AUTOMORPHISM GROUPS OF CERTAIN HYPERBOLIC MANIFOLDS

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§1. Introduction.

The complex structure we consider on the tangent bundle of a real-analytic Riemannian manifold M is the unique complex structure which turns every Riemannian foliation into a holomorphic curve. We call this complex structure the *adapted complex structure.* This structure might only be defined on a proper subset of the tangent bundle; it could be defined on the whole tangent bundle only if all sectional curvatures of M are everywhere positive. It was shown by S. Halverscheid that the converse does not hold. The example he considered is the ellipsoid in $\mathbb{R}^3$ with the induced metric.

The subject we are interested in is the disk bundle $T^rX = \{(x,v) : x \in X, v \in T_xX, |v| < r\}$ endowed with the adapted complex structure. We call T^rM a *Grauert tube* of radius r over the Riemannian manifold M.

When M is compact, there exists a maximal radius $r_{max}(M) > 0$ such that the adapted complex structure exists in T^rM for all $r < r_{max}(M)$, but not for any $r \geq r_{max}(M)$. There is a strictly plurisubharmonic exhaustion function for the Grauert tube T^rM, hence T^rM is a complete hyperbolic Stein manifold. Burns-Hind have shown the rigidity, i.e. $Aut(T^rM) = Isom(M)$, holds for all compact M for any $r \leq r_{max}$. So, we will only deal with the non-compact cases. We will assume from now on that that M is non-compact.

In the non-compact cases, the $r_{max}(M)$ could be zero and there is no guarantee on the complete hyperbolicity of such T^rM. However, when M is homogeneous then $r_{max}(M) > 0$ and we claimed in [K] the complete hyperbolicity and the rigidity for Grauert tube over Riemannian homogeneous spaces. The techniques there heavily

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depend on the homogeneity of the center manifold M. In this note, we will claim a "loose" rigidity version of Grauert tubes over general non-compact Riemannian manifolds such that $r_{max}(M) > 0$. We prove the following two theorems in this article.

Theorem 1. *Let M be a real-analytic Riemannian manifold such that T^rM is not covered by the ball. Then $Aut_0(T^rM) = Isom_0(M)$ for any $r < r_{max}$.*

Theorem 2. *Let X be a real-analytic Riemannian manifold such that $X/Isom(X)$ is compact. Then for any $r < r_{max}$, T^rX is complete hyperbolic.*

§2. Proof of Theorem 1.

Before we start, let's quote some results from [K] that we are going to use for the proof of Theorem 1.

Theorem 3.1 of [K]. *Let M be a domain in a complex manifold $\hat{M}$. Suppose that there exist a point $p \in M$ and a sequence of automorphisms $\{f_j\} \subset Aut(M)$ such that $f_j(p) \to q \in \partial M$, a C^2-smooth strictly pseudoconvex point. Then M is biholomorphic to the unit ball.*

Theorem 4.1 of [K]. *Let $f \in Aut(T^rX)$, $r \leq r_{max}$. Then $f = du$ for some $u \in Isom(X)$ if and only if $f(X) = X$.*

We also make the following observation that in each Grauert tube T^rM, the length square function $\rho(x, v) := |v|^2$ is strictly plurisubharmonic. Then by the Theorem 3 of [S], T^rM is hyperbolic at every point $p \in T^rM$ and hence is hyperbolic. Therefore, the automorphism group of T^rM is a Lie group.

Let σ be the anti-holomorphic involution

$$\sigma : T^rX \to T^rX, \ \ (x, v) \to (x, -v),$$

and $T^r_pM := \{(p, v) : v \in T_pM, |v| < r\}$ be the fiber though $p \in M$. Let I denote the isometry group of M and G be the automorphism group of T^rM. Then the group

$$\hat{G} := G \cup \sigma \cdot G$$

is again a Lie group. Let $\hat{\mathcal{G}}$ be the Lie algebra of $\hat{G}$; each element of $\hat{\mathcal{G}}$ could be viewed as a vector field in T^rM.

Lemma 1. *For each $\xi \in \hat{\mathcal{G}}$, there is an $\eta \in \hat{\mathcal{G}}$ corresponding to ξ such that $\exp t\eta : T_p^r M \to T_p^r M,\ \forall p \in M$.*

Proof. Let $p \in M$, define $\eta \in \hat{\mathcal{G}}$ as

$$\eta = \xi - \frac{d}{dt}|_{t=0}(\sigma \cdot (expt\xi)).$$

In local coordinates centered at p, $\sigma(z) = \bar{z}$. Then

$$\eta(iy) = 2i \text{ Im } \xi(iy).$$

It follows that η is tangent to the fiber $T_p^r X$ and the result follows. □

Fix a point $p \in M$ and take U being a small neighborhood of p in M. Let V be the domain $T^r U \subset T^r M$. The domain V is equipped with the metric d induced from the Kobayashi metric d_K of $T^r M$, i.e., $d(z, w) := d_K(z, w), \forall z, w \in V$.

Consider the restriction of the mappings $\exp(t\eta)$ to V, which are continuous mappings from V to $T^r M$. Denote

$$\mathcal{F} = \{\exp(t\eta)|_V; t \in \mathbb{R}\} \subset C(V, T^r M).$$

Lemma 2. *Suppose $T^r M$ is not biholomorphic to the ball, then $\mathcal{F}$ is a compact subset of $C(V, T^r M)$.*

Proof. It is clear that $\mathcal{F}$ is closed in $C(V, T^r M)$. As d_K is an invariant metric and $\exp(t\eta) \in Aut(T^r M), \forall t \in \mathbb{R}$,

$$d(\exp(t\eta)(z), \exp(t\eta)(w)) = d_K(\exp(t\eta)(z), \exp(t\eta)(w)) = d_K(z, w) = d(z, w).$$

This shows that $\mathcal{F}$ is equicontinuous. We then claim that for every $z = (p, v) \in V$, the set $\mathcal{F}(z) := \{\exp(t\eta)(z) : t \in \mathbb{R}, z \in V\}$ is relatively compact in $T^r M$. Suppose not, $\exp(t\eta)(z)$ approach to the boundary of $T_p^r M$ which is a smooth strictly pseudoconvex point. By Theorem 3.1 of [K], this forces $T^r M$ to be the ball which is a contradiction. Therefore $\mathcal{F}(z)$ is a relative compact subset of $T^r M$. By the Ascoli theorem (c. f. [W]), $\mathcal{F}$ is compact in $C(V, T^r M)$. □

Lemma 3. *For each $\xi \in \hat{\mathcal{G}}$, the vector field ξ is tangent to $T^r M$.*

Proof. By Lemma 1 and 2, we have a compact subfamily $\mathcal{F}$ of $C(V, T^r M)$. Each element of $\mathcal{F}$ is of the form $\exp(t\eta)$ for some real number t. If η is not identically zero then $\mathcal{F}$ is isomorphic to $\mathbb{R}$ which is non-compact. Therefore, η is identically zero. By the construction of η in Lemma 1, this implies that

$$\xi = \frac{d}{dt}|_{t=0}(\sigma \cdot (expt\xi)).$$

Hence the imaginary part of $\xi(p)$ is zero for all $p \in M$. Therefore the tangent vector ξ is tangent to M. □

Proof of Theorem 1. Applying Theorems 3.1 and 4.1 of [K], an immediate corollary of Lemma 3 is that $G_0 \subset \hat{G}_0 \subset I_0$, the identity component of the isometry group of X. But, $I_0 \subset G_0$ from the construction of Grauert tubes. Therefore $G_0 = I_0$. □

§3. Proof of Theorem 2.

In general, it is difficult to see whether a hyperbolic manifold Ω is complete or not. Fornaess and Sibony proved in [F-S] that if the quotient $\Omega/Aut(\Omega)$ is compact then Ω is complete hyperbolic.

When the center is co-compact, the Grauert tube is complete hyperbolic since it is the universal covering of a complete hyperbolic manifold. In [K], we also proved that T^rM is complete hyperbolic provided that M is homogeneous. Inspired from Fornaess-Sibony's work, we will prove in this section that the Grauert tube T^rM is complete hyperbolic when M/I is compact where I is the isometry group of M.

Let d_K be the Kobayashi metric on the hyperbolic manifold T^rM and d be the restriction of d_K to M. Notice that I is a subgroup of the automorphism group of T^rM and hence any $h \in I$ is an isometry for the metric d.

Proof of Theorem 2. We would prove the theorem in three steps.

(I). d is a complete metric in M. In general the convergence of any Cauchy sequence does not necessarily imply the completeness of the metric. However, this does work for the Kobayashi metric and hence it works for the induced metric d as well. It is therefore sufficient to show that every Cauchy sequence actually converges.

Since M/I is compact, there is a compact set K such that for any point $p \in M$ there exists a $g \in I$ such that $g(p) \in K$. Let $\{p_n\} \subset M$ be a Cauchy sequence in M with respect to the metric d. That is, there exists a large m such that $d(p_k, p_l) < \epsilon, \forall k, l \geq m$. Let $g \in I$ such that $g(p_m) \in K$. Then

$$d(g(p_n), g(p_m)) = d_K(g(p_n), g(p_m)) = d_K(p_n, p_m) = d(p_n, p_m) \leq \epsilon, \ \forall n > m.$$

Following the same argument in [F-S], the Cauchy sequence $\{g(p_n)\}$ converges and therefore $\{p_n\}$ converges.

(II). Let $q = (x, v), |v| = r$. Then the local estimate of the Kobayashi metric near a smooth strictly pseudoconvex boundary point as we did in [K] shows that $d_K(z, q) = \infty$ for any $z \in T^r M$.

(III). Fix $z = (p_1, v_1) \in T^r M$ and let $w = (p_2, v_2)$ varies in $T^r M$. Let $h \in I$ such that $h(p_2) \in K$, then $d_K((p_2, 0), (p_2, v_2)) = d_K((h(p_2), 0), (h(p_2), h_*(v_2))$ is finite. Similarly, $d_K((p_1, 0), (p_1, v_1))$ is finite as well. Now,

$$\begin{aligned} d_K(z, w) &\geq d_K((p_1, 0), ((p_2, v_2)) - d_K((p_1, 0), ((p_1, v_1)) \\ &\geq d_K((p_1, 0), ((p_2, 0)) - d_K((p_2, 0), ((p_2, v_2)) - d_K((p_1, 0), ((p_1, v_1)). \end{aligned}$$

The last two terms are finite for any z and w. As p_2 approaches to the infinity, $d_K(z, w)$ goes to infinity.

(II) and (III) show that d_K is complete. □

§4. Grauert tubes of product manifolds.

Let M and N be two real-analytic Riemannian manifolds such that the Grauert tubes $T^r M$ and $T^r N$ exist for some $r > 0$. Let H be the product manifold of M and N with the product metric. Then both M and N are totally geodesic submanifolds of H and hence both $T^r M$ and $T^r N$ are complex submanifolds of $T^r H$ where

$$T^r H = \cup_{r \geq \delta \geq 0} T^{r-\delta} M \times T^\delta N \subset T^r M \times T^r N.$$

If both $T^r M$ and $T^r N$ are complete hyperbolic, so is $T^r M \times T^r N$. In this case, $T^r H$ is complete hyperbolic as well since $T^r H$ is a closed submanifold of $T^r M \times T^r N$. Therefore, when H is the product of a homogeneous manifold and a compact manifold then the complete hyperbolicity of $T^r H$ is guaranteed.

The above could also be concluded as an immediate corollary of Theorem 2 since $M \times N / Isom(M \times N)$ is compact when M is homogeneous and N is compact.

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