Weighted Homogeneous Polynomial CR Manifolds and Tanaka-Hodge Numbers

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1. Let M be a compact Sasakian (i.e., normal strongly pseudo-convex CR) $(2n+1)$-manifold, $n \geq 2$. An explicit formula for Tanaka-Hodge number $h^{p,q}$ of a Sasakian manifold M which is defined as the zero locus of a weighted homogeneous polynomial f is represented in terms of the graded Milnor algebra associated to f.

2. A complex polynomial f in $n+2$-complex variables, $n \geq 2$

$$f = f(z_1, z_2, \cdots, z_{n+2}) \;=\; \sum c_{a_1 \cdots a_{n+2}}\, z_1^{a_1} z_2^{a_2} \cdots z_{n+2}^{a_{n+2}}$$

is called weighted homogeneous, of degree d and with weight $\mathrm{w} = (w_1, w_2, \cdots, w_{n+2}) \in \mathbb{N} \times \cdots \times \mathbb{N}(\ n+2$ times), if it satisfies

$$w_1 a_1 + \cdots + w_{n+2} a_{n+2} \;=\; d$$

for each monomial $z_1^{a_1} z_2^{a_2} \cdots z_{n+2}^{a_{n+2}}$ appearing in f

Example (Brieskorn-Pham polynomial)

$$f(z) = z_1^{a_1} + z_2^{a_2} \cdots + z_{n+2}^{a_{n+2}}$$

Its degree is $d = a_1 a_2 \cdots a_{n+2}$ and weights are $w_i = \frac{d}{a_i}$, $1 \leq i \leq n+2$.

3. The zero locus of $f = f(z)$

$$V_f \;=\; \{z \in \mathbb{C}^{n+2} \mid f(z) = 0\}, \quad n \geq 2$$

is a hypersurface of $\mathbb{C}^{n+2}$ and admits in general the origin as a singularity.

Assume that the origin is an isolated singularity.

Intersection of V_f with an ε-hypersphere $S_\varepsilon^{2n+3} = \{z \in \mathbb{C}^{n+2} \mid |z| = \varepsilon\}$ centered at 0 yields a smooth $(2n+1)$-manifold $M (= K_f) \;=\; V_f \cap S_\varepsilon^{2n+3}$, called a link of the singularity.

Proposition. The link K_f carries an S^1-action and is equipped with a Sasakian structure (η, ξ, φ, g) or equivalently a normal strongly pseudo convex CR structure. The Sasakian structure on K_f is almost regular in the sense of C. B. Thomas[17].

In fact, let $\mathbb{C}^\times \times \mathbb{C}^{n+2} \longrightarrow \mathbb{C}^{n+2}$,

$$(t, z) \mapsto t \cdot z = (t^{w_1} z_1, \cdots, t^{w_{n+2}} z_{n+2})$$

be the weighted $\mathbb{C}^\times$-action of $\mathbb{C}^{n+2}$. Then f satisfies

$$f(t \cdot z) = t^d f(z)$$

so that V_f is $\mathbb{C}^\times$-invariant. Further K_f is invariant under the action of $S^1 = \{e^{is} \in \mathbb{C}^\times \mid s \in \mathbb{R}\}$.

Assume that $\mathrm{GCD}\{w_1, \cdots, w_{n+2}\} = 1$. This guarantees effectiveness of the $\mathbb{C}^\times$-action.

To define a Sasakian structure on K_f we notice K_f is a real hypersurface of a smooth $n+1$-dimensional complex manifold $V_f \backslash \{0\}$ around K_f. We set ξ, η, φ as

$$\begin{aligned} \xi &= J\nu, \\ JX &= \varphi(X) - \eta(X)\nu, \;\; X \in T_z K_f, \end{aligned}$$

where ν is the unit normal vector field along K_f, obtained by taking the tangent vector of the real orbit of the $\mathbb{C}^\times$-action and $\varphi(X)$ is tangential part of JX. Note ξ coincides with the infinitesimal vector field of the S^1-action. The metric g is induced from the Euclidean metric of $\mathbb{C}^{n+2}$. So, (η, ξ, φ, g) yields a Sasakian structure on K_f admitting isometric S^1-action[2].

Since the isotropy group G_z, $z \in K_f$ is

$$G_z = \{\zeta \in S^1 \mid \zeta^{w(z)} = 1\},$$

where $w(z) = \mathrm{GCD}\{w_i \mid z_i \neq 0\}$ for $z = (z_1, \cdots, z_{n+2})$. So this action has no fixed points and admits additionally finitely many distinct isotropy groups contained in some common finite subgroup Γ of S^1. By Theorem 1 in [17] this implies that the Sasakian structure is almost regular. Notice that the set of points of free S^1-action is open dense in K_f.

Remark that a regular Sasakian manifold is an S^1-fiber bundle over a Hodge complex manifold([3],[5]).

4. The harmonic theory of a Sasakian manifold. The tangent bundle TM of a Sasakian manifold M splits as $TM = D \oplus \mathbb{R}\xi$, where $D = \mathrm{Ker}\ \eta = \xi^\perp$ is a rank $2n$ real subbundle, called the contact bundle. It admits the almost complex structure $\varphi|_D$ and also the $\varphi|_D$-hermitian metric $g|_D E$. Then $D \otimes \mathbb{C} = D^{(1,0)} \oplus D^{(0,1)}$ relative to $\varphi|_D$ and it holds for the space $\Gamma(D^{(0,1)})$ of smooth sections in $D^{(0,1)}$

$$[\Gamma(D^{(0,1)}), \Gamma(D^{(0,1)})] \subset \Gamma(D^{(0,1)}) \ \text{(integrability of the CR structure)} \qquad (1)$$

$$[\xi, \Gamma(D^{(0,1)})] \subset \Gamma(D^{(0,1)}) \ \text{(normality of the CR structure)} \qquad (2)$$

so that $(D^{(0,1)}, \xi, d\eta)$ yields on M a normal strongly pseudo convex CR strucure.

Remark. It is not difficult to see that a normal strongly pseudo convex CR structure induces conversely a Sasakian structure so that a Sasakian structure is a synonym with a normal strongly pseudo convex CR structure([9]).

Theorem([15]) Let $(M, (\eta, \xi, \varphi, g))$ be a compact Sasakian $(2n+1)$-manifold M. Then a harmonic k-form θ on M, $k \le n$, satisfies

$$\begin{aligned} i(\xi)\theta &= 0, \\ \Lambda\theta &= 0 \end{aligned}$$

where $\Lambda : \Lambda^{k+2}(M) \longrightarrow \Lambda^k(M)$ is the adjoint of $L = e(d\eta)$, the exterior product of $d\eta$.

From this theorem any harmonic k-form θ is in $\Gamma(M, \Lambda^k(D^*))$ for $k \le n$. By splitting $\Lambda^k(D^* \otimes \mathbb{C})$ into $\otimes_{p+q=k}\Lambda^{p,q}(D^*)$, one has $\theta = \sum_{p+q=k} \theta^{p,q}$ for which each $\theta^{p,q}$ is $\overline{\partial}_D$-harmonic, where $\overline{\partial}_D$ is the operator : $\Gamma(M, \Lambda^{p,q}(D^*)) \longrightarrow \Gamma(M, \Lambda^{p,q+1}(D^*))$, defined in [16] so that on a compact Sasakian manifold one has the Hodge decomposition, like on a Kähler manifold. Namely, denote by $\mathbb{H}^k(M)$ and $\mathbb{H}^{p,q}(M)$ the space of harmonic forms and the space of $\overline{\partial}_D$-harmonic (p,q)-forms on M, respectively and set $h^{p,q}(M) = \dim_{\mathbb{C}} \mathbb{H}^{p,q}(M)$, which we call Tanaka-Hodge number. Then

Theorem([16]). Let $(M, (\eta, \xi, \varphi, g))$ be a compact Sasakian $(2n+1)$-manifold, $n \ge 2$. For $k \le n$

$$\begin{aligned} \mathbb{H}^k(M) &= \oplus_{p+q=k} \mathbb{H}^{p,q}(M), \quad k \le n, \\ \overline{\mathbb{H}}^{q,p}(M) &= \mathbb{H}^{p,q}(M). \end{aligned}$$

5.Milnor fibration. Milnor showed that by using the Milnor fibration the link K_f associated to a weighted homogeneous polynomial f is $(n-1)$-connected so that non-trivial Betti numbers of K_f are only $b_n = b_{n+1}$(Theorem 5.2, [10]). These can be computed as ([11],[12])

$$b_n(K_f) = \sum_{s=0}^{n+2} \sum_I (-1)^{n-s} \frac{q_{i_1} \cdots q_{i_s}}{[u_{i_1}, \cdots, u_{i_s}]}$$

Here $I = (i_1, \cdots, i_s)$ is an s-tuple of indices $1 \le i_1 < \cdots < i_s \le n+2$ and $[u_{i_1}, \cdots, u_{i_s}]$ is the LCM of integers $u_{i_1}, \cdots, u_{i_s}$(where $q_i = d/w_i = u_i/v_i$, $(u_i, v_i) = 1$, $i = 1, \cdots, n+2$).

6. Define a graded algebra $\mathbb{M}_f = \mathbb{C}(z_1, \cdots, z_{n+2})/(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{n+2}})$, called the Milnor algebra where each z_i has degree w_i. Then

Theorem 1([9]). $h^{p,q}$ of the link K_f is given for $p+q=n$

$$h^{p,q}(K_f) = \dim_{\mathbb{C}} (\mathbb{M}_f)_\ell$$

Here $(\mathbb{M}_f)_\ell$ is the linear subspace of $\mathbb{M}_f$ consisting of degree ℓ elements, where $\ell = (p+1)d - |\mathrm{w}|$. ($|\mathrm{w}| = w_1 + \cdots + w_{n+2}$ is the total sum of all the weights).

Corollary. The Betti number b_n of the link K_f is given in two ways, one in an arithmetical way and another in an algebraic way:

$$\begin{aligned} b_n(K_f) &= \sum_{s=0}^{n+2} \sum_I (-1)^{n-s} \frac{q_{i_1} \cdots q_{i_s}}{[u_{i_1}, \cdots, u_{u_s}]} && (3) \\ &= \sum_{k=0}^{n} \dim_{\mathbb{C}} (\mathbb{M}_f)_{(k+1)d-|w|} && (4) \end{aligned}$$

We can pose the following problem. Does $h^{p,q}(K_f)$ depend only on the degree and the weights ?

7. **Sketch of a proof of Theorem**. The Sasakian contact structure of K_f is almost regular so that one has the double fibration([17]), first by dividing by the common finite subgroup Γ of S^1 and then by S^1/Γ:

$$K_f \xrightarrow{\pi_1} K_f/\Gamma \xrightarrow{\pi_2} K_f/S^1 = (V \setminus \{0\})/\mathbb{C}^\times$$

where π_1 is a branched covering, π_2 is a projection by a free $S^1/\Gamma(\cong S^1)$-action, and $(V\setminus\{0\})/\mathbb{C}^\times$, which we denote by V_f^*, is a hypersurface in the $(n+1)$-dim weighted projective space $\mathbb{P}^{n+1}(\mathrm{w})$.

The space K_f/Γ is a finite quotient of a smooth manifold. So its cohomology group $H^n(K_f/\Gamma,\mathbb{R})$ is isomorphic to $\mathbb{H}^n(K_f)$ through π_1, since $H^n(K_f/\Gamma,\mathbb{R})\cong\mathbb{H}^n(K_f)^\Gamma$, the space of Γ-invariant harmonic n-forms on K_f, and the S^1-action is isometric so a harmonic form on K_f is Γ-invariant.

Apply the Gysin exact sequence to the $S^1(\cong S^1/\Gamma)$-fibration $\pi_2: K_f/\Gamma\longrightarrow K_f/S^1=V_f^*$ to have

$$\begin{array}{ccccc} \longrightarrow H^{n-1}(K_f/\Gamma) & \overset{\int}{\longrightarrow} & H^{n-2}(V_f^*) & \overset{b}{\longrightarrow} & H^n(V_f^*) \\ \longrightarrow H^n(K_f/\Gamma) & \overset{\int}{\longrightarrow} & H^{n-1}(V_f^*) & \overset{b}{\longrightarrow} & \end{array}$$

where the map $b:[\alpha]\mapsto[\alpha]\wedge[\Omega]$ $([\Omega]=\iota^*[\omega]\in H^2(V_f^*))$ is induced by the embedding $\iota: V_f^*\longrightarrow\mathbb{P}^{n+1}(\mathrm{w})$ (ω denotes the curvature form representing the fibration π_2, namely the Kähler form of $\mathbb{P}^{n+1}(\mathrm{w})$). The map $\int$ is integration along fibres. Remark that $\int$ reduces to zero map, since harmonic forms in K_f/Γ have no fibre directional part. Then b is injective and one has

$$H^n(V_f^*)=H_0^n(V_f^*)\oplus H^{n-2}(V_f^*) \tag{5}$$

where $H_0^n(V_f^*)$ is called the primitive cohomology group of V_f^*, isomorphic to $H^n(V_f^*)/\mathrm{Im}b$. So the Gysin sequence yields an isomorphism

$$\pi_2^*: H_0^n(V_f^*)\ \overset{\cong}{\longrightarrow}\ H^n(K_f/\Gamma)(\cong\mathbb{H}^n(K_f))$$

Moreover, V_f^* is a V-manifold so the Hodge-de Rham-Kodaira harmonic thoery is applicable([1]). So $H_0^n(V_f^*)$ splits into $\oplus_{p+q=n}\mathbb{H}_0^{p,q}(V_f^*)$, where $\mathbb{H}_0^{p,q}(V_f^*)$ is the space of $\overline{\partial}$-harmonic (p,q)-forms $\psi^{p,q}$ on V_f^* satisfying the primitive condition $\Lambda_\omega\psi^{p,q}=0$.

Claim. For $p,q,\ p+q=n$

$$\mathbb{H}_0^{p,q}(V_f^*)\cong\mathbb{H}^{p,q}(K_f)$$

In fact, let ψ be a primitive harmonic n-form of V_f^*. Then $\psi = \sum_{p+q=n} \psi^{p,q}$, $\psi^{p,q} \in \mathbb{H}_0^{p,q}(V_f^*)$. So $(\pi_2 \circ \pi_1)^*(\psi) = \sum_{p+q=n}(\pi_2 \circ \pi_1)^*(\psi^{p,q})$. Here $(\pi_2 \circ \pi_1)^*(\psi^{p,q})$ is a (p,q)-forms on K_f. On the other hand, by Tanaka's Hodge decomposition on a Sasakian manifold, we have $(\pi_2 \circ \pi_1)^*(\psi) = \sum_{p+q=n} \varphi^{p,q}$ in terms of harmonic (p,q)-forms on K_f. Therefore $(\pi_2 \circ \pi_1)^*(\psi^{p,q}) = \varphi^{p,q}$, for each p,q so that $h_0^{p,q}(V_f^*) \leq h^{p,q}(K_f)$. However, from the above argument one has $\dim H_0^n(V_f^*) = \dim H^n(K_f)$. So $h_0^{p,q}(V_f^*) = h^{p,q}(K_f)$, from which the claim follows.

Theorem([13], [14]). The primitive Hodge number $h_0^{p,q}(V_f^*) = \dim_{\mathbb{C}} \mathbb{H}_0^{p,q}(V_f^*)$ is

$$h_0^{p,q}(V_f^* = \dim_{\mathbb{C}}(\mathbb{M}_f)_\ell.$$

See also Appendix B.34 in [4].

Applying these results, our Theorem is obtained.

8. Example

Let f be a polynomial of Brieskorn-Pham type:

$$f = z_1^2 + z_2^3 + z_3^3 + z_4^4 + z_5^6.$$

f has degree $d = 12$ and weights $\mathrm{w} = (6,4,4,3,2)$.

The link of the singularity K_f is 7-dimensional and $b_3(K_f) = 2$ and $h^{3,0}(K_f) = 0$ and $h^{2,1}(K_f) = 1$.

In fact, to compute the Betti number b_3 we have $(q_1, q_2, q_3, q_4, q_5) = (2,3,3,4,6)$ and $q_i = u_i$, $i = 1, \cdots, 5$. So, from the formula of Milnor-Orlik, Orlik

$$b_3 = -\sum_{s=0}^{5} \sum_{i_1 < \cdots < i_s} \frac{q_{i_1} \times \cdots \times q_{i_s}}{[q_{i_1}, \cdots, q_{i_s}]}$$

whose value is $-\{1 - 5 + 19 - 47 + 66 - 36\} = 2$.

On the other hand,

$$\left(\frac{\partial f}{\partial z_i}\right) = \left(2z_1, 3z_2^2, 3z_3^2, 4z_4^3, 6z_5^5\right)$$

so that $\mathbb{M}_f$ is the algebra generated by z_2, z_3, z_4, z_5 with degrees $\deg(z_2) = \deg(z_3) = 4, \deg(z_4) = 3, \deg(z_5) = 2$. Then $\mathbb{M}_f$ is spanned by monimials

$$z_2^{b_2} z_3^{b_3} z_4^{b_4} z_5^{b_5},$$

$0 \leq b_2, b_3 \leq 1, 0 \leq b_4 \leq 2, 0 \leq b_5 \leq 4$. From our Theorem $h^{3,0} = \dim(\mathbb{M}_f)_\ell$, where $\ell = 29$. But $(\mathbb{M}_f)_{29} = \{0\}$ since the highest degree element in the algebra is $z_2 z_3 z_4^2 z_5^4$ whose degree is 22 and so $h^{3,0} = 0$. For $h^{2,1}$ we compute $\dim(\mathbb{M}_f)_\ell$, $\ell = (2+1)d - |\mathrm{w}| = 17$ and $(\mathbb{M}_f)_{17} = \mathbb{C} z_2 z_3 z_4 z_5^3$ so $h^{2,1} = 1$. It holds $b_3 = 2(h^{3,0} + h^{2,1})$.

9. Sasakian 5-manifolds

Now assume that a compact Sasakian manifold M is 5-dimensional. Then the contact bundle D is equipped with the Hodge star operator $\star$ so that D splits into $D_+ \oplus D_-$ in terms of self-dual and anti-self-dual subbundles. From theorems of Tachibana and Tanaka each harmonic 2-form α splits as self-dual and anti-self-dual harmonic forms as $\alpha = \alpha^+ + \alpha^-$ so that $b_2(M) = b^+(M) + b^-(M)$ and $b^+(M) = 2h^{2,0}(M)$ and $b^-(M) = h^{1,1}(M)$([6]).

Further the Riemannian curvature tensor R satisfies $i(\xi)R = 0$. This means that the curvature operator R maps $\Lambda^2 D^*$ into itself. Now representing R as $W + S + \frac{\rho}{12} \cdot id$ in terms of Weyl curvature, Ricci curvature and scalar curvature, one has the Block decomposition of the Weyl curvature.

$$W = \begin{pmatrix} W^+ & * \\ * & W^- \end{pmatrix}$$

Definition([6]). A Sasakian 5-manifold is *self-dual* (*anti-self-dual*) if $W^- = 0$ (respectively $W^+ = 0$).

Theorem 2. Let $(M, (\eta, \xi, \phi, g))$ be a compact Sasakian 5-manifold of positive scalar curvature $\rho > 0$. If M is self-dual, then $b^-(M) = 0$.

Another phenomena similar to self-dual 4-manifolds is the following which concerns with twistor space. On a Sasakian 5-manifold one can define a twistor space Z over M as the unit sphere bundle of the anti-self-dual 2-form bundle $\Lambda^2_-(D^*)$, which admits canonical almost CR structure.

Theorem 3([7],[8]). Let $(M, (\eta, \xi, \phi, g))$ be a Sasakian 5-manifold. If it is self-dual, then the almost CR structure of the twistor space Z is integrable.

Typical example of CR twistor space Z over 5-sphere S^5 is given, explicitly represented, as $Z = \{(z, [w]) \in S^5 \times \mathbb{C}P^2 \mid \sum_i z_i w_i = 0\}$, which is also a $\mathbb{C}P^1$-fibration over the flag manifold F, twistor space, of self-dual 4-manifold $\mathbb{C}P^2$.

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