Logarithmic Surfaces and Hyperbolicity

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We first introduce some notations: Let $\bar{X}$ be an algebraic surface, and let $D \subset \bar{X}$ be a divisor with simple normal crossings only. We call the pair $(\bar{X}, D)$ a log surface and denote $X = \bar{X} \setminus D$. Let $T^*\bar{X}$ be its cotangent bundle and $\bar{T}^*X$ its log cotangent bundle. We denote $q_{\bar{X}} = \dim_{\mathbf{C}} H^0(\bar{X}, T^*\bar{X})$ its irregularity and $\bar{q}_X = \dim_{\mathbf{C}} H^0(\bar{X}, \bar{T}^*X)$ its log irregularity. It is called of log general type if its log canonical bundle $\bar{K}X = \bigwedge^2 \bar{T}^*X$ is big. Finally let $\alpha_X : X \to \mathcal{A}_X$ be the quasi-Albanese map. It is a holomorphic map which extends to a rational map $\bar{\alpha}_X : \bar{X} \to \bar{\mathcal{A}}_X$ (Iitaka '76 [8]).

It is known that for any log surface such that $\bar{q}_X > 2$, any entire holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerated (i.e. contained in a proper algebraic subvariety of $\bar{X}$). More generally, by results of Noguchi '81 [10] and Noguchi-Winkelmann '02 [11] one has:

Theorem 1 (Noguchi, Noguchi-Winkelmann) *Let $\bar{X}$ be a compact Kähler manifold and D be a hypersurface in $\bar{X}$. If $\bar{q}_X > \dim \bar{X}$, then entire holomorphic curve $f : \mathbf{C} \to X$ is analytically degenerated.*

In the following we are interested in the case of log surfaces $(\bar{X}, D)$ with $\bar{q}_X = 2$.

Theorem 2 *Let $(\bar{X}, D)$ a log surface of log general type with log irregularity $\bar{q}_X = 2$. Furthermore, in the case $q_{\bar{X}} = 0$, suppose that $\bar{\alpha}_X : \bar{X} \to \bar{\mathcal{A}}_X$ is a morphism or that D has at most 3 irreducible components. Then any entire holomorphic Brody curve $f : \mathbf{C} \to X$ is algebraically degenerated.*

Corollary 3 *If X doesn't contain any non-hyperbolic algebraic curve and if D is hyperbolically stratified (i.e. every irreducible component of D minus all the others is a hyperbolic curve), then X is complete hyperbolic and hyperbolically imbedded.*

Remark 4 *It is very likely that the last condition in the theorem is not needed. In fact, without this condition one still gets in the case* $q_{\bar{X}} = 0$ *that*

$$\mathrm{rank}_{\mathbb{Z}}\mathrm{NS}(\bar{X}) \geq \mathrm{rank}_{\mathbb{Z}}\{c_1(D_i)\}_{i=1}^{k} = k-2$$

where $D_1, ..., D_k$ *are the irreducible components of* D *and* $NS(\bar{X})$ *denotes the Neron-Severi group of* $\bar{X}$*. This is easily obtained from Hodge theory due to Deligne [3] and from Catanese '84 [2]. But there does not seem to be an easy way to profit from this, unless one assumes some bounds on this Neron-Severi group of* $\bar{X}$*.*

Remark 5 *However, the condition that the log surface is of log general type is essential: Let* $\bar{X} = (\mathbf{P}_1)^2$ *and* $X = \bar{X} \setminus D = (\mathbf{C}^*)^2$*. Then* $\bar{q}_X = 2$ *and* $\bar{\alpha}_X : (\mathbf{P}_1)^2 \to (\mathbf{P}_1)^2$ *is the identity, but* $\bar{K}X = 0$ *is trivial. The curve*

$$f : \mathbf{C} \to (\mathbf{C}^*)^2; t \to (e^t, e^{at}) \text{ with } a \in \mathbb{Q} \setminus \mathrm{R}$$

is Brody w.r.t. the Fubini-Study metric of $(\mathbf{P}_1)^2$*, but not algebraically degenerated.*

Remark 6 *Much is known already about the hyperbolicity of particular log surfaces, e.g. about log tori or log surfaces having small Neron-Severi groups, under the additional condition that all irreducible components* D_i *of* D *are ample (Noguchi-Winkelmann '02 [11]). The case where* $\bar{X}$ *is the projective plane is particularily well understood (e.g. Demailly-ElGoul [4], [7]).*

In the following we sketch a proof of Theorem 2.

Basic Proof Idea of Theorem 2: We prove that under the conditions of the theorem $f'(\mathbf{C})$ is a leaf of a log foliation of $\bar{X}$, i.e. there exists a hypersurface $Y_1 \subset \mathbf{P}(\bar{T}X)$ which is generically one to one over $\bar{X}$, such that

$$\mathbf{P}(f')(\mathbf{C}) \subset Y_1 \subset \mathbf{P}(\bar{T}X) \tag{1}$$

This is easy to prove in the case that $q_{\bar{X}} = 2$, which includes the compact case (i.e. if $D = \emptyset$), but not so easy to obtain in the non-compact case because of the possible presence of points of indeterminacy of the map $\bar{\alpha}_X$. Then we conclude the proof by results of McQuillan [9] and ElGoul [7], which state that under condition (1) f is algebraically degenerated.

(I) The case $q_{\bar{X}} = 2$**:** In this case the Albanese map is a morphism. The euclidean metric of the universal cover $\mathbf{C}^2$ of the Albanese torus $\mathcal{A}_{\bar{X}}$ descends to a metric h on it. So we have:

$$\mathbf{C} \xrightarrow{f} (\bar{X}, g) \xrightarrow{\alpha_{\bar{X}}} (\mathcal{A}_{\bar{X}}, h) \leftarrow (\mathbf{C}^2, \text{eucl.})$$

Now since f is a Brody curve, we have $g(f') \leq C$, and by composing with the Albanese map, we get

$$h((\alpha_{\bar{X}} \circ f)') \leq C'$$

After lifting to $\mathbf{C}^2$, the components of $(\alpha_{\bar{X}} \circ f)'$ are bounded holomorphic functions. Hence, by Liouville's theorem, they are constant. So we have

$$\mathbf{P}((\alpha_{\bar{X}} \circ f)') \subset \mathbf{C}^2 \times [v_0] \subset \mathbf{C}^2 \times \mathbf{P}(\mathbf{C}^2) = \mathbf{P}(T\mathbf{C}^2) \tag{2}$$

By trivializing $T\mathcal{A}_{\bar{X}}$ with the trivialization obtained by the one of $T\mathbf{C}^2$, we obtain from (2):

$$\begin{array}{ccccc}
Y_1 & & \mathcal{A}_{\bar{X}} \times [v_0] & & \mathbf{C}^2 \times [v_0] \\
\cap & & \cap & & \cap \\
\mathbf{P}(T\bar{X}) & \overset{\mathbf{P}(\alpha'_{\bar{X}})}{- \rightarrow} & \mathbf{P}(T\mathcal{A}_{\bar{X}}) & \leftarrow & \mathbf{P}(T\mathbf{C}^2) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{X} & \overset{\alpha_{\bar{X}}}{\rightarrow} & \mathcal{A}_{\bar{X}} & \leftarrow & \mathbf{C}^2
\end{array}$$

Here $Y_1 \subset \mathbf{P}(T\bar{X})$ is the pull back of the (linear) foliation $\mathcal{A}_{\bar{X}} \times [v_0]$ by the dominant rational map $\mathbf{P}(\alpha'_{\bar{X}})$, hence it is itself a foliation s.th. $\mathbf{P}(f')(\mathbf{C}) \subset Y_1$. $\square$

(II) The case where $\bar{\alpha}_X : \bar{X} \to \bar{\mathcal{A}}_X$ is a morphism: The compactification $\bar{\mathcal{A}}_X$ of the quasi Albanese torus involves projective spaces. In order to obtain the linearity of the differential of the map $\bar{\alpha}_X \circ f$ we need the following result due to Bertheloot-Duval '01 [1]:

Theorem 7 (Bertheloot-Duval) *A Brody curve $f : \mathbf{C} \to (\mathbf{C}^*)^t \subset \mathbf{P}_t$ is linear w.r.t. the coordinates coming from the universal cover $\mathbf{C}^t \to (\mathbf{C}^*)^t$.*

This result suffices to treat the case $q_{\bar{X}} = 0$**:** In this case we have $\mathcal{A}_X = (\mathbf{C}^*)^2$ and $\bar{\mathcal{A}}_X = (\mathbf{P}_1)^2$ or $= \mathbf{P}_2$ (where we have to distinguish these two possibilities since there is no dominant morphism between these two different compactifications). As in (I), we have that $(\alpha_{\bar{X}} \circ f)'$ is bounded w.r.t. the Fubini Study metric on $\bar{\mathcal{A}}_X$. After lifting to the universal cover $\mathbf{C}^2 \to (\mathbf{C}^*)^2$ and by Theorem 7, the components of $\alpha_{\bar{X}} \circ f$ are linear. Hence, the components of $(\alpha_{\bar{X}} \circ f)'$ are constant.

Next, we treat the case $q_{\bar{X}} = 1$: Then $\bar{\mathcal{A}}_X$ is a $\mathbf{P}_1$-bundle over the torus $\mathcal{A}_{\bar{X}}$, and we cannot apply Theorem 7 directly. But we have the following result due to Noguchi-Winkelmann '02 [11], which states that the transition functions of the $\mathbf{P}^1$-bundle $\bar{\mathcal{A}}_X$ can be chosen to be isometries w.r.t. the Fubini-Study metric on $\mathbf{P}^1$:

Proposition 8 (Noguchi-Winkelmann) *There exists a metric h on $\bar{\mathcal{A}}_X$ s.th. the universal cover*

$$(\mathbf{C} \times \mathbf{P}_1, \text{eucl.} \times \text{FS}) \to (\bar{\mathcal{A}}_X, h)$$

is a local isometry.

As in (I), $(\alpha_{\bar{X}} \circ f)'$ is bounded w.r.t. h. So after lifting to $\mathbf{C} \times \mathbf{P}_1$, we get that the components of $(\alpha_{\bar{X}} \circ f)'$ are holomorphic and bounded (for the $\mathbf{C}$-factor) resp. constant (for the $\mathbf{P}^1$-factor w.r.t. to the universal cover $\mathbf{C} \to \mathbf{C}^*$, again by Theorem 7). Hence, they are constant, w.r.t. the coordinates on the universal cover $\mathbf{C}^2 \to \bar{\mathcal{A}}_X$.

We conclude as in (I), replacing the tangent bundles by the log tangent bundles, taking into account that the coordinates of the universal cover $\mathbf{C}^2 \to (\mathbf{C}^*)^2$ trivialize the log tangent bundle $\bar{T}\mathcal{A}_X$ (see e.g. Dethloff-Lu '01 [5], where this was shown more generally in the case of jet bundles). $\square$

(III) The case $q_{\bar{X}} = 1$: We have the following diagram:

$$\begin{array}{ccccc} \mathbf{C} & \xrightarrow{f} & \bar{X} & \overset{\bar{\alpha}_X}{-\!\!\!\rightarrow} & \bar{\mathcal{A}}_X \\ & & & \overset{\alpha_{\bar{X}}}{\searrow} & \downarrow \\ & & & & \mathcal{A}_{\bar{X}} \end{array}$$

Like in (I), we get that $\alpha_{\bar{X}} \circ f$ is linear w.r.t. the coordinates from the universal covering $\mathbf{C} \to \mathcal{A}_{\bar{X}}$. Denote $\Phi : \mathbf{C} \to \mathbf{C}^* \subset \mathbf{P}_1$ the fiber component of the map $\bar{\alpha}_X \circ f$, i.e. (after lifting to the universal cover $\mathbf{C} \times \mathbf{P}^1 \to \bar{\mathcal{A}}_X$) we have $\bar{\alpha}_X \circ f = (\alpha_{\bar{X}} \circ f, \phi)$.

Let $I \subset D \subset \bar{X}$ be the (finite) set of points of indeterminacy of $\bar{\alpha}_X$, $U \subset \mathcal{A}_{\bar{X}}$ a small neighborhood of $\alpha_{\bar{X}}(I)$, $V = \alpha_{\bar{X}}^{-1}(U)$ and $W = f^{-1}(V)$.
$\bar{\alpha}_X$ is a morphism on the compact set $\bar{X} \setminus V$, so $(\bar{\alpha}_X \circ f)'$ is bounded on $\mathbf{C} \setminus W$. As in (II), we get: After lifting to the universal cover $(\mathbf{C} \times \mathbf{P}_1) \to \bar{\mathcal{A}}_X$, the differential Φ' is holomorphic and bounded on $\mathbf{C} \setminus W$. But we cannot use any more Theorem 7. Hence, our strategy will be to estimate Φ' not pointwise, but in the integral mean, and then to get the claim by a generalization of a proof of Theorem 7 for the special case $\mathbf{P}^1$.

Brody curves are of order ≤ 2 (in the sense of Value Distribution Theory). Rational maps preserve order ≤ 2 (see e.g. Dethloff-Schumacher-Wong '95 [6]). Hence, $\Phi : \mathbf{C} \to \mathbf{C}^* \subset \mathbf{P}_1$ is of order ≤ 2, so $\Phi(z) = [1 : e^{P(z)}]$ with $\deg P \leq 2$. If $\deg P \leq 1$, we finish the proof as before. Assume $\deg P = 2$. We get, by a linear coordinate change in $\mathbf{C}$, that

$$P(z) = a_0 + ib_0 + z^2 \ , \quad a_0, b_0 \in \mathrm{R}$$

Since $U \subset \mathcal{A}_{\bar{X}}$ is a small neighborhood of the finite number of points $\alpha_{\bar{X}}(I)$ and the map $\alpha_{\bar{X}} \circ f$ is *linear*, so it reproduces the fundamental domains of $\bar{\mathcal{A}}_X$ with constant dilation, only, there exists a sequence on the diagonal

$$(z_v = x_v + ix_v)_{v \to \infty} \subset \mathbf{C} \setminus W \text{ with } x_v \to \infty$$

We have

$$|\Phi'|_{FS}(z) = \frac{|2z| e^{a_0 + x^2 - y^2}}{1 + e^{2(a_0 + x^2 - y^2)}}$$

and hence, since Φ' is bounded on $\mathbf{C} \setminus W$:

$$C \geq |\Phi'|_{FS}(z_v) = \frac{|2z_v| e^{a_0}}{1 + e^{2a_0}} \to \infty$$

This is a contradiction. $\square$

(IV) The case $q_{\bar{X}} = 0$ and $\mathrm{pr}_1 \circ \bar{\alpha}_X$ a morphism: This means

$$\begin{array}{ccccc} \mathbf{C} & \stackrel{f}{\to} & \bar{X} & \stackrel{\bar{\alpha}_X}{-\to} & (\mathbf{P}_1)^2 \\ & & & \searrow & \downarrow{\scriptstyle pr_1} \\ & & & & \mathbf{P}_1 \end{array}$$

As in (II), $\mathrm{pr}_1 \circ \bar{\alpha}_X$ is linear w.r.t. the coordinates coming from the universal cover $\mathbf{C} \to \mathbf{C}^*$. Now we ague as in (III). $\square$

(V) The case $q_{\bar{X}} = 0$ and $D = D_1 ... D_k$ with $k \leq 3$: Our proof idea is as follows: We observe that by a suitable choice of the basis of $H^0(\bar{X}, \bar{T}^* X)$, either

$$\bar{\alpha}_X : \bar{X} \to \mathbf{P}_2$$

becomes a morphism or

$$\mathrm{pr}_1 \circ \bar{\alpha}_X : \bar{X} \to \mathbf{P}_1$$

becomes a morphism. In the first case the proof is finished by (II), in the second case it is finished by (IV).

Proposition 9 (e.g. Noguchi-Winkelmann '02, [11]) *We can choose a basis $\omega_1, \omega_2 \in H^0(\bar{X}, \bar{T}^*X)$ s.th. for all irreducible component D_j of D there exist $a_{1j}, a_{2j} \in \mathbb{Z}$ s.th. a_{ij} is the residue of ω_j along D_i. The matrix obtained like that has rank 2:*

$$\begin{array}{ccccc} & D_1 & D_2 & \dots & D_k \\ \omega_1 & a_{11} & a_{12} & \dots & a_{1k} \\ \omega_2 & a_{21} & a_{22} & \dots & a_{2k} \end{array} \tag{3}$$

Let $\phi_i : X \to \mathbf{C}^*$ be the component of α_X corresponding to ω_i. It extends to a rational function $\bar{\phi}_i : \bar{X} \to \mathbf{P}_1$. Let e.g. $P \in D_1$ (resp. $P \in D_1 \cap D_2$) and let z_1, z_2 be local coordiantes around P s.th. $D_1 = \{z_1 = 0\}$ (resp. $D_1 = \{z_1 = 0\}$ and $D_2 = \{z_2 = 0\}$). Then there exists a holomorphic function $h : U(P) \to \mathbf{C}^*$ on a neighborhood $U(P)$ of P s.th.

$$\bar{\phi}_i(z_1, z_2) = z_1^{a_{i1}} h(z_1, z_2)$$

$$(\text{resp. } \bar{\phi}_i(z_1, z_2) = z_1^{a_{i1}} z_2^{a_{i2}} h(z_1, z_2)$$

Proposition 10 .
i) (see Noguchi-Winkelmann '02 [11]) $\bar{\phi}_i : \bar{X} \to \mathbf{P}_1$ is a morphism outside the points of intersection of the irreducible components of D.

ii) $\bar{\phi}_i : \bar{X} \to \mathbf{P}_1$ is a morphism iff $a_{ij_1} a_{ij_2} \geq 0$ for all irreducible components D_{j_1}, D_{j_2} s.th. $D_{j_1} \cap D_{j_2} \neq \emptyset$. This is certainly the case if in the matrix (3) all entries of the i-th line have the same sign.

iii) We add a third line to the matrix (3): For $j = 1, ..., k$, let $a_{3j} = \min(0, a_{1j}, a_{2j})$:

$$\begin{array}{ccccc} & D_1 & D_2 & \dots & D_k \\ \omega_1 & a_{11} & a_{12} & \dots & a_{1k} \\ \omega_2 & a_{21} & a_{22} & \dots & a_{2k} \\ & a_{31} & a_{32} & \dots & a_{3k} \end{array} \tag{4}$$

Then the map $\alpha_X = (\phi_1, \phi_2) : X \to (\mathbf{C}^*)^2$ extends to a morphism to $\mathbf{P}_2$ iff for any couple of irreducible components D_{j_1}, D_{j_2} s.th. $D_{j_1} \cap D_{j_2} \neq \emptyset$, the 3×2 matrix which consists of the j_1-th and the j_2-th column of the matrix (4) has the following property: Its third line is either identically zero or it is equal to the first or it is equal to the second line. This is certainly the case if this is true for any couple of indices (j_1, j_2).

By a change of basis over $\mathbb{Z}$, we always can obtain one of the cases (ii) or (iii) of Proposition 10. □

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