \ . \tag{3}$$

The inequality (3) is the approximation counterpart of the characterization (1) of the exact state.

The third step is to incorporate the geometry of the space where the Wronskian $W(f)$ lives (indeed, $W(f)$ takes values in the total space of the anticanonical bundle $K^{-1}_{\mathbb{P}^n(\mathbb{C})} \to \mathbb{P}^n(\mathbb{C})$) into the inequality (3). We then end up with the inequality in Theorem 1.

The Diophantine analogue of Theorem 1 is Roth-Schmidt's SST (subspace theorem). Let $k$ be a number field and $S$ a finite set of places of $k$ including all Archimedean ones.

**Theorem 2.** *Let $D = \{D_1, \ldots, D_q\}$ be a collection of hyperplanes in general position in $\mathbb{P}^n(k)$ and $\varepsilon$ any positive number. Then there exists a finite number of proper linear subspaces $Z$ such that the set of the solutions outside of $Z$ for the Diophantine inequality*

$$m_S(x, D) + \mathrm{ht}_{K_{\mathbb{P}^n}}(x) \ > \ \varepsilon\, \mathrm{ht}_{\mathcal{O}(1)}(x) \tag{4}$$

*for points of $\mathbb{P}^n(k) - D$ is finite.*

Clearly, the "exact state" (resp. "the approximation state") in Theorem 2 is to consider rational points in $D$ (resp. those outside of $D$). In [V, Theorem 6.4.3], Vojta defined the Diophantine analogue $x \mapsto x'$ (defined in $\mathcal{O}^{n+1}_{k,S}$) of the derivative of a holomorphic curve (lifted to $\mathbb{C}^{n+1} - \{0\}$) using the geometry of numbers on adèles. Vojta used Ahlfors' variant of the Lemma on logarithmic derivative as a geometric model for the choice of the length function[1]. The point is that the Diophantine analogue $x \mapsto x'$ of the derivative is a relative notion which makes no sense without the approximation target $D$. In [V, Chapt. 6], Vojta incorporated the role of the Diophantine analogue of the derivative into Schmidt's proof of Theorem 2. However, the 3-steps structure characterizing Nevanlinna-Cartan theory is not so recognizable. It is then natural to try to identify the 3-steps structure in the proof of Theorem 2.

We note that the ramification counting function $N_{W(f),0}(r)$ exists in (1) and no Diophantine counterpart exists in (4). Although Vojta proposed of the Diophantine analogue [V, Theorem 6.4.3] of the derivative, we cannot recover the Diophantine counter part of the ramification counting function from its proof. The next section is an attempt toward recovering it.

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[1] In fact, he used the Diophantine analogue of Ahlfors' variant of the Lemma on logarithmic derivative as the "defining equation" of $x'$.

## §2. Truncated Counting Function and Its Diophantine Analogue.

Throughout this section, we let $\mathbb{P}^n$ and $D$ be as in Theorem 1 or 2. Vojta's definition of the Diophantine analogue $x \mapsto x'$ of the derivative depends essentially on the target $D$. This motivates the following consideration. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve s.t. $f(\mathbb{C}) \not\subset D$. Although counting zeros of $f'$ (which has nothing to do with the target $D$) has no Diophantine analogue, counting those of $f'$ only at $z \in \mathbb{C}$ s.t. $f(z) \in D$ does. Indeed, this is just to associate to each $x$ the set of finite places on which the Zariski closures (over the ring of intergers $\mathcal{O}_k$) of $x$ and $D$ intersect. Let $N^n_{f,D}(r)$ be defined by replacing $\deg_z(f^*D)$ in the usual counting function by $\max\{\deg_z(f^*D) - n, 0\}$ (we note that the difference $N_{f,D}(r) - N^n_{f,D}(r)$ is the usual notion of the level $n$ truncated counting function). Since $D$ is linear and $W(f)$ is defined w.r.to affine coordinates, the condition $\deg_z(f^*D) \geq n$ implies $\deg_z(f^*D) - n \leq \deg_z W(f)$ (where the equality holds "generically"). This implies

$$N^n_{f,D}(r) \leq N_{W(f),0}(r) \ .$$

We can now define the Diophantine analogue of $N^n_{f,D}(r)$ (via Vojta's dictionary). This consists of the association $x \mapsto S^n_x$, $S^n_x$ being the set of all finite places where the Zariski closures of $x$ and $D$ intersect with multiplicity $\geq n$, together with the "counting function" $N^n(x, D)$ which counts the intersection of the Zariski closures of $x$ and $D$ over the places in $S^n_x$ with the multiplicity $\deg_v(x^*D)$ (in the usual counting function) replaced by $\max\{\deg_v(x^*D) - n, 0\}$ $(\forall v \in S^n_x)$.

In contrast to working with a fixed set $S$ of places (as in Theorem 2), we cannot fix the set of finite places if we try to define the the Diophantine analogue of the truncated counting function.

The conclusion of this lecture is the following conjectural analogue of the Lemma on logarithmic derivative in the setting of varying $S = S(x)$ where $S(x) = S_\infty \cup S^n_x$ ($S_\infty$ being the set of all Archimedean places).

**Conjecture 3.** *Let $F_0, \dots, F_q$ be a set of linear forms in $k^{n+1}$ in general position and $\varepsilon$ any positive number. Then there exists a finite set $\mathcal{S}$ of proper linear subspaces of $k^{n+1}$ with the following property. If $x \in k^{n+1}$ is not a vector in the union of the linear subspaces in $\mathcal{S}$, then we can inductively construct a sequence $x^{(1)}, \dots, x^{(n)} \in \mathcal{O}_k^{n+1}$ of vectors with the following properties:*
*(i) $x, x^{(1)}, \dots, x^{(n)}$ are linearly independent:*

$$x \wedge x^{(1)} \wedge \cdots \wedge x^{(n)} \neq 0 \ .$$

*(ii) $\mathrm{ord}_v(x^{(t)} \cdot F_i)$ decreases $1$ as $t$ increases $1$, i.e., if $\mathrm{ord}_v(x \cdot F_i) \geq n$, we have*

$$\mathrm{ord}_v(x^{(t)} \cdot F_i) \geq \mathrm{ord}_v(x \cdot F_i) - t$$

*for $\forall t = 1, 2, \dots, n$.*
*(iii) If we set $x^{\leq p-1} = x \wedge x^{(1)} \wedge \cdots \wedge x^{(p-1)}$ and $F_{i,p} = F_i \wedge F_{n-p+2} \wedge \cdots \wedge F_n$ for $p = 1, \dots, n$, we have the following inequality: after suitably re-ordering the $F$'s for each $v \in S(x)$ (according to the $v$-adic approximation of $x$ to $D$), we have*

$$\sum_{v \in S(x)} \log \frac{||(x^{\leq p-1} \wedge (x^{\leq p-2} \wedge x^{(p)})) \cdot F_{i,p}||_v}{||x^{\leq p-1}||_v ||x^{\leq p-1} \cdot F_{i,p}||_v^{\mathrm{modified}}} < \varepsilon \, \mathrm{ht}(x) \tag{5}$$

*for* $\forall i = 0, 1, \ldots, q$ *and* $\forall x$ *such that* $x^{\leq p-1} \cdot F_{i,p} \neq 0$. *If* $x^{\leq p-1} \cdot F_{i,p} = 0$ *then* $(x^{\leq p-2} \wedge x^{(p)}) \cdot F_{i,p} = 0$. *Here,* $||x^{\leq p-1} \cdot F_{i,p}||_v^{\text{modified}}$ *means that if* $v \in S_x^n$ *we replace* $\operatorname{ord}_v(x^{\leq p-1} \cdot F_{i,p})$ *in the original definition by* $\max\{\operatorname{ord}_v(x^{\leq p-1} \cdot F_{i,p}) - 1, 0\}$, *and if* $v \in S_\infty$, *we need no modification.*

To prove Conjecture 3, we must incorporate the association $x \mapsto S_x^n$ into Vojta's interpretation [V, Chapt. 6] of Schmidt's proof of SST (Theorem 2). The point of the proof in this direction is the choice of the length function (which reflects on the left hand side of (5)) used in the successive minima. The inequality (5) implies that we may replace the inequality in the condition $\operatorname{ord}_v(x^{(t)} \cdot F_i) \geq \operatorname{ord}_v(x \cdot F_i) - t$ by the equality.

We can put the inequality (5) in more geometric form.

**Conjecture 4.** *Let* $D$ *be a linear divisor of* $\mathbb{P}^n(k)$ *in general position. Let* $D^{(p)}$ *denote the union of the* $p$*-th jet space of all irreducible components of* $D$. *Then there exists a finite union* $\mathcal{S}$ *of proper linear subspaces of* $\mathbb{P}^n(k)$ *such that, if* $x \notin \mathcal{S}$, *then there exist* $\overline{x}^{(1)}, \ldots, \overline{x}^{(n)} \in T_{[x]}\mathbb{P}^n(\mathcal{O}_k)$ *which satisfy the inequalities*

$$m_{S_\infty}(x, D) \leq m_{S_\infty}(\overline{x}^{(p)}, D^{(p)}) + \varepsilon \operatorname{ht}(x)$$
$$m_{S_\infty}(\overline{x}^{(p)}, \infty) \leq \varepsilon \operatorname{ht}(x)$$

*and the condition*

$$\operatorname{ord}_v(x^{(t)} \cdot F_i) = \operatorname{ord}_v(x \cdot F_i) - t \quad \forall v \in S_x$$

*for* $\forall p = 1, 2, \ldots, n$ *(up to uniform error). Here,* $S(x)$ *is the finite set of places of* $k$ *defined by* $S(x) = S_\infty \cup S_x^n$ *where* $S_x^n$ *is the set of non-Archimedean places of* $k$ *over which the section* $x : \operatorname{Spec}(\mathcal{O}_k) \to \mathbb{P}^n(\mathcal{O}_k)$ *and the linear divisor* $D$ *in* $\mathbb{P}^n(\mathcal{O}_k)$ *intersect with multiplicity* $m \geq n$.

We have an equivalence Conjecture 3 $\Leftrightarrow$ Conjecture 4. The following is the direct consequence of Conjecture 4:

**Corollary 5.** *Suppose that Conjecture 3 is true. We then have*

$$N^n(x, D) \leq N_{S_\infty}(x, D)(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) - N_{S(x)}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) + \varepsilon \operatorname{ht}(x)$$

*outside a finite union* $\mathcal{S}$ *of proper linear subspaces of* $\mathbb{P}^n(k)$, *where the counting functions measure the* $v$*-adic approximation of* $x^{(1)} \wedge \cdots \wedge x^{(n)}$ *to* $0$ *for appropriate finite places (*$N_S$ *measures the* $v$*-adic approximation for* $v$ *outside of* $S$*) in the total space of the anticanonical bundle of* $\mathbb{P}^n(k)$.

The "exact state" in the Diophantine setting is characterized by

$$x^{(1)} \wedge \cdots \wedge x^{(n)} = 0 \qquad \text{in } K^{-1}_{\mathbb{P}^n(k)} .$$

This is reasonable, because if we perform the successive minima restricted to a hyperplane, the sequence of linearly independent vectors $x, x^{(1)}, \ldots, x^{(t)}$ ends at

$t = n-1$ and therefore $x^{(1)} \wedge \cdots \wedge x^{(n-1)} \wedge x^{(n)} = 0$. It follows from Conjecture 4 and Corollary 5 that the corresponding "approximation state" is

$$
\begin{aligned}
(6) \qquad & m_{S_\infty}(x, D) + N^n(x, D) \\
& \leq m_{S_\infty}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) + N_{S_\infty}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) \\
& \quad - N_{S(x)}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) + \varepsilon\, \mathrm{ht}(x)\ .
\end{aligned}
$$

The right hand side of (6) is bounded above by

$$-\mathrm{ht}_{K_{\mathbb{P}^n}}(x) + m_{S_\infty}(x^{(1)} \wedge \cdots \wedge x^{(n)}, \infty) + \varepsilon\, \mathrm{ht}(x)\ .$$

Using Conjecture 4 again, we conclude that this is bounded above by

$$-\mathrm{ht}_{K_{\mathbb{P}^n}}(x) + \varepsilon\, \mathrm{ht}(x)\ .$$

Therefore Conjecture 4 impplies Schmidt's Subspace Theorem with truncated counting functions:

**Corollary 6.** *Suppose that Conjecture 3 is true. Then the following improvement of Schmidt's SST is true: Let $D = \{D_1, \ldots, D_q\}$ be a collection of hyperplanes in general position in $\mathbb{P}^n(k)$ and $\varepsilon$ any positive number. Then there exists a finite number of proper linear subspaces $Z$ such that the set of the solutions outside of $Z$ for the Diophantine inequality*

$$m_S(x, D) + N^n(x, D) + \mathrm{ht}_{K_{\mathbb{P}^n}}(x)\ >\ \varepsilon\, \mathrm{ht}_{\mathcal{O}(1)}(x)$$

*for points of $\mathbb{P}^n(k) - D$ is finite.*

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