COMPACT KÄHLER SURFACES

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This article is a reasonably close approximation to the talk I gave at the Hayama Symposium 2002. References have been added, and typographical and other errors have hopefully been removed.

Although the subject of this talk is that of compact Kähler surfaces, it should be admitted that I am not an expert in this field and my mathematical background is more closely tied to *gauge theory*. I was drawn into the area of compact complex surfaces through the work of Simon Donaldson. Recall that in the 1980's he introduced some invariants of the differentiable structures of smooth oriented 4-manifolds, and when the 4-manifold in question happened to be an algebraic surface, he showed how the invariants could be "easily" calculated using algebraic geometry.

My initial interest was in extending the ideas to Kähler surfaces rather than just algebraic surfaces, but as Donaldson pointed out, there is a theorem of Kodaira which states that every Kähler surface is a deformation of an algebraic surface, so from the differentiable viewpoint there is nothing to be gained. Consequently I turned my attention to the non-Kähler case, but that is another story.

I just mentioned the eminent Japanese mathematician Kunihiko Kodaira, whose name is intimately tied to the development of the theory of compact complex surfaces, a subject on which he worked largely in the 1960's. One of the other main areas of Kodaira's work was in the area of *deformations* of compact complex manifolds, and much of his work in this area was achieved during the 1950's. (It should be added that Kodaira received his Fields Medal in 1954, *before* much of his work on deformation theory and on complex surfaces, which reflects his breadth and prowess as a mathematician.)

Kodaira proved a large number of important results concerning compact surfaces, and many of these can all be bracketed together to form one of the crowning achievements of twentieth century mathematics, namely the *classification* of compact complex surfaces. Although I will only be addressing the question of classification of complex manifolds in a very superficial way, it is in fact a sub-theme of this talk and provides much of the underlying motivation for it. This I shall now endeavour to explain.

Recall that classification of compact Riemann surfaces is classical, dating back to the second half of the 19^{th} century. Some of the key names associated with this classification are Riemann himself, Abel, and Jabobi. At a crude level, the classification is essentially just the topological classification:

κ	g	Description
$-\infty$	0	$\mathbb{P}_1$
0	1	Tori
1	> 1	"The rest"

Here g is the genus and κ is the *Kodaira dimension*: For any compact complex manifold X of dimension n with canonical bundle K_X, $\kappa = \limsup_m \frac{\ln\left(h^0(X, K_X^m)\right)}{\ln(m)}$, which takes values in $\{-\infty, 0, 1, \ldots, n\}$.

In the next dimension up, the *algebraic* compact surfaces were largely classified particularly by the Italian school of geometers which flourished around the turn of the last century, with the central figures in this case being Castelnuovo, Enriques, and Severi. The essentially complete classification—including the non-algebraic surfaces—was given by Kodaira in a series of papers in the 1960's.

At its broadest level, the classification again makes the Kodaira dimension the most fundamental distinguishing invariant. In outline, the scheme is as follows: Every compact complex surface has a minimal model (the latter being a surface with no exceptional curves), and this model is unique if $\kappa \geq 0$. The mimimal surfaces are described by

κ	b_1 even	b_1 odd
$-\infty$	Rational, ruled	Class VII
0	$K3$, Enriques, Tori, Hyperelliptic	Kodaira
1	Elliptic	
2	General type ("the rest")	

For compact complex manifolds of dimension greater than two, at least in the algebraic category there is something of a similar picture emerging. In this case much progress has been made by Mori and his coworkers, particularly for the case of algebraic 3-manifolds, for which the classification is sometimes cited in the literature as "nearly complete".

For compact complex manifolds which are not algebraic however, the situation very unclear. Indeed, the following theorem of Taubes from 1992 shows just how far from a good understanding we are:

Theorem. [Taubes.] *If X is a smooth compact oriented 4-manifold, the connected sum $X\#n\overline{\mathbb{P}}_2$ admits a metric with anti-self-dual Weyl curvature for n sufficiently large.*

The twistor space of an anti-self-dual 4-manifold Y is a complex 3-manifold fibred over $\overline{Y}$ by $\mathbb{P}_1$'s. Donaldson's invariants are not affected by taking connected sums with $\overline{\mathbb{P}}_2$ (as opposed to $S^2 \times S^2$ for example), so it can be seen that distinct diffeomorphism classes of smooth 4-

manifolds give rise to (collections of) holomorphically distinct complex 3-manifolds. As Taubes once put it at a talk describing his result:

> *The classification of complex* 3*-manifolds is at least as complicated as the classification of smooth* 4*-manifolds.*

In spite of this somewhat challenging outlook, in recent years there has been much progress at least in the case of compact *Kähler* manifolds using techniques from differential geometry and hard analysis (as opposed to strictly algebraic geometry), and there are signs that we shall arrive at a good understanding of such manifolds in the not-to-distant future. One of the leaders in this field is Professor J.-P. Demailly, who has also spoken at this symposium and whose talk provides an excellent illustration of some of these recent advances.

It is usually true in mathematics that in order to understand the higher-dimensional case one should first fully understand the lower dimensional situation, and in the past few years there have been several results shedding light on aspects of the theory of compact complex surfaces. The primary goal of my talk is to describe these results, with the underlying aim of seeing them in the broader context of classification of compact complex manifolds of any dimension.

It has always been a great pleasure to read the papers of Kodaira, conveniently collected together in his three-volume Collected Works. Although most of the papers were written 40 or more years ago, they have a very contemporary feel to them—this is surely a testament to the deep influence Kodaira has had on modern mathematics. In particular, Kodaira was a leader in using analytical and differential-geometric techniques (for example, in his embedding theorem) and whenever the opportunity presented itself, he made much use of Kähler metrics in his calculations.

It is not true that every compact complex surface admits a Kähler metric: For any compact complex manifold X which does admit such a metric, the Hodge identities imply that $b_q(X)$ must be even for each odd q. This is not a sufficient condition to guarantee the existence of a Kähler metric—for example, $S^3 \times S^3$ admits integrable complex structures but cannot be Kähler.

However, for complex surfaces, Kodaira conjectured in his book with Morrow that this condition should be sufficient; i.e., that a compact complex surface with even first Betti number should admit a Kähler metric; ($b_3 = b_1$ by duality).

This conjecture attracted a great deal of attention over the years until it was finally settled in the affirmative in 1983. The proof was achieved on a case-by-case treatment using classification: the difficult cases are those of elliptic surfaces and $K3$ surfaces, with the former being solved by Miyaoka in 1974 and the latter by Siu in 1983. It should also be noted that a second proof of the case of elliptic surfaces appeared that year given by Harvey and Lawson, this proof using completely different and innovative methods, specifically the theory *currents*.

In 1999, two independent short proofs of conjecture not invoking classification appeared, one due to author and other to A. Lamari:

Theorem. *A compact complex surface with even first Betti number admits a Kähler metric.*

Although the two proofs are at first sight quite different, at heart both made essential use of results of Demailly from 1992 on the smoothing of positive closed $(1,1)$-currents.

Given a surface X with $b_1(X)$ even, it is of considerable interest to know which classes in $H^2(X,\mathbb{R}) \cap H^{1,1}(X)$ can represent Kähler metrics. Such a class c must satisfy $c \cdot c > 0$ and $c \cdot [D] > 0$ for any effective divisor $D \subset X$, and if $c \in H^2(X,\mathbb{Q}) \cap H^{1,1}(X)$, the Nakai-Moishezon criterion implies that these conditions are sufficient, for then a multiple of c is the first Chern class of a very ample line bundle on X.

The results of Lamari and the author actually go considerably further than just establishing the existence of Kähler metrics. They also prove a generalized form of Nakai-Moishezon criterion, doing away with condition that c be rational. Specifically:

Theorem. [Buchdahl, Lamari.] *If $c \in H^{1,1}_{\mathbb{R}}(X)$ satisfies $c \cdot c > 0$, $c \cdot [D] > 0$ for all effective $D \subset X$ and $c \cdot [\omega_0] > 0$ for some Kähler ω_0, then $c = [\omega]$ for some Kähler form ω on X.*

The proofs follow on from the proofs of the earlier result, again making use of the key results of Demailly.

Very recently, there has been a major improvement on this result, generalizing it to all dimensions. Demailly and Paun have proved the following:

Theorem. [Demailly & Paun.] *The Kähler cone of a compact Kähler manifold X is one component of the set of classes $c \in H^2(X,\mathbb{R}) \cap H^{1,1}(X)$ satisfying $\int_{D_p} c^p > 0$ for every effective $D_p \subseteq X$, $1 \leq p \leq n$.*

Based on the utility of the result in the case of surfaces (to be illustrated shortly), it can be expected that this result will have very significant consequences in a variety of settings. Some important applications have already begun to appear: A recent preprint of Huybrechts makes essential use of the result to prove a projectivity criterion for hyper-Kähler manifolds.

The result of Siu proving the existence of Kähler metrics on $K3$ surfaces actually completed another program, this one initiated by André Weil in the 1950's. Weil made a number of conjectures on $K3$ surfaces (which he attributed also to Andreotti) and which over the years stimulated an extraordinary amount of research.

To state the conjectures, it is best to adopt the same viewpoint as Weil regarding $K3$ surfaces, which I shall now outline. Recall that a $K3$ surface X is by definition a compact complex surface with $b_1(X) = 0$ and with canonical bundle K_X trivial. As such, it carries a nowhere vanishing holomorphic two-form κ which is necessarily closed and satisfies $\kappa \wedge \kappa = 0$ and $\kappa \wedge \overline{\kappa} > 0$.

Conversely, if X_0 is a compact oriented smooth 4-manifold which possesses a complex 2-form κ satisfying the conditions $\kappa \wedge \kappa = 0$, $\kappa \wedge \overline{\kappa} > 0$ and $d\kappa = 0$, the kernel of $\kappa \wedge : \Lambda^1_{\mathbb{C}} \to \Lambda^3_{\mathbb{C}}$ has rank two at each point and has 0 intersection with the complex conjugate subspace, and it therefore defines an almost complex structure. The condition that κ be closed implies that this almost complex structure is integrable, and hence defines a compact complex surface with trivial canonical bundle. Two such forms κ, κ' define biholomorphic structures if and only if there is a self-diffeomorphism $f : X_0 \to X_0$ satisfying $f^*\kappa = \lambda\, \kappa'$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.

Such a form κ defines a point in the *period domain* $\mathcal{P} = \mathbb{P}\big(\{x \in H^2(X_0,\mathbb{C}) \mid x{\cdot}x = 0, x{\cdot}\bar{x} > 0\}\big)$, mapping κ to the image of the cohomology class $[\kappa]$.

If X_0 is the smooth oriented 4-manifold underlying some $K3$ surface, the Weil-Andreotti conjectures were:

W1 All $K3$ surfaces form one connected family;

W2 All $K3$ surfaces admit Kähler metrics;

W3 The period map is surjective;

W4 The period map is 1–1. ("Torelli theorem for $K3$'s")

The affirmation of these conjectures has a history even longer and more involved than that of Kodaira's conjecture mentioned earlier: Kodaira proved W1 in 1964, and more—that they are all deformation equivalent. Conjecture W4 was proved first in the algebraic category by Piatetskii-Shapiro and Shafarevic in 1971, then in the Kähler case by Burns and Rapoport in 1975, and finally the general case was proved by Looijenga and Peters in 1981. Conjecture W3 was proved first in the algebraic case by Kulikov in 1977, then by Todorov in the general case in 1980. The proof of W2 was completed by Siu in 1983, as already mentioned.

Todorov's proof of the surjectivity of the period map made very elegant use of Yau's (then) recently-proved results on the Calabi conjecture. Using Ricci-flat Kähler metrics, he was able to make isometric deformations of the complex structure; i.e., varying the complex structure but keeping the same (compatible) metric. Complications arose from the possibility of non-Kähler $K3$ surfaces.

When fore-armed with the knowledge that every $K3$ surface admits a Kähler metric—and indeed, of precisely which classes are Kähler classes, the proof of the surjectivity of the period map is very simple, boiling down to a couple of paragraphs (still using the isometric deformations). This then makes it a simple matter to show that every $K3$ surface is deformation equivalent to a particular Kummer surface, a special type of $K3$ surface. Combined with the identification of the Kähler classes, it also leads to a reasonably simple proof of the Torelli theorem for $K3$ surfaces. That is,

Theorem. *All the conjectures of Weil and Andreotti are true (in refined form).*

The "refined form" of the conjectures refers specifically to conjectures W3 and W4, which can be sharpened to include statements about Kähler classes. Weil's original conjecture W4 stated that two structures κ, κ' defining the same period point should be related by a self-diffeomorphism of X_0 which is homotopic to the identity.

It turns out that this is *not* always the case, and there are some subtleties involving the determination of the image of the self-diffeomorphism group in the automorphism group of $H^2(X_0, \mathbb{Z})$ preserving the cup product. Weil was aware of the difficulties and asked if the group G of such isometries coincided with the group $\overline{G}$ of all isometries, a fact which would be true by a result of Wall if X_0 could be written as the connected sum of $S^2 \times S^2$ with another smooth manifold.

In 1986 it was shown by Matumoto and independently by Borcea that that the group G contains all isometries in a certain index two subroup G_+ of $\overline{G}$, but it was not until 1990 that G was finally identified when Donaldson used gauge theory to prove that G in fact coincides with G_+.

Early in this talk I mentioned one of Kodaira theorems stating that every compact surface admitting a Kähler metric is a deformation of an algebraic surface. Kodaira proved this by using classification together with his proof of the deformation equivalence of $K3$ surfaces and also his classification of elliptic surfaces.

My results and those of A. Lamari appeared in 1999 but were proved earlier. Lamari announced some of his results at a conference in honour of François Norguet in Paris in 1998, and, as was reported to me by a colleague, Siu asked the question whether the same techniques could be applied to deduce Kodaira's result on deforming Kähler surfaces into algebraic surfaces (to which the response was negative, at that time at least).

A Kähler metric is essentially the same thing as a positive closed $(1,1)$-form, and if this form defines an integral cohomology class, the form can be viewed as the curvature of a connection on a line bundle which, by the Kodaira Embedding theorem, is ample.

The same statement obviously applies when the cohomology class is rational rather than integral, simply by multiplying by a sufficiently large integer. Since the rationals are dense in the reals, it is then reasonable to hope that the complex structure of an arbitrary compact Kähler manifold should be deformable into an algebraic manifold—indeed, that the Kähler manifold should have arbitrarily small deformations which are algebraic.

Question: *Does every compact Kähler manifold have (arbitrarily close) deformations which are algebraic?*

This question does not seem to have received much attention in the literature, although it is clearly a very important question from the point of classification theory. Obviously every compact Riemann surface is algebraic, and in addition to the result mentioned above, Kodaira also proved in 1964 that every compact surface with even first Betti number is a deformation of an algebraic surface (again using a case-by-case approach).

A result which is related to this question is due to Tjurin in 1965, proving that the algebraic deformations of an algebraic manifold X lie on a countable union of analytic sets in the deformation spaces, each of codimension at most $h^{2,0}(X)$. The paper of Demailly and Paun already mentioned also contains some results on the behaviour of the Kähler cone under deformations.

A Kähler manifold (X,ω) can always be viewed as a smooth oriented Riemannian manifold (X_0,g) together with a compatible symplectic form ω which is covariantly constant. Any closed form ω' near ω will be symplectic, and one can look for the condition that it should be covariantly constant with respect to some compatible metric g' near g.

By a theorem of Moser, if ω' is cohomologous and sufficiently close to ω, there is a self-diffeomorphism $f\colon X_0 \to X_0$ such that $f^*\omega = \omega'$, so for such forms ω' it *is* possible to find such metrics.

At the infinitesimal level, some straight-forward tensor analysis reveals that the question is equivalent to the following: The closed form $\omega \in \Lambda^{1,1}(X)$ defines a class in $H^1(X,\Omega^1)$, this corresponding to a (non-split) extension $0 \to \mathcal{O} \to \mathcal{S} \to \Theta_X \to 0$. If the connecting homomorphism $H^1(X,\Theta_X) \to H^2(X,\mathcal{O})$ is surjective, it will follow from the Implicit Function Theorem that X has arbitrarily nearby deformations which are algebraic.

Unfortunately, it is easy to see that this homomorphism is not always surjective; for example, take X to be the product of two Riemann surfaces both of genus at least 5. In spite of this however, not all is lost, for the failure of this homomorphism to be surfective implies the existence of forms on X with some particular properties. In the case that X is a complex surface, using the kinds of methods employed to prove the results described earlier, it can be shown that if the homomorphism is not surjective, X must in fact already be algebraic. Hence:

Theorem. *Every compact Kähler surface has arbitrarily nearby algebraic deformations.*

Although the technique does not immediately generalise to higher dimensions, there are certainly a number of aspects of the analysis which are dimension-independent and there are some grounds for believing that it may be possible to prove the higher-dimensional version of this last theorem.

References

This list of references is only intended to cover all the papers mentioned explicitly in the talk. Many other authors made major contributions to the development of the ideas presented but were not cited directly in the talk. The bibliography in the book of Barth, Peters and Van de Ven is very comprehensive.

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