Anti-self-dual Hermitian Structures on Inoue Surfaces via Twistor Method

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The following is an exposition of the joint work with Massimiliano Pontecorvo (Rome III). The detail will appear elsewhere.

The purpose is to construct via twistor method a family of anti-self-dual hermitian metrics on Inoue surfaces with positive second betti number. We shall give statements and an outline of the proofs here. In the first section we recall the definition of anti-self-dual metrics and the associated twistor spaces and state our main result in the second section. In the third section we explain the method of Donaldson-Friedman [3]and its variation, which actually are our fundamental methods. In the fourth section we shall set up the situation where Inoue surfaces arise in the previous framework. Finally in the fifth section we sketch how to complete the proof by using Joyce twistor spaces.

1 Anti-self-dual metric and twistor space

1.1. *(Anti-)self-dual metrics*

Let M be an oriented 4-dimensional C^∞ manifold, and g a Riemannian metric on M. Then the Riemannian curvature tensor R of g is naturally decomposed as a sum of two tensors

$$R = W + \rho,$$

where W is the Weyl conformal curvature tensor of g and ρ is a tensor determined by the Ricci tensor of g. This really holds true in any dimension, but as a special phenominon in 4-dimension, W further decomposes in two parts

$$W = W_+ + W_-.$$

where W_+ and W_- are called respectively the anti-self-dual part and the self-dual part of W.

Definition (Atiyah-Hitchin-Singer [1]) When $W_+ \equiv 0$ (resp. $W_- \equiv 0$), the Riemannian metric g is called *self-dual* (resp. *anti-self-dual*) metric and (M, g) the *self-dual* (resp. *anti-self-dual*) manifold.

Remark. 1) In general $W \equiv 0$ if and only if g is conformally flat, namely g is locally written as a product of a positive function and the Euclidian metric with respect to suitable local coordinates. In particular if g is conformally flat, it is both self-dual and anti-self-dual.

2) W and $W_\pm$ depends only on the conformal class $[g]$ of g Hence, the (anti-)self-duality is the notion defined for a conformal manifold $(M, [g])$.

3) If we change the oritentation of M, $W_\pm$, and hence also the self-duality and the anti-self-duality, are interchanged. In this manuscript, we consider the case where M is the (underlying C^∞ manifold) of a complex surface S and construct anti-self-dual hermitian metrics on M with respect to its natural oritentation.

1.2. *Twistor spaces*

Now here we do not deal the metric directly, but study a complex geometric object called a twistor space. Namely the following is known.

Theorem (Penrose correspondence) (cf. [1]) *Let $(M, [g])$ be an anti-self-dual manifold. Then there exists a 3-dimensional complex manifold Z, called the* twistor space *of $(M, [g])$, with the following properties:* 1) *Z has the structure of a C^∞ fiber bundle $t : Z \to M$ (*twistor fibration*) over M such that each fiber $L_x := t^{-1}(x), x \in M$, is a complex partmanifold of Z which is isomorphic to the complex projective line $\boldsymbol{P}^1$. Moreover, the normal bundle $N_{L_x/Z}$ is isomorphic to $O(1) \oplus O(1)$. (L_x are called* twistor lines.*)*

2) *There exists a fixed-point-free anti-holomorphic involution σ of $Z, \sigma^2 = id_Z$, which preserves each fiber of t; $\sigma(L_x) = L_x, x \in M$. ($\sigma$ is called the* real structure *of Z.)*

Conversely, given a pair $(t : Z \to M, \sigma)$ with the above properties, an anti-self-dual conformal class $[g]$ on M is naturally determined.

Example. 1. Suppose that M is a 4 dimensional sphere S^4 and, the metric is the canonical metric $g = g_{std}$. In this case g is conformally flat, and hence anti-self-dual. If we identify S^4 with the quaternionic projective line $\boldsymbol{P}^1(\boldsymbol{H}) = \boldsymbol{H} \cup \{\infty\}$, the twistor fibration is given by

$$t : \boldsymbol{P}^3 = (\boldsymbol{C}^4 - \{0\})/\boldsymbol{C}^* = (\boldsymbol{H}^2 - \{0\})/\boldsymbol{C}^* \to (\boldsymbol{H}^2 - \{0\})/\boldsymbol{H}^* =: \boldsymbol{P}^1(\boldsymbol{H}),$$

where $\boldsymbol{H} = \{quaternionion\}$ and $\boldsymbol{H}^* = \boldsymbol{H} - \{0\}$.)

2. With respect to the identification $S^4 = \boldsymbol{R}^4 \cup \{\infty\}$, g is conformal to the Euclidian metric g_{eucl} and on $\boldsymbol{R}^4$ the induced map $t : \boldsymbol{P}^3 - L_\infty \to \boldsymbol{R}^4$ is the twistor fibration associated to $(\boldsymbol{R}^4, g_{eucl})$. $\boldsymbol{P}^3 - L_\infty$ is identified with the total space of the vector bundle $O(1) \oplus O(1)$ over $\boldsymbol{P}^1$.

3. Over $S^4 - \{0, \infty\} = \boldsymbol{R}^4 - \{0\}$, the metric is further conformally equivalent to g_{eucl}/ρ^2 with ρ the distance from the origin 0. This metric descends to the Hopf surface $(\boldsymbol{R}^4 - \{0\})/ < r > \cong S^1 \times S^3$ $(r > 1)$ 上に and gives a conformally flat metric there. The metrics in 2 and 3 are considered as an anti-self-dual hermitian metric with respect to a suitable complex structure.

2 Main Result

2.1. *Inoue surfaces*

First we recall some basic facts about Inoue surfaces. A compact complex analytic surface is called a surface of class VII if its first betti number $b_1(S)$ is equal to 1. In particular it non-Kähler.

Example 1) The most typical surfaces of class VII are the (primary) Hopf surfaces $S_{\alpha,\beta}$ determined for any pair of complex numbers $\alpha, \beta \in \boldsymbol{C}$ with $1 < |\alpha|, |\beta|$ by

$$S = S_{\alpha,\beta} := (\boldsymbol{C}^2 - \{0\})/\langle g \rangle, \quad g : (z, w) \to (\alpha z, \beta w), \quad (z, w) \in \boldsymbol{C}^2 - \{0\}.$$

S is thus diffeomorphic to $S^1 \times S^3$ and hence the second betti number $b_2(S) = 0$.

2) A blown-up Hopf surface S is surface obtained by blowing-up a Hopf surface at a finite number of points. In this case we have $b_2(S) > 0$, but S is not minimal.

3). The first minimal examples with $b_2(S) > 0$ are one of the surfaces in the following three families of Inoue surfaces:

a) parabolic Inoue surfaces, b) hyperbolic Inoue surfaces, c) half (hyperbolic) Inoue surfaces.

It is known that these surfaces are characterized by the curves which exist on them. We shall state this in the following:

Theorem (Nakamura [14]) *Let S be a minimal surface of class VII. Then the following holds.*

a) *S is parabolic $\Leftrightarrow$ there exist on S a nonsingular elliptic curve $E = E_\omega$ and a cycle of nonsingular rational curves.*

b) *S is hyperbolic $\Leftrightarrow$ there exist two cycles of nonsingular rational curves on S.*

c) *S is half hyperbolic $\Leftrightarrow$ there exist a cycle of nonsingular rational curves on S whose number of irreducible components coincides with the second betti number of S.*

Here, by a cycle of nonsingular rational curves we mean a curve $C = C_1 \cup \ldots \cup C_k$ with irreducible components $C_i, 1 \leq i \leq k$, such that $C_i \cong \boldsymbol{P}^1$ (complex projective line), C_i and C_{i+1} $(1 \leq i \leq k, C_{k+1} := C_1)$ intersect transversally at one point and that $C_i \cap C_j = \emptyset, |i - j| \geq 2$. When $k = 1$, we understand that C is just a rational curve with one node. If we set $a_i = C_i^2$ (selfintersection number), the sequence of weights $a(C) = (a_1, \ldots, a_k)$ is an invariant of the cycle.

In Case a) the surface admits a continuous parameter corresponding to the period ω, $\mathrm{Im}\ \omega > 0$ of the elliptic curve E ; $S = S_\omega$, while in Cases b), c) they have only discrete parameters. Correspondingly, in the latter cases their deformation as Inoue surfaces are rigid. In the parabolic case we always have $a_i = -2$ $(\forall i)$, and the number k of irreducible components of the cycle of rational curves is its unique discrete invariant. In the cases of hyperbolic or half-hyperbolic cases, the weight sequence of the cycles gives the discrete parameter in question. The characterization of these weights is also given by Nakamura.

2.2. Now our main theorem is stated as follows.

Main Theorem.

1) *On every hyperbolic or half hyperbolic Inoue surface there exists a continuous family of anti-self-dual hermitian metric.*

2) *On any "real "parabolic Inoue surface S_ω with a sufficiently large Im ω there exists a continuous family of anti-self-dual hermitian metrics.*

That Im ω is sufficiently large means that E_ω is a small deformation of a rational curve with a node. We omit the explanation of the meaning of "real " here. In our method, there also arises a certain real constraint on the period ω and we have the anti-self-dual metrics on S_ω only for a certain real one-dimensional family of ω. This reality is not yet identifed explicitly, but one plausible answer would be that the period ω would be pure imaginary or of absolute value one.

2.3. *Known results*

We next say a few words as to why we consider the surfaces of class VII. Let (S, h) be a compact anti-self-dual hermitian surface. In this case obviously we have the following two cases.

Case A. h is conformally equivalent to some Kähler metric k.

Case B. h is never conformally equivalent to any Kähler metric.

In general for a Kähler metric k it is known that

anti-self-dual $\Leftrightarrow$ scalar curvature $\equiv 0$

Hence the following relation holds:

Calabi-Yau (Ricci-flat) Kähler metric $\Rightarrow$ anti-self-dual Kähler metric

$\Rightarrow$ Calabi's extremal Kähler metric.

The importance of the study of Case A is thus clear.

On the other hand, in Case B of our interest, the significance of the anti-self-dual metrics are not yet clear enough. In any case in this case the surface S is non-Kähler and its plurigenera all vanish. In particular S is a surface of class VII (Boyer [2]). This is the reason why we consider the surface of class VII.

Now on the existence of anti-self-dual hermitian metrics on surfaces of class VII very little is known so far.

1) Case $b_2(S) = 0$:

For the Hopf surface $S = S_{\alpha,\beta}$, suppose that $|\alpha| = |\beta|$. Then, the hermitian metric

$$\tilde{h} := (dz \cdot d\bar{z} + dw \cdot d\bar{w})/(|z|^2 + |w|^2), \tag{1}$$

on $\boldsymbol{C}^2 - \{0\}$ induces a conformally flat, and hence, anti-self-dual hermitian metric h on S.

Conversely, if an anti-self-dual metric exists on the surface S of class VII with $b_2(S) = 0$, then we necessarily have $S \cong S_{\alpha,\beta}, |\alpha| = |\beta|$ with the metric conformally equivalent to the above one (Pontecorvo). Here we just note that the condition $|\alpha| = |\beta|$ is considered a reality constraint on the "period" of S.

2) The unique example known so far in the case $b_2(S) > 0$ is the example given by LeBrun [10]. They are constructed explicitly by using the socalled Hyperbolic Ansatz. The surface S is a blown-up Hopf surface, or more concretely S is a surface obtained by blowing-up a Hopf surface $S_r := S_{r,r}, r > 0$, at a finite number of points on the nonsingular elliptic curve $E = \{z = 0\}/\langle h \rangle$. The method of LeBrun can also be applied to parabolic Inoue surfaces S_ω with ω pure imaginary [10].

3 Method of Donaldson-Friedman

Hereafter all the manifolds are assumed compact.

3.1. *Method of Donaldson-Friedman*

Given two compact anti-self-dual manifolds $(M_i, [g_i]), i = 1, 2$, Donaldson-Friedman [3] discovered a method of constructing a new anti-self-dual structure $[g]$ on the connected sum $M_1 \# M_2$ of the two manifolds[1]. When the metric is conformally flat, such a theory is classical. They interpreted this method in the conformally flat in terms of the twistor space, and then generalized the latter in the case of anti-self-dual metrics. The method is as follows.

Let $Z_i, i = 1, 2$, be the twistor spaces of $(M_i, [g_i])$. Take points $x_i \in M_i$ as above and blow-up Z_i with center the twistor lines $L_i := L_{x_i}$. Let $\mu_i : \tilde{Z}_i \to Z_i$ be the blowing-down map. The exceptional set $Q_i := \mu^{-1}(L_i) \subseteq \tilde{Z}_i$ is, by the relation $N_{L_i/Z_i} \cong O(1) \oplus O(1)$ isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and $\mu_i | Q_i : Q_i \to L_i$ is identified with the first projection $p_1 : \boldsymbol{P}^1 \times \boldsymbol{P}^1 \to \boldsymbol{P}^1$. The real structure σ_i of Z_i lifts to $\tilde{Z}_i$, which we shall still denote by σ_i. Now fix a (σ_1, σ_2)-equivariant isomorphism $\varphi : Q_1 \to Q_2$ which interchanges the first and the second factors of the both spaces. Moreover, identifying subspaces Q_i of Z_i by φ we obtain a new space $\hat{Z} := Z_1 \cup_\varphi Z_2$. $\hat{Z}$ is a compact complex space which has only normal crossings along the subspace Q which is the identifed $Q_1 \cong Q_2$.

Now we consider the Kuranishi family (semiuniversal deformation) of $\hat{Z}$ $\{f : \mathcal{Z} \to T, o \in T, Z_o \cong \hat{Z}\}$. (Thus in particular $\mathcal{Z}, T$ is a complex space, f is a proper and flat holomorphic map, and T is identified with the germ it defines at the base point o. Moreover, $Z_o = f^{-1}(o)$.) Then the main result of [3] is stated as follows.

Theorem(Donaldson-Friedman). *Suppose that $H^2(Z_i, \Theta_i) = 0, i = 1, 2,$ where Θ_i is the*

[1]Take a point x_i on M_i and a small open ball B_i around x_i for each i, then identify $M_1 - B_1$ and $M_2 - B_2$ along the two boundaries $bB_i \cong S^3$. The resulting space, after a suitable smoothing, is the connected sum $M_1 \# M_2$.

sheaf of holomorphic vector fields on Z_i. Then T is nonsingular, and for a general point t of T the fiber $Z_t := f^{-1}(t)$ of f is nonsingular. Moreover, for any real point t, Z_t is the twistor space associated to some anti-self-dual metric on $M_1 \# M_2$.

Remark. We omit the precise formulation of the 'real point' here for the brevity of the exposition. The real structure σ_i of Z_i induces a real structure $\hat{\sigma}_i$ of $\hat{Z}$. If the Kuranishi family above is universal, the latter in turn induces the real structure on T and the real points are precisely its fixed points on T. However, in many cases of interest $\hat{Z}$ admits a positive dimensional automorphism group, and therefore f is not universal.

3.2. *The Hermitian condition on the metric*

The above construction, however, in general gives little information on the properties of the anti-self-dual metrics and the corresponding twistor spaces thus constructed. To remedy this point various variations of the method of [3] has been considered. In order to explain the necessary variation in our case we first explain the condition on the twistor space for the resulting anti-self-dual metric to be hermitian.

Let $(M, [g])$ be the anti-self-dual manifold, and Z the corresponding twistor space. Let S be the complex surface embedded in Z. In general the intersection number $L_x \cdot S$ is positive, and when $L_x \cdot S = 1$, we call S *elementary.* Then $\bar{S} := \sigma(S)$ also is anelementary complex surface. In this case the following is known:

Proposition (Poon [15]) *An elementary surface S is nonsingular. We have two types according as S contains a twistor line L or not.*

Type 1*: $\exists L \subseteq S$. Then twistor line L contained in S is unique, and for its self-intersection number in S we have $L^2 = 1$. (Hence S is isomorphic to a blown-up complex projective plane $\boldsymbol{P}^2$ and L is a proper transform of a line.) Furthermore $S \cap \bar{S} = L$ and the intersection is transversal.*

Type 2*: $\forall L \not\subseteq S$. Then $t|S$ gives an oritentation preserving diffeomorphism $S \to M$ and $(t|S)^* g$ becomes an anti-self-dual hermitian metric on S ($\cong M$).*

Conversely, any anti-self-dual hermitian metric on M with respect to some complex structure on M is obtained from some naturally determined elementary surface S in Z as above.

Thus the elementary surfaces of Type 2 are of our main interest, but actually in our construction the elementary surface of Type 1 also play an important role. Namely, here by a suitable variation of the method of [3] we try to produce from a twistor space which contains an elementary surface of Type 1 a twistor space which contains elementary surface of Type 2. The method below is a variation of the method of [8].

3.3. *A variation of the method of Donaldson-Friedman*

One of the main points is, instead of considering the connected sum of two manifolds, to consider, the self connected sum of one manifold M. Namely, we delete two points

$x_i, i = 1, 2$ from M and a small 4dimensional open ball B_i with center x_i, and identify on $M - B_1 - B_2$ the two boundaries $bB_i \cong S^3$ via a suitable diffeomorphism $bB_1 \to bB_2$; then the resulting manifold is called the *self connected sum* of M.

Let $(M, [g])$ be an anti-self-dual manifold, and Z the corresponding twistor space. We assume that Z contains an elementary surface S of Type 1. If we put $L_1 = S \cap \bar{S}$, then by the proposition above, L_1 is a twistor line. If we take any twistor line L_2 other than L_1, then again by the above proposition L_2 intersect with S (resp. $\bar{S}$) exactly at one point p (resp. $\bar{p}$) and transversally. We now blow up Z along the disjoint union $L_1 \cup L_2$: $\mu : \tilde{Z} \to Z$. In the same way as above we have $Q_i := \mu^{-1}(L_i) \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and then we fix an σ-invariant isomorphism $\varphi : Q_1 \to Q_2$ which interchanges the first and the second factors. Idenfitying Q_1 and Q_2 via φ denote by $\hat{Z}$ the resulting complex space, which has normal crossing singularities along a complex surface Q which is isomorphic to Q_i.

Let $\tilde{S}$ and $\tilde{\bar{S}}$ be the proper transforms in $\tilde{Z}$ of S and $\bar{S}$ respectively. Since $L_1 = S \cap \bar{S}$ is the center of the blowing-up, $\tilde{S}$ does not intersect with $\tilde{\bar{S}}$. Therefore, their images $\hat{S}$ and $\hat{\bar{S}}$ in $\hat{Z}$ have no common points $\hat{S}$ and $\hat{\bar{S}}$. Here σ induces a real structure $\hat{\sigma}$ on $\hat{Z}$ with $\hat{\bar{S}} = \sigma(\hat{S})$.

Now we put more restictions on φ. The natural map $\mu_S : \tilde{S} \to S$ is nothing but the blowing-up at the point p.

If we put $E = \mu_S^{-1}(p)$, then we have $E = \tilde{S} \cap Q_2$, and E is a fiber of the first projection defined on $Q_2 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. On the other hand, since $\tilde{L} := \tilde{S} \cap Q_1$ is mapped isomorphically onto L_1 by μ_S, it is a fiber of the second projection. The situation is the same for $\bar{E} := \sigma(E) = \tilde{\bar{S}} \cap Q_2$ and $\tilde{\bar{L}} =: \sigma(\tilde{L}) = \tilde{\bar{S}} \cap Q_1$. Hence, we may impose on φ the additional condition that $\varphi(L) = E, \ \varphi(\bar{L}) = \bar{E}$. (Namely such a φ exists.)

With this assumed, we now consider the Kuranishi family of the pair of the complex spaces $(\hat{Z}, \hat{S} \coprod \hat{\bar{S}})$

$$\{f : (\mathcal{Z}, \mathcal{S} \coprod \bar{\mathcal{S}}) \to T, o \in T, (Z_o, S_o \coprod \bar{S}_o) \cong (\hat{Z}, \hat{S} \coprod \hat{\tilde{\bar{S}}})\},$$

where $\mathcal{Z} \to T$ and $\mathcal{S} \coprod \bar{\mathcal{S}} \to T$ are proper flat holomorphic maps and Z_o, S_o and $\bar{S}_o$ are the fibers over o. Then as an analogue of [3] we obtain the following:

Theorem 1. *Suppose that $H^2(Z, \Theta(-\log(S \cup \bar{S})) = 0$, where $\Theta(-\log(S \cup \bar{S}))$ is the sheaf of holomorphic vector fields on Z which are tangent to both S and $\bar{S}$ (which is locally free since the intersection $S \cap \bar{S}$ is transversal). Then T is nonsingular, and for a general point of T the corresponding fiber Z_t, S_t and $\bar{S}_t$ are all nonsingular. Moreover, if t is a "real point", Z_t is the twistor space associated to some anti-self-dual metric g over the self-connected sum M, of which S_t is an elementary surface of Type 1. Hence in particular Z_t becomes a twistor space associated to some anti-self-dual hermitian metric on S_t, which is a surface of class VII.*

Example . Suppose that $M = S^4$ and $Z = \boldsymbol{P}^3$ as in Example of 1.2. We may take S

to be a plane $\boldsymbol{P}^2$ containing L_∞. Then $\tilde{S}$ becomes the Hirzebruch surface Σ_1 of degree 1. Then if we choose φ suitably, in the above smoothing $\hat{S} \to S_t$, S_t is isomorphic to the Hopf surface S_r. Thus Z_t turns out to be the twistor space of the Hopf surface S_r.

4 How to obtain Inoue surfaces

We would like to consider the case where the surface S_t of class VII surface obtained in Theorem 1 becomes one of the Inoue surfaces mentioned in §2. In the notation of 3.3, the curves $\tilde{L}$ and E in $\tilde{S}$ have selfintersection numbers $+1$ and -1 respectively. Hence $\hat{S}$ is obtained from the rational surface $\tilde{S}$ by identifying a $(+1)$-curve and a (-1)-curve which are mutually disjoint in it, and moreover, S_t is obtained as a smoothing of $\hat{S}$. The deformation $\Sigma_1 \to S_r$ in the above example coincides with the famouse example by Kodaira [9]. More generally a general method to obtain a surface of class VII from a rational surface has been developped by Nakamura [11], [13]. More precisely, let us call a rational surface $\tilde{S}$ *admissible* when it contains mutually disjoint nonsingular rational curves C_+ and C_- with $C_+^2 = +1$ and $C_-^2 = -1$. Then given an admissible surface as above, by a small deformation of the non-normal rational surface $\hat{S}$ with a double curve C ($\cong C_+ \cong C_-$) obtained by identifying C_+ and C_- via some isomorphism $\psi : C_+ \to C_-$ in $\tilde{S}$ one obtains a surface of class VII containing a global spherical shell. Conversely, every such surface is obtained from some admissible rational surface in this way. In particular starting from a toric rational surface and using the method of toric degeneration Nakamura [12] constructed Inoue surfaces. Here, with the relation to twistor spaces in mind, we work in a slightly different formulation.

Let S be a nonsingular toric rational surface. In particular S admits an effective algebraic $\boldsymbol{C}^{*2}$-action. If U is the open orbit, its complement $C := S - U$ is a cycle of nonsingular rational curves. Write this as $C = C_1 \cup \ldots \cup C_k, C_i \cong \boldsymbol{P}^1$. Let $p_i = C_i \cap C_{i+1}, 1 \le i \le k-1, C_k \cap C_1 = p_k$ be the intersection points of the irreducible components of C. We assume the following:

(A) Among C_j there exists an irreducible component C_i with $C_i^2 = 1$.

We may then assume that $C_1^2 = 1$ changing the numbering if necessary. Let $\nu : \tilde{S} \to S$ be the blowing-up of the point $p_j \in S$ $(j \neq 1, k)$ and $E = \nu^{-1}(p_j)$ the exceptional (-1)-curve. Then $\tilde{S}$ is again a toric surface, and clearly is admissible in the sense defined above. Denote by $\tilde{C}_i$ the proper transform of C_i in $\tilde{S}$.

Now we take an isomorphism $\psi : \tilde{C}_1 \to E$ such that

$$a)\psi(p_1) = p_{j-1},\ \psi(p_k) = p_j \quad \text{or} \quad b)\ \psi(p_1) = p_j,\ \psi(p_k) = p_{j-1}.$$

By ψ we identify C_1 and E and obtain from $\tilde{S}$ a non-normal surface $\hat{S} := S/\psi$. If we denote by $\hat{C}_i$ the image of $\tilde{C}_i$ in $\hat{S}$, $\hat{S}$ contains two disjoint cycles $\hat{C}_2 \cup \ldots \cup \hat{C}_{j-1}$ and

$\hat{C}_{j+1} \cup \ldots \cup \hat{C}_{k-1}$ of nonsingular rational curves. We denote the union of these two cycles by $\hat{C}$, which is Cartier divisor on $\hat{S}$. Then the following hold.

Proposition 2. *Consider the Kuranishi family* $(h : (\mathcal{S}, \mathcal{C}) \to T,\ o \in T,\ (S_o, C_o) = (\hat{S}, \hat{C}))$ *of the pair* $(\hat{S}, \hat{C})$ *of the complex surface* $\hat{S}$ *and the curve* $\hat{C}$ *on it. Then the following hold:*

1) *dim* $T = 1$ *and* T *is nonsingular.*
2) *For* $t \neq o$ $\hat{S}_t$ *is nonsingular.*
3) *If we assume further that*
   (B) *if* $C_l^2 = -1$, *then* $l = j$ *or* $j = \pm 1$,

then the following hold.

In Case (a):

(1) *If* $j = 2$ *or* $k - 1$, $S_t, t \neq o$, *is a parabolic Inoue surface.*

(2) *Otherwise* S_t, $t \neq o$, *is a hyperbolic Inoue surface. Then* S_t, $t \neq o$, *are all isomorphic. Further any hyperbolic Inoue surface is obtained from a toric surface by the above method.*

In Case (b): $S_t, t \neq o$, *is a half Inoue surface. Then* $S_t, t \neq o$, *are all isomorphic. Moreover, every half Inoue surface is obtained from some toric surface by the above method.*

Remark. In the general case where (B) does not hold true, S_t is a non-minimal surface obtained from an Inoue surface by blowing-up some nodes of cycle of nonsingular rational curves.

5 Joyce twistor space

By Proposition 2 and Theorem 1, the remaining task for the proof of the main theorem is to find for any toric surface S satisifyng the condition (A) a twistor space Z which contains S as an elementary surface of Type 1 and for which we have $H^2(Z, \Theta(-\log(S \cup \bar{S})) = 0$. In fact In this case in the course of the construction $(Z, S) \to (\hat{Z}, \hat{S})$ the construction $S \to \hat{S}$ is precisely as in the previous section, the vanishing of the cohomology groups implies that the smoothing $\hat{S} \to S_t$ in Proposition 2 is extendible to the smoothing $\hat{Z} \to Z_t$ of Z_t with respect to the inclusion $\hat{S} \subseteq \hat{Z}$. Hence Main Theorem follows from Theorem 1.

Let S be a toric surface satisifying the condition (A). We take as Z a Joyce twistor space. First we shall explain this. Let $M = m\boldsymbol{P}^2 := \boldsymbol{P}^2 \# \ldots \# \boldsymbol{P}^2, m \geq 0$, be the connected sum of m copies of complex projective plane $\boldsymbol{P}^2$, where we understand $0\boldsymbol{P}^2 = S^4$. We consider the two-dimensional torus $K := S^1 \times S^1$ as a compact Lie group and fix a smooth action on M. (Each time we fix m there are up to diffeomorphisms only a finite number of choices.) Let $m' = \max(m, 1)$. Then the following hold.

Theorem (Joyce [7]) There exists K-invariant self-dual structures $[g] = [g]_s$ on M depending on m'-dimensional continuous real parameter.

Denote by $\bar{M}$ the manifold M with oritentation reversed. $[g]$ is then an anti-self-dual structure on $\bar{M}$. We call any of the corresponding twistor spaces Z a *Joyce twistor space*. Z then admits the induced bi-holomorphic action of $G := \boldsymbol{C}^{*2}$. The structure of Z has been studied in detail in [5]. From the results in [**?**] it is not difficult to deduce 1) of the following theorem.

Theorem 3. Let S be a toric surface satisifying the condition (A). Set $m = b_2(S) - 1$. Then there exists a unique smooth K-action on $m\boldsymbol{P}^2$ such that

1) any of the associated Joyce twistor spaces Z contains S as an elementary surface of Type 1, and

2) $H^2(Z, \Theta(-\log(S \cup \bar{S})) = 0$.

The vanishing result in 2) holds in general for any Moishezon twistor space Z and an elementary surface S contained in it (which is necessarily of Type 1). In fact, the latter follows easily from the vanishing theorem $H^2(Z, \Theta(-(S+\bar{S})) = 0$ shown under this general assumption in [6].

Remark. 1) The above description actually is a bit simplified, and for a rigorous treatment it is necessary to consider the deformation of the triple $(\hat{Z}, \hat{S} \coprod \bar{\hat{S}}, \hat{C} \coprod \bar{\hat{C}})$ of complex spaces studied by Honda in [4].

2) By the above method we can show that anti-self-dual hermitian metrics exist even on many surfaces of class VII other than Inoue surfaces

3) Joyce twistor spaces are known to have rich structures and from this a number of interesting problems arise; for instance it seems quite likely that the anti-self-dual metrics constructed by our method coincides with those constructed by LeBrun [10] (cf. (2.3, 2)).

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