On \( CR \) manifolds with constant Levi-signature

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1 Introduction

Our paper is motivated by the problem of the nonexistence of compact Levi-flat \( CR \) manifolds in complex projective spaces. The nonexistence of smooth Levi-flat real hypersurfaces in \( \mathbb{CP}^n, n \geq 3 \), was proved in [S2]. Here the Levi-flatness of \( M \) means that \( M \) is foliated by local complex manifolds of dimension \( n - 1 \). Later, Siu proved that the same statement also holds true for \( n = 2 \) [S3].

The interest in such a nonexistence result was motivated by problems in dynamical systems in \( \mathbb{CP}^n \) (see [CLS]). Indeed, the well known theorem of Poincaré-Bendixson says the following: Consider a foliation on \( \mathbb{RP}^2 \), defined by a polynomial differential equation on \( \mathbb{R}^2 \). Then the closure of each leaf of the foliation contains either a compact leaf or a singularity of the foliation. It is therefore natural to ask whether the same holds true for holomorphic foliations on \( \mathbb{CP}^2 \). A more general problem is the following: Does there exist a holomorphic foliation \( \mathcal{F} \) on \( \mathbb{CP}^n \) which admits a minimal set \( M \) (i.e. \( M \) is compact in \( \mathbb{CP}^n \setminus \text{Sing}(\mathcal{F}) \), \( \mathcal{F} \)-invariant and minimal with respect to the inclusion \( \subset \))? The connection between the existence of minimal sets and Levi-flat hypersurfaces was investigated in [C] and [LN].

A natural problem seems to extend Siu’s result to higher codimensional Levi-flat \( CR \) manifolds, and our main result is the following

**Theorem 1.1**

In \( \mathbb{CP}^n \), there exists no smooth Levi-flat \( CR \) manifold \( M \) with real codimension \( n - s \) and \( CR \) dimension \( s \geq 2 \) such that the determinant \( \mathbb{C} \)-line bundle of \( N^{1,0}_{M,X} \) is smoothly trivial.

Here the Levi-flatness of \( M \) means that \( M \) is foliated by local complex manifolds of dimension \( s \). Following [S2], the main idea of the proof of Theorem 1.1 is to get a contradiction to the existence of such a Levi-flat \( M \) from the positivity of the \( \mathbb{C} \)-line bundle \( \det(N^{1,0}_{M,X}) \). Indeed, the main goal is to show that the curvature of \( \det(N^{1,0}_{M,X}) \) is equal to \( \partial \bar{\partial} \psi \) restricted to each local holomorphic leaf of the foliation of \( M \), where \( \psi \) is a smooth function on \( M \). A contradiction occurs at a point where \( \psi \) achieves its maximum.
For this, one needs to prove a result on the solvability of the tangential Cauchy-Riemann equation on $M$ in degree $(0,1)$, namely: Let $f$ be a smooth $(0,1)$-form along the holomorphic foliation of $M$, such that $\overline{\partial} f = 0$ on each local holomorphic leaf of the foliation of $M$. Then there exists a smooth function $u$ on $M$ satisfying $\overline{\partial} u = f$ on each local holomorphic leaf of the foliation of $M$.

Our method to prove this fact is to extend $f$ to a smooth $\overline{\partial}$-closed $(0,1)$-form on $X$ by means of $L^2$-estimates on $X \setminus M$. Our $L^2$-estimates involve weights equal to some high negative power of the distance function to $M$. The main difficulty comes from the fact that when the real codimension of $M$ is greater than 1, the eigenvalues of the Levi-form of $-\log(\text{distance to } M)$ go to $-\infty$ of order 2 in the directions normal to $M$ as the point approaches $M$. In the good directions tangential to $M$, however, the sum of an appropriate number of eigenvalues has a positive lower bound as the point approaches $M$ (due to the nondegeneracy condition of $X$). In order to set up good $L^2$-spaces, we must therefore construct a hermitian metric on $X \setminus M$ making the negative eigenvalues very small. This metric, however, will not be Kähler; thus we must be able to control the torsion of this metric. Another difficulty is due to the fact that the positivity of sufficiently many eigenvalues of the Levi-form of $-\log(\text{distance to } M)$ holds only at points very close to $M$, making it impossible to get good $L^2$-estimates on an exceptional compact set of $X \setminus M$. We overcome this difficulty by invoking Ohsawa’s method of pseudo-Runge pairs. In this paper, we have only sketched the ideas of the proofs; the details can be found in [B2].

Finally we mention that our method breaks down when the $CR$ dimension of $M$ is equal to 1. However, as another byproduct of our analysis, we can generalize the nonexistence statement in Theorem 1.1 to certain non necessarily Levi-flat $CR$ submanifolds in $\mathbb{C}^p$ (admitting at least one Levi-flat direction). As a further application of our methods, we prove the validity of the Hartogs phenomenon in certain weakly 2-convex-concave $CR$ submanifolds of Stein manifolds (thereby generalizing the results of [B1]).

### 2 Basic definitions

For tangent vectors $\sigma$ and $\tau$ of a Kähler manifold $X$ of complex dimension $n$ the bisectional curvature of $\sigma$ and $\tau$ is

$$R(\sigma, \tau, \tau, \sigma) + R(\sigma, J\tau, J\tau, \sigma),$$

where $R(\cdot, \cdot, \cdot, \cdot)$ is the Riemann curvature tensor and $J$ is the complex structure operator of the tangent bundle $TX$ of $X$. When $\sigma = 2\text{Re}\xi$ and $\tau = 2\text{Re}\eta$
for tangent vectors $\xi = \sum \xi^\alpha \frac{\partial}{\partial z^\alpha}$ and $\eta = \sum \eta^\alpha \frac{\partial}{\partial z^\alpha}$, the bisectional curvature of $\sigma$ and $\tau$ is equal to

$$4 \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\beta\gamma\delta}\xi^{\alpha}\overline{\xi}^{\beta}\eta^{\gamma}\overline{\eta}^{\delta},$$

where $R_{\alpha\beta\gamma\delta}$ are the components of the Riemann curvature tensor with respect to the local holomorphic coordinates $z^1, \ldots, z^n$ of $X$. This leads to the following concept of nondegeneracy, which was introduced in [S1].

**Definition 2.1** The bisectional curvature of a Kähler manifold $(X, \omega)$ is said to be $s$-nondegenerate if the following holds: If $k$ is a positive integer and $\xi^{(1)}, \ldots, \xi^{(k)}$ are $\mathbb{C}$-linearly independent tangent vectors of $X$ of type $(1,0)$ and $\eta$ is a nonzero tangent vector of $X$ of type $(1,0)$ such that

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\beta\gamma\delta}\xi^{\alpha}_{(\mu)}\overline{\xi}^{\beta}_{(\mu)}\eta^{\gamma}\overline{\eta}^{\delta} = 0$$

for $1 \leq \mu \leq k$, then $k \leq s - 1$.

### 3 $L^2$-estimates

An important ingredient in our method is the estimation the eigenvalues of the Levi-form of $-\log(\text{distance to } M)$, where $M$ is a Levi-flat submanifold. When the real codimension of $M$ is greater than 1, in the directions normal to $M$ the eigenvalues go to $-\infty$ of order 2 as the point approaches $M$. In the good directions tangential to $M$, however, the sum of at least $(s - 1)$ eigenvalues has a positive lower bound as the point approaches $M$.

**Proposition 3.1**

Let $X$ be an irreducible compact Hermitian symmetric manifold of complex dimension $n$ whose bisectional curvature is $(s - 1)$-nondegenerate. Let $M$ be a smooth Levi-flat CR submanifold of $X$ of real codimension $n - s$ and of CR dimension $s$. Let $\omega$ be the standard Kähler form on $X$, and let $d_M(z)$ be the distance of $z \in X \setminus M$ to $M$ with respect to $\omega$, defined in a neighborhood of $M$ in $X$. Let $\lambda_1 \leq \ldots \leq \lambda_n$ denote the eigenvalues of $\partial\overline{\partial}(\log d_M)$ with respect to $\omega$. Then there exist a compact subset $K$ of $X \setminus M$ and constants $c, c' > 0$ such that

(a) $-\frac{c}{d_M^s} \leq \lambda_1 \leq \ldots \leq \lambda_{n-s-1} \leq -\frac{c'}{d_M^s}$,

(b) $0 \leq \lambda_{n-s} \leq \ldots \leq \lambda_{n-1} \leq c'$, $\lambda_{n-s} + \ldots + \lambda_{n-2} \geq c$,

(c) $\frac{c}{d_M^s} \leq \lambda_n \leq \frac{c'}{d_M^s}$
at every point of $X \setminus (M \cup K)$.

The following technical lemma permits to obtain $L^2$-vanishing theorems on $X \setminus M$ with powers of the boundary distance as weight functions. We recall that $K$ will be the compact of $X \setminus M$ as in Lemma 3.1.

**Lemma 3.2**

There exists a hermitian metric $\omega_M$ on $X \setminus M$ with the following properties:

(i) Let $\gamma_1 \leq \ldots \leq \gamma_n$ be the eigenvalues of $i\overline{\partial}(-\log d_M)$ with respect to $\omega_M$. There exists $\sigma > 0$ such that $\gamma_1 + \ldots + \gamma_{n-2} > \sigma$ on $X \setminus K$.

(ii) There are constants $a, b > 0$ such that $a \omega \leq \omega_M \leq b d_M^{-2} \omega$.

(iii) There is a constant $C > 0$ such that $|\partial \omega_M|_{\omega_M} \leq C$.

Let $K$ be the compact as in Lemma 3.1. Fix $\alpha > 0$ such that $K \subset \{z \in X \setminus M \mid d_M(z) > \alpha\}$ and set $X_1 = \{z \in X \setminus M \mid d_M(z) > \alpha\}$, $X_2 = X \setminus M$.

**Lemma 3.3**

There exists a complete hermitian metric $\omega_o$ on $X_1$, a weight function $\varphi_o$ on $X_1$, a sequence of complete hermitian metrics $\omega_k$ ($k = 1, 2, \ldots$) on $X_2$ and a sequence of weight functions $\varphi_k$ on $X_2$ satisfying the following properties:

(i) $\omega_k$ (resp. $\varphi_k$) converges uniformly with all its derivatives to $\omega_o$ (resp. $\varphi_o$) on every compact subset of $X_1$ as $k \to +\infty$.

(ii) $\omega_k \leq \omega_{k+1}$, $\varphi_k \leq \varphi_{k+1}$ for every $k = 1, 2, \ldots$ and $\omega_k \leq \omega_o$, $\varphi_k \leq \varphi_o$ on $X_1$.

(iii) Let $\gamma_k^1 \leq \ldots \leq \gamma_k^n$ be the eigenvalues of $i\overline{\partial}(-\log d_M)$ with respect to $\omega_k$. There exists $\tau > 0$ independent of $k$ such that $\gamma_k^1 + \ldots + \gamma_k^{n-2} > \tau$ on $X_2 \setminus X_1$.

(iv) Let $\xi_k^1 \leq \ldots \leq \xi_k^n$ be the eigenvalues of $i\overline{\partial}\varphi_k$ with respect to $\omega_k$. Then there is a constant $A > 0$ independent of $k$ such that $\xi_k^1 + \ldots + \xi_k^{n-2} - 2|\partial \varphi_k|_{\omega_k} \geq -A$ on $X_2$.

(v) For every $k$ there are constants $a, b > 0$ such that $a \omega \leq \omega_k \leq b d_M^{-2} \omega$ and $a \leq e^{\varphi_k} \leq b d_M^2$.

Let $X_1, X_2$ and $\omega_o, \omega_k$ be as in the preceding lemma, and let $N \in \mathbb{N}$. We denote by $L^2_{p,q}(X_2, N, \omega_k)$ the Hilbert space of $(p, q)$-forms $u$ on $X_2$ which satisfy

$$\|u\|_{N,k}^2 := \int_{X_2} |u|_{\omega_k}^2 d_M^N e^{-\varphi_k} dV_{\omega_k} < +\infty$$
We denote by $\ll, \gg_{N,k}$ the global inner product on forms of $L^2_{p,q}(X_2, N, \omega_k)$ and by $\overline{\mathcal{D}}_{N,k}$ the Hilbert space adjoint of $\overline{\mathcal{D}}$ with respect to $\ll, \gg_{N,k}$. Analogously we let $L^2_{p,q}(X_1, N, \omega_o)$ be the Hilbert space of $(p,q)$-forms $u$ on $X_1$ which satisfy
\[ \|u\|_{N,o}^2 := \int_{X_1} |u|_{\omega_o}^2 d_M e^{-\varphi_o} dV_{\omega_o} < +\infty \]

The global inner product $\ll, \gg_{N,o}$ and the Hilbert space adjoint $\overline{\mathcal{D}}_{N,o}$ are defined as before.

We observe that the curvature of the metric $d_M^N \exp(-\varphi_k) = \exp(-\{-N \log d_M + \varphi_k\})$ is equal to $-N \partial \overline{\mathcal{D}} \log d_M + \partial \overline{\mathcal{D}} \varphi_k$. Then it follows from the generalized Bochner-Kodaira-Nakano inequality (see [O], [D1]) and standard computations from [D1] that
\[ \frac{3}{2} (\|\nabla u\|_{N,k}^2 + \|\overline{\mathcal{D}}_{N,k} u\|_{N,k}^2) \geq N \ll (\gamma_1^k + \ldots + \gamma_q^k)u, u \gg_{N,k} \]
+ $\ll (c_1^k + \ldots + c_q^k)u, u \gg_{N,k} - \frac{1}{2} (\|\tau_k u\|_{N,k}^2 + \|\overline{\mathcal{D}}_{N,k} u\|_{N,k}^2 + \|\tau_k u\|_{N,k}^2 + \|\overline{\mathcal{D}}_{N,k} u\|_{N,k}^2)$
for every smooth $(n,q)$-form $u$ with compact support in $X_2$, where $\Lambda_k$ is the adjoint of multiplication by $\omega_k$ and $\tau_k = [\Lambda_k, \partial \omega_k]$. Let $K'$ be any compact of $X_1$ containing $K$ in its interior. Using (iii) and (iv) of Lemma 3.3, we then have the following basic estimate:
There exist $A', N_o > 0$ such that for every $N \geq N_o$ and every $k \in \mathbb{N}$ we have
\[ \|u\|_{N,k}^2 \leq \|\nabla u\|_{N,k}^2 + \|\overline{\mathcal{D}}_{N,k} u\|_{N,k}^2 + A'' \int_{K'} |u|_{\omega_k}^2 \frac{1}{d_M^N} dV_{\omega_k} \]
for every smooth $(n,q)$-form $u$ with compact support in $X_2$, $q \geq n - 2$. Due to the completeness of the metrics $\omega_k$, this estimate remains true for forms $u \in \text{Dom}(\overline{\mathcal{D}}) \cap \text{Dom}(\overline{\mathcal{D}}_{N,k})$. Combining this basic estimate (which is uniform in $k$) with (i) and (ii) of Lemma 3.3, the pair $(X_1, X_2)$ satisfies the conditions of [O, Definition 2.1], i.e. $(X_1, X_2)$ is a pseudo-Runge pair at bidegree $(n,q)$, $q \geq n - 2 + 1$. The following result then follows immediately from [O, Lemma 2.2].

**Lemma 3.4**
For every $N \geq N_o$ there exists an integer $k_o$ and a constant $C > 0$ such that for any $k \geq k_o$
\[ \|u\|_{N,k}^2 \leq C (\|\nabla u\|_{N,k}^2 + \|\overline{\mathcal{D}}_{N,k} u\|_{N,k}^2) \]
for any $u \in L^2_{n,q}(X_2, N, \omega_k)$ satisfying $u|_{X_1} \perp \ker \overline{\mathcal{D}} \cap \ker \overline{\mathcal{D}}_{N,o}^*$, $q \geq n - 2$. 

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Lemma 3.4 implies the following theorem.

**Theorem 3.5**

For every \( N \geq N_0 \) there exists an integer \( k_0 \) such that for any \( k \geq k_0 \) the following holds: Let \( f \) be a \((0,q)\)-form on \( X_2 \) with \( q \leq 2 \),

\[
\int_{X_2} |f|^2 \frac{1}{\omega_k} d_N^{\omega_k} dV_{\omega_k} < +\infty
\]

and \( f|_{X_1} \equiv 0 \). Then there exists a \((0,q-1)\)-form \( u \) on \( X_2 \) such that \( \bar{\partial} u = f \) in the sense of distributions and

\[
\int_{X_2} |u|^2 \frac{1}{\omega_k} d_N^{\omega_k} dV_{\omega_k} < +\infty
\]

### 4 Nonexistence of Levi-flat CR manifolds

**Theorem 4.1**

Let \( X \) be an irreducible compact Hermitian symmetric manifold of complex dimension \( n \) whose bisectional curvature is \((s-1)\)-nondegenerate. Assume that \( M \) is a smooth Levi-flat CR manifold of real codimension \( n-s \) and CR dimension \( s \geq 2 \). Let \( f \in \Lambda^1(T^{0,1}M)^* \) satisfy \( \bar{\partial} f = 0 \) restricted to each leaf of the foliation of \( M \), and let \( k \in \mathbb{N} \). Then there exists \( u \in \mathcal{C}^k(M) \) satisfying \( \bar{\partial} u = f \) on each leaf of the foliation of \( M \).

**Proof.** Let \( f \) be as in the theorem. It follows from [S2, Lemma 5.2] that there exists a smooth \((0,1)\)-form \( \tilde{f} \) on \( X \) which extends \( f \) as a class (in the sense that the restriction of \( \tilde{f} \) to any leaf of the foliation of \( M \) agrees with \( f + \bar{\partial} h \) for some smooth function \( h \) on \( M \)) such that \( \bar{\partial} \tilde{f} \) vanishes to infinite order along \( M \).

Keeping the notations of Theorem 3.5, it is no loss of generality to assume that \( \bar{\partial} \tilde{f}|_{X_1} \equiv 0 \). Then Theorem 3.5 yields a \((0,1)\)-form \( g \) on \( X \setminus M \) satisfying \( \bar{\partial} g = \bar{\partial} \tilde{f} \) and

\[
\int_{X \setminus M} |g|^2 \frac{1}{d_M^m} < +\infty, \; m \gg 1.
\]

Set \( F = \tilde{f} - g \in L^2_{0,1}(X) \). It is then easy to prove that we have \( \bar{\partial} F = 0 \) on \( X \).

Let \( G \) be the Green operator on \( X \). We set \( U = \bar{\partial}^* GF \). Due, for example, to the simple connectedness of \( X \) we have \( H^{0,1}(X) = 0 \). Since \( \bar{\partial} F = 0 \), this implies that \( \bar{\partial} U = F \) (by basic properties of the Green operator \( G \)). Set \( u = U|_M \). Then one can show that \( u \) belongs to \( \mathcal{C}^k(M) \) (if \( m \) is sufficiently big). We claim that we have \( \bar{\partial} u = f \) on each leaf of the foliation of \( M \).
Indeed, if $L_z$ is the holomorphic leaf of the foliation of $M$ passing through $z \in M$ and if $(\overline{\partial} u - f)(z) \neq 0$, there exists a neighborhood $V_z$ of $z$ and a constant $C > 0$ such that $|g|^2 \geq C$ on $V_z$ (recall that $\overline{\partial} U = F = f - g$). But this implies $|g| d^{m}_{M} \geq C d^{m}_{M} \notin L^{2}(V_z)$ if $2m \geq n - s + 1$, and we obtain a contradiction. \hfill $\square$

We can now prove the following conjecture by Siu [S2].

**Theorem 4.1**

Let $X$ be an irreducible compact Hermitian symmetric manifold of complex dimension $n$ whose bisectional curvature is $(s - 1)$-nondegenerate. Then in $X$ there exists no smooth Levi-flat CR manifold $M$ with real codimension $n - s$ and CR dimension $s \geq 2$, such that the determinant $\mathbb{C}$-bundle of $N^{1,0}_{M,X}$ is smoothly trivial.

**Proof.** Our arguments follow [S2]. Let $N^{1,0}_{M,X}$ be the holomorphic normal bundle of $M$ and set $L = \det N^{1,0}_{M,X}$. We endow $L$ with the metric induced from the standard Kähler form on $X$. Let $\theta = i \Theta(L)$ be the $(1,1)$-form on $M$ which is the curvature of that metric of $L$. Since $L$ is by assumption smoothly trivial over $M$, there exists a smooth real-valued 1-form $\alpha$ on the holomorphic leaves of the foliation of $M$ (i.e., $\alpha$ is a function on $T^{1,0}M \oplus T^{0,1}M$) such that $\theta = d\alpha$ as functions on $\Lambda^{2}(T^{1,0}M \oplus T^{0,1}M)$. Let $\alpha = \alpha^{1,0} + \alpha^{0,1}$ so that $\alpha^{1,0}$ is a function on $T^{1,0}M$ and $\alpha^{0,1}$ is a function on $T^{0,1}M$. From type considerations we conclude that $\overline{\partial} \alpha^{0,1} = 0$ on each local holomorphic leaf of the foliation of $M$. By Theorem 4.1 we can find a smooth function $u$ on $M$ such that $\overline{\partial} u = \alpha^{0,1}$ on each local holomorphic leaf of the foliation of $M$. Hence $\theta = \overline{\partial} \alpha^{1,0} + \partial \alpha^{1,0} = \overline{\partial} \psi$ on each local holomorphic leaf of the foliation of $M$, where $\psi = u - \overline{u}$.

Let $p_o$ be the point on $M$ where $\psi$ achieves its maximum value. Let $z_1, \ldots, z_s$ be the local holomorphic coordinates at $p_o$ of the holomorphic leaf through $p_o$ of the foliation of $M$ such that $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_s}$ are mutually orthogonal unit tangent vectors of type $(1,0)$ of $X$ at $p_o$. Furthermore let $e_1, \ldots, e_{n-s}$ be mutually orthogonal unit tangent vectors of $X$ of type $(1,0)$ at $p_o$ which are orthogonal to $T^{1,0}M$. Thus, since the bisectional curvature of $X$ is $(s - 1)$-nondegenerate (and in particular $s$-nondegenerate), it follows that

$$
\sum_{j=1}^{s} \frac{\partial^2 \psi}{\partial z_j \partial \overline{z}_j} = \sum_{j=1}^{s} \theta(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}) = \sum_{j=1}^{s} \sum_{\lambda=1}^{n-s} i \Theta(N^{1,0}_{M,X})(\frac{\partial}{\partial z_j} \otimes e_\lambda, \frac{\partial}{\partial \overline{z_j}} \otimes \overline{e}_\lambda) \\
\geq \sum_{j=1}^{s} \sum_{\lambda=1}^{n-s} i \Theta(T^{1,0}X)(\frac{\partial}{\partial z_j} \otimes e_\lambda, \frac{\partial}{\partial \overline{z_j}} \otimes \overline{e}_\lambda) > 0
$$
at $p_0$, which contradicts the maximum principle.

5 Nonexistence results for $CR$ manifolds with constant Levi-rank

Using the same methods described before, the nonexistence result in Theorem 4.1 may be generalized to non necessarily Levi-flat $CR$ manifolds.

Let $M$ be a smooth $CR$ manifold with real codimension $n - s$ and $CR$ dimension $s$. The characteristic bundle $H^oM$ is defined to be the annihilator of $\text{Re}(T^{1,0}M \oplus T^{0,1}M)$ in $T^*M$. Given $\xi \in H^o_xM$, $X \in T^{1,0}_xM$, let us choose $X' \in \Gamma(M,T^{1,0}M)$ such that $X'(x) = X$. In this way we associate to $\xi \in H^o_xM$ the Levi form at $\xi$:

$$L(\xi, X) = \xi([X', \overline{X'}]).$$

Theorem 5.1

In $\mathbb{CP}^n$ there exists no smooth $CR$ manifold $M$ with real codimension $n - s$ and $CR$ dimension $s$ such that the determinant $\mathbb{C}$-line bundle of $N^1_{M,X}$ is smoothly trivial and such that there exists $q, \ell \in \mathbb{N}$ with $2q + \ell = s$, $\ell \geq 1$, $q + \ell \geq 2$ so that for every $x \in M$ and each $\xi \in H^o_xM \setminus \{0\}$ the Levi form $L(\xi, \cdot)$ has $q$ negative, $\ell$ zero and $q$ positive eigenvalues.

In the hypersurface situation one can show a better result:

Theorem 5.2

In $\mathbb{CP}^n$ there exists no smooth real hypersurface $M$ such that the Levi form of $M$ has constant signature $(p^-, p^o, p^+)$ along $M$ with $p^- + p^o \geq 2$, $p^+ + p^o \geq 2$ and $p^o \geq 1$.

6 The Hartogs phenomenon for $CR$ manifolds with constant Levi-rank

Let $X$ be a Stein manifold of complex dimension $n$, and let $M$ be a smooth, closed $CR$ submanifold of $X$ with real codimension $n - s$ and $CR$ dimension $s$. We assume that there exist $q, \ell \in \mathbb{N}$ with $2q + \ell = s$ such that for every $x \in M$ and each $\xi \in H^o_xM \setminus \{0\}$ the Levi form $L(\xi, \cdot)$ has $q$ negative, $\ell$ zero and $q$ positive eigenvalues.

Let $D \subset X$ be a smoothly bounded, strictly pseudoconvex domain such that $M$ intersects $\partial D$ transversally. Using the same methods as described before again (see also [B1]) we can show the following:
Let $f$ be a smooth $\overline{\partial}$-closed $(0,2)$-form on $X$ such that $\text{supp} f \subset \overline{D}$ and $f|_M = 0$. Then there exists a smooth $(0,1)$-form $u$ on $X$ such that $\overline{\partial}u = f$, $\text{supp} u \subset \overline{D}$ and $u|_M = 0$.

This implies that the tangential Cauchy-Riemann cohomology group with compact support $H^{0,1}_c(M)$ is zero. As a consequence we can prove the following result on the Hartogs phenomenon in $CR$ manifolds:

**Theorem 6.1**

Let $M$ be as above. Let $K$ be a compact subset of $M$ such that $M \setminus K$ is connected and globally minimal. Then every smooth $CR$ function on $M \setminus K$ extends to a smooth $CR$ function on $M$.

The assumption of global minimality is needed in order to assure that the weak analytic continuation principle holds for $CR$ function. However, this assumption is always satisfied as long as the Levi form is not identically zero (or if $M$ is of finite bracket type).

**References**


