NonClassical Constructions of Analytic Cohomology

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There are two classical constructions of analytic cohomology, namely Čech and Dolbeault. In some applications, however (for example, concerned with the Penrose transform and certain representations), it is convenient to use some nonclassical constructions.

Consider, for example, the cohomology of $\mathbb{C}^n \setminus \mathbb{R}^n$. For Čech cohomology, it is best to use a Stein covering. There are many choices of such coverings and, in this particular case though not at all in general, finite Stein coverings are available. Such finite coverings, however, are not so symmetric. Perhaps the most natural Stein covering is by tubes of over the general half-space:–

$$Z_\xi = \{x + iy \in \mathbb{C}^n \text{ s.t. } \langle \xi, y \rangle > 0\} \subset \mathbb{C}^n \setminus \mathbb{R}^n.$$

Certainly, one could use this cover to construct Čech cohomology in the usual manner but this construction ignores the fact that the parameter space,

$$\Xi = S^{n-1} = \{\xi \in \mathbb{R}^n \text{ s.t. } |\xi| = 1\}$$

for this covering, is a smooth manifold rather than just a discrete set. Instead, in the spirit of de Rham cohomology versus simplicial cohomology, one might expect to construct a complex of differential forms

$$\omega(z, \xi, d\xi), \text{ a smooth form in } \xi \text{ depending holomorphically on } z$$

with differential being exterior derivative in ξ and, having done this, find that $H^r(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O})$ is realised as the r^{th} cohomology of this complex. This is the construction suggested in [4]. Our aim is to make precise and generalise this construction but, before doing so, it is worth remarking that this realisation of $H^{n-1}(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O})$ is well adapted to Sato's theory of hyperfunctions [7]. In this theory, suitably well-behaved functions f on $\mathbb{R}^n$ are represented inside $H^{n-1}(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O})$. Writing the Fourier inversion formula in polar coördinates,

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\langle \xi, x \rangle} d\xi = \int_{\xi \in S^{n-1}} \left(\int_{r=0}^{\infty} \hat{f}(r\xi) e^{ir\langle \xi, x \rangle} r^{n-1} dr \right) d\Omega$$

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where $d\Omega$ is the volume form on the unit sphere, suggests replacing x by $z = x + iy$ in the integrand of this expression, to obtain

$$\omega(z, \xi, d\xi) = \left(\int_{r=0}^{\infty} \hat{f}(r\xi) e^{-r\langle \xi, y \rangle + ir\langle \xi, x \rangle} r^{n-1} dr \right) d\Omega$$

as a representative differential form. Certainly, this expression is holomorphic in z provided $\langle \xi, y \rangle > 0$.

This prototype, though not yet well-formulated, suggests a construction of the analytic cohomology of a complex manifold Z based on a double fibration

$$\begin{array}{ccccc} & & M & & \\ & \eta \swarrow & & \searrow \tau & \\ Z & & & & \Xi \end{array}$$

in which the fibres of τ are Stein manifolds. It turns out, however, that the parameter space Ξ is unnecessary in the general formulation, as follows.

A mixed manifold M of type (m, n) is a smooth manifold of dimension $m+2n$ equipped with a Levi flat CR structure of codimension m. In other words, M is equipped with a foliation of dimension $2n$ with smoothly varying complex structure on the leaves. These are manifolds locally modelled on $\mathbb{R}^m \times \mathbb{C}^n$ with transition functions of the form

$$(t, z) \mapsto (s(t), w(t, z)),$$

where $w(t, z)$ is holomorphic in z. Partially holomorphic functions on a mixed manifold are smooth functions whose restriction to the leaves of the foliation are holomorphic. Write Œ for the sheaf of germs of partially holomorphic functions. The partially holomorphic hull $\widehat{K}$ of a compact subset $K \subset M$ is defined to be

$$\widehat{K} = \{x \in M \text{ s.t. } |f(x)| \leq \sup_K |f| \ \ \forall f \in \Gamma(M, \text{Œ})\}.$$

A Cartan manifold in the sense of Jurchescu is a mixed manifold satisfying the following three conditions:–

- if $K \subset M$ is compact, then so is $\widehat{K}$;
- the partially holomorphic functions on M separate points;
- the partially holomorphic functions on M provide local coördinates.

In particular, a Cartan manifold of type $(0, n)$ is a Stein manifold.

Suppose Z is a complex manifold whose cohomology $H^r(Z, \mathcal{O})$ we wish to describe. Suppose we have a Cartan manifold M and a partially holomorphic submersion $\eta : M \to Z$ with contractible fibres.

Theorem There is a complex of sheaves $\text{Œ}(B^\bullet)$ on M so that

$$H^r(Z, \mathcal{O}) \cong H^r(\Gamma(M, \text{Œ}(B^\bullet))).$$

If we take M to be Z as a smooth manifold, then we arrive at Dolbeault cohomology. At the other extreme, we may always take M to be Stein: this is the holomorphic realisation of [2]. The smoothly parameterised Čech cohomology of [3] and occurring in our prototype is another special case.

To formulate this theorem more precisely, let Λ^1_M denote the bundle of complex-valued 1-forms on M and $\Lambda^{1,0}_M$ denote the sub-bundle annihilating the $(0, 1)$-vectors of the CR structure. Then, the ingredients of the proof may be gathered into the following commutative diagram of vector bundles on M with exact rows and columns.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & \eta^*\Lambda^{0,1}_Z & \to & \Lambda^{0,1}_M & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \to & \eta^*\Lambda^1_Z & \to & \Lambda^1_M & \to & \Lambda^1_\eta & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \eta^*\Lambda^{1,0}_Z & \to & \Lambda^{1,0}_M & \to & B^1 & \to & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

In particular, $\eta^*\Lambda^1_Z \hookrightarrow \Lambda^1_M$ is precisely that η is a submersion and the middle row defines Λ^1_η. That η is partially holomorphic then gives $\eta^*\Lambda^{1,0}_Z \hookrightarrow \Lambda^{1,0}_M$ and the bottom row defines B^1. The theorem is a consequence of spectral sequences obtained from this diagram. One of these collapses owing to a theorem of Buchdahl [1]. It allows us to deduce a spectral sequence

$$E_1^{p,q} = H^q(M, \text{Œ}(B^p)) \Longrightarrow H^{p+q}(Z, \mathcal{O})$$

computing the analytic cohomology of Z. When M is a Cartan manifold, Jurchescu's vanishing theorem [5] implies that

$$H^q(M, \text{Œ}(B^p)) = 0 \quad \text{for } q \geq 1$$

and the spectral sequence collapses to the isomorphism stated in the theorem.

As another example of this theorem, we may construct the cohomology of

$$Z = \{[z_1, z_2, z_3] \in \mathbb{CP}_2 \text{ s.t. } |z_1|^2 + |z_2|^2 > |z_3|^2\}$$

in terms of the complex manifold

$$M = \{(z, \zeta) \in \mathbb{CP}_2 \text{ s.t. } z \neq \zeta \text{ and the line joining them lies entirely in } Z\}.$$

It is shown in [2] that M is Stein and $\eta : M \to Z$ defined by $(z, \zeta) \mapsto z$ is a submersion with contractible fibres. Notice that the natural group action of $\mathrm{SU}(2,1)$ on Z lifts to an action on M. It is shown in [2] that this is quite a general phenomenon: representations on cohomology such as $H^1(Z, \mathcal{O})$ can often be expressed in a purely holomorphic language. This language is well adapted to the Penrose transform. One may easily deduce, for example, the isomorphism

$$H^1(Z, \mathcal{O}(-2)) \xrightarrow{\cong} \Gamma(W, \mathcal{O}(-1))$$

where

$$W = \{[w_1, w_2, w_3] \in \mathbb{CP}_2^* \text{ s.t. } |w_1|^2 + |w_2|^2 < |w_3|^2\}.$$

This isomorphism may be regarded as either a simple example of the Penrose transform, a complex analogue of the Radon transform, or an instance of projective duality due to Martineau [6].

As a final example, consider

$$Z = \{z = x + iy \in \mathbb{C}^3 \text{ s.t. } {y_1}^2 + {y_2}^2 > {y_3}^2\},$$

a tube over a non-convex cone in $\mathbb{R}^3$. Let

$$M = \{(\theta, z) \in S^1 \times Z \text{ s.t. } |y_1 \cos\theta + y_2 \sin\theta| > |y_3|\}$$

with $\eta : M \to Z$ given by $\eta(\theta, z) = z$. It is easily verified that M is a Cartan manifold of type $(1,3)$. The typical fibre of η is a semicircle, therefore contractible. The complex $\Gamma(M, \text{Œ}(B^\bullet))$ is

$$\Gamma(M, \text{Œ}(B^0)) \ni f \longmapsto \frac{\partial f}{\partial \theta}\, d\theta \in \Gamma(M, \text{Œ}(B^1))$$

where f is a smooth function of $(\theta, z) \in M$ holomorphic in z. This suggests that elements of $H^1(Z, \mathcal{O})$ should have boundary values on $\mathbb{R}^3$, formally given by boundary values of $\omega(\theta, z)\, d\theta$ integrated over $\theta \in S^1$. For a construction of $H^r(Z, \mathcal{O})$ better suited to the natural action of $\mathrm{SO}(2,1)$, we may take

$$M = \left\{ (\theta, \phi, x + iy) \in S^1 \times S^1 \times \mathbb{C}^3 \ \text{ s.t. } \begin{array}{l} y_1 \cos\theta + y_2 \sin\theta > y_3, \\ y_1 \cos\phi + y_2 \sin\phi > -y_3 \end{array} \right\}.$$

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