On CR manifolds with constant Levi-signature

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1 Introduction

Our paper is motivated by the problem of the nonexistence of compact Levi-flat CR manifolds in complex projective spaces. The nonexistence of smooth Levi-flat real hypersurfaces in $\mathbb{CP}^n$, $n \geq 3$, was proved in [S2]. Here the Levi-flatness of M means that M is foliated by local complex manifolds of dimension $n-1$. Later, Siu proved that the same statement also holds true for $n = 2$ [S3].

The interest in such a nonexistence result was motivated by problems in dynamical systems in $\mathbb{CP}^n$ (see [CLS]). Indeed, the well known theorem of Poincaré-Bendixson says the following: Consider a foliation on $\mathbb{RP}^2$, defined by a polynomial differential equation on $\mathbb{R}^2$. Then the closure of each leaf of the foliation contains either a compact leaf or a singularity of the foliation. It is therefore natural to ask whether the same holds true for holomorphic foliatons on $\mathbb{CP}^2$. A more general problem is the following: Does there exist a holomorphic foliation $\mathcal{F}$ on $\mathbb{CP}^n$ which admits a minimal set M (i.e. M is compact in $\mathbb{CP}^n \setminus \mathrm{Sing}(\mathcal{F})$, $\mathcal{F}$-invariant and minimal with respect to the inclusion $\subset$)? The connection between the existence of minimal sets and Levi-flat hypersurfaces was investigated in [C] and [LN].

A natural problem seems to extend Siu's result to higher codimensional Levi-flat CR manifolds, and our main result is the following

Theorem 1.1
In $\mathbb{CP}^n$, there exists no smooth Levi-flat CR manifold M with real codimension $n-s$ and CR dimension $s \geq 2$ such that the determinant $\mathbb{C}$-line bundle of $N^{1,0}_{M,X}$ is smoothly trivial.

Here the Levi-flatness of M means that M is foliated by local complex manifolds of dimension s. Following [S2], the main idea of the proof of Theorem 1.1 is to get a contradiction to the existence of such a Levi-flat M from the positivity of the $\mathbb{C}$-line bundle $\det(N^{1,0}_{M,X})$. Indeed, the main goal is to show that the curvature of $\det(N^{1,0}_{M,X})$ is equal to $\partial\overline{\partial}\psi$ restricted to each local holomorphic leaf of the foliation of M, where ψ is a smooth function on M. A contradiction occurs at a point where ψ achieves its maximum.

For this, one needs to prove a result on the solvability of the tangential Cauchy-Riemann equation on M in degree $(0,1)$, namely: Let f be a smooth $(0,1)$-form along the holomorphic foliation of M, such that $\overline{\partial} f = 0$ on each local holomorphic leaf of the foliation of M. Then there exists a smooth function u on M satisfying $\overline{\partial} u = f$ on each local holomorphic leaf of the foliation of M.

Our method to prove this fact is to extend f to a smooth $\overline{\partial}$-closed $(0,1)$ -form on X by means of L^2-estimates on $X \setminus M$. Our L^2- estimates involve weights equal to some high negative power of the distance function to M. The main difficulty comes from the fact that when the real codimension of M is greater than 1, the eigenvalues of the Levi-form of -log(distance to M) go to $-\infty$ of order 2 in the directions normal to M as the point approaches M. In the good directions tangential to M, however, the sum of an appropriate number of eigenvalues has a positive lower bound as the point approaches M (due to the nondegeneracy condition of X). In order to set up good L^2-spaces, we must therefore construct a hermitian metric on $X \setminus M$ making the negative eigenvalues very small. This metric, however, will not be Kähler; thus we must be able to control the torsion of this metric. Another difficulty is due to the fact that the positivity of sufficiently many eigenvalues of the Levi-form of -log(distance to M) holds only at points very close to M, making it impossible to get good L^2-estimates on an exceptional compact set of $X \setminus M$. We overcome this difficulty by invoking Ohsawa's method of pseudo-Runge pairs. In this paper, we have only sketched the ideas of the proofs; the details can be found in [B2].

Finally we mention that our method breaks down when the CR dimension of M is equal to 1. However, as another byproduct of our analysis, we can generalize the nonexistence statement in Theorem 1.1 to certain non necessarily Levi-flat CR submanifolds in $\mathbb{CP}^n$ (admitting at least one Levi-flat direction). As a further application of our methods, we prove the validity of the Hartogs phenomenon in certain weakly 2-convex-concave CR submanifolds of Stein manifolds (thereby generalizing the results of [B1]).

2 Basic definitions

For tangent vectors σ and τ of a Kähler manifold X of complex dimension n the bisectional curvature of σ and τ is

$$R(\sigma,\tau,\tau,\sigma) + R(\sigma,J\tau,J\tau,\sigma),$$

where $R(\cdot,\cdot,\cdot,\cdot)$ is the Riemann curvature tensor and J is the complex structure operator of the tangent bundle TX of X. When $\sigma = 2\mathrm{Re}\xi$ and $\tau = 2\mathrm{Re}\eta$

for tangent vectors $\xi = \sum \xi^\alpha \frac{\partial}{\partial z^\alpha}$ and $\eta = \sum \eta^\alpha \frac{\partial}{\partial z^\alpha}$, the bisectional curvature of σ and τ is equal to

$$4 \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^\alpha \overline{\xi}^\beta \eta^\gamma \overline{\eta}^\delta,$$

where $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ are the components of the Riemann curvature tensor with respect to the local holomorphic coordinates $z^1, \dots, z^n$ of X. This leads to the following concept of nondegeneracy, which was introduced in [S1].

Definition 2.1 *The bisectional curvature of a Kähler manifold (X, ω) is said to be s-nondegenerate if the following holds: If k is a positive integer and $\xi_{(1)}, \dots, \xi_{(k)}$ are $\mathbb{C}$-linearly independent tangent vectors of X of type $(1,0)$ and η is a nonzero tangent vector of X of type $(1,0)$ such that*

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^\alpha_{(\mu)} \overline{\xi}^\beta_{(\mu)} \eta^\gamma \overline{\eta}^\delta = 0$$

for $1 \le \mu \le k$, then $k \le s-1$.

3 L^2-estimates

An important ingredient in our method is the estimation the eigenvalues of the Levi-form of -log(distance to M), where M is a Levi-flat submanifold. When the real codimension of M is greater than 1, in the directions normal to M the eigenvalues go to $-\infty$ of order 2 as the point approaches M. In the good directions tangential to M, however, the sum of at least $(s-1)$ eigenvalues has a positive lower bound as the point approaches M.

Proposition 3.1
Let X be an irreducible compact Hermitian symmetric manifold of complex dimension n whose bisectional curvature is $(s-1)$-nondegenerate. Let M be a smooth Levi-flat CR submanifold of X of real codimension $n-s$ and of CR dimension s. Let ω be the standard Kähler form on X, and let $d_M(z)$ be the distance of $z \in X \setminus M$ to M with respect to ω, defined in a neighborhood of M in X. Let $\lambda_1 \le \dots \le \lambda_n$ denote the eigenvalues of $\partial\overline{\partial}(-\log d_M)$ with respect to ω. Then there exist a compact subset K of $X \setminus M$ and constants $c, c' > 0$ such that

- *(a)* $-\frac{c'}{d_M^2} \le \lambda_1 \le \dots \le \lambda_{n-s-1} \le -\frac{c}{d_M^2}$,
- *(b)* $0 \le \lambda_{n-s} \le \dots \le \lambda_{n-1} \le c'$, $\lambda_{n-s} + \dots + \lambda_{n-2} \ge c$,
- *(c)* $\frac{c}{d_M^2} \le \lambda_n \le \frac{c'}{d_M^2}$

at every point of $X \setminus (M \cup K)$.

The following technical lemma permits to obtain L^2-vanishing theorems on $X \setminus M$ with powers of the boundary distance as weight functions. We recall that K will be the compact of $X \setminus M$ as in Lemma 3.1.

Lemma 3.2
There exists a hermitian metric ω_M on $X \setminus M$ with the following properties:

(i) *Let $\gamma_1 \leq \ldots \leq \gamma_n$ be the eigenvalues of $i\partial\overline{\partial}(-\log d_M)$ with respect to ω_M. There exists $\sigma > 0$ such that $\gamma_1 + \ldots + \gamma_{n-2} > \sigma$ on $X \setminus K$.*

(ii) *There are constants $a, b > 0$ such that $a\,\omega \leq \omega_M \leq b\; d_M^{-2}\omega$.*

(iii) *There is a constant $C > 0$ such that $|\partial\omega_M|_{\omega_M} \leq C$.*

Let K be the compact as in Lemma 3.1. Fix $\alpha > 0$ such that $K \subset \{z \in X \setminus M \mid d_M(z) > \alpha\}$ and set $X_1 = \{z \in X \setminus M \mid d_M(z) > \alpha\}$, $X_2 = X \setminus M$.

Lemma 3.3
There exists a complete hermitian metric ω_o on X_1, a weight function φ_o on X_1, a sequence of complete hermitian metrics ω_k $(k = 1, 2, \ldots)$ on X_2 and a sequence of weight functions φ_k on X_2 satisfying the following properties:

(i) *ω_k (resp. φ_k) converges uniformly with all its derivatives to ω_o (resp. φ_o) on every compact subset of X_1 as $k \to +\infty$.*

(ii) *$\omega_k \leq \omega_{k+1}$, $\varphi_k \leq \varphi_{k+1}$ for every $k = 1, 2, \ldots$ and $\omega_k \leq \omega_o$, $\varphi_k \leq \varphi_o$ on X_1*

(iii) *Let $\gamma_1^k \leq \ldots \leq \gamma_n^k$ be the eigenvalues of $i\partial\overline{\partial}(-\log d_M)$ with respect to ω_k. There exists $\tau > 0$ independent of k such that $\gamma_1^k + \ldots + \gamma_{n-2}^k > \tau$ on $X_2 \setminus \overline{X}_1$.*

(iv) *Let $c_1^k \leq \ldots \leq c_n^k$ be the eigenvalues of $i\partial\overline{\partial}\varphi_k$ with respect to ω_k. Then there is a constant $A > 0$ independent of k such that $c_1^k + \ldots + c_{n-2}^k - 2|\partial\omega_k|_{\omega_k} \geq -A$ on X_2.*

(v) *For every k there are constants $a, b > 0$ such that $a\,\omega \leq \omega_k \leq b\; d_M^{-2}\omega$ and $a \leq e^{\varphi_k} \leq b\; d_M^{-2}$.*

Let X_1, X_2 and ω_o, ω_k be as in the preceding lemma, and let $N \in \mathbb{N}$. We denote by $L^2_{p,q}(X_2, N, \omega_k)$ the Hilbert space of (p,q)-forms u on X_2 which satisfy

$$\|u\|^2_{N,k} := \int_{X_2} |u|^2_{\omega_k} d_M^N e^{-\varphi_k} dV_{\omega_k} < +\infty$$

We denote by $\ll\, ,\gg_{N,k}$ the global inner product on forms of $L^2_{p,q}(X_2,N,\omega_k)$ and by $\overline{\partial}^*_{N,k}$ the Hilbert space adjoint of $\overline{\partial}$ with respect to $\ll\, ,\gg_{N,k}$. Analogously we let $L^2_{p,q}(X_1,N,\omega_o)$ be the Hilbert space of (p,q)-forms u on X_1 which satisfy

$$\|u\|^2_{N,o} := \int_{X_1} |u|^2_{\omega_o} d_M^N e^{-\varphi_o} dV_{\omega_o} < +\infty$$

The global inner product $\ll\, ,\gg_{N,o}$ and the Hilbert space adjoint $\overline{\partial}^*_{N,o}$ are defined as before.

We observe that the curvature of the metric $d_M^N \exp(-\varphi_k) = \exp(-\{-N\log d_M + \varphi_k\})$ is equal to $-N\partial\overline{\partial}\log d_M + \partial\overline{\partial}\varphi_k$. Then it follows from the generalized Bochner-Kodaira-Nakano inequality (see [O], [D1]) and standard computations from [D1] that

$$\frac{3}{2}(\|\overline{\partial}u\|^2_{N,k} + \|\overline{\partial}^*_{N,k}u\|^2_{N,k}) \ \geq\ N \ll (\gamma_1^k + \ldots + \gamma_q^k)u, u \gg_{N,k}$$

$$+ \ll (c_1^k + \ldots + c_q^k)u, u \gg_{N,k} -\frac{1}{2}(\|\tau_k u\|^2_{N,k} + \|\overline{\tau}_k u\|^2_{N,k} + \|\tau_k^* u\|^2_{N,k} + \|\overline{\tau}_k^* u\|^2_{N,k})$$

for every smooth (n,q)-form u with compact support in X_2, where Λ_k is the adjoint of multiplication by ω_k and $\tau_k = [\Lambda_k, \partial\omega_k]$. Let K' be any compact of X_1 containing K in its interior. Using (iii) and (iv) of Lemma 3.3, we then have the follwing basic estimate:
There exist $A'', N_o > 0$ such that for every $N \geq N_o$ and every $k \in \mathbb{N}$ we have

$$\|u\|^2_{N,k} \leq \|\overline{\partial}u\|^2_{N,k} + \|\overline{\partial}^*_{N,k}u\|^2_{N,k} + N\,A'' \int_{K'} |u|^2_{\omega_k} \frac{1}{d_M^N} dV_{\omega_k}$$

for every smooth (n,q)-form u with compact support in X_2, $q \geq n-2$. Due to the completeness of the metrics ω_k, this estimate remains true for forms $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_{N,k})$. Combining this basic estimate (which is uniform in k) with (i) and (ii) of Lemma 3.3, the pair (X_1, X_2) satisfies the conditions of $[\mathrm{O}, \mathrm{Definition\ 2.1}]$, i.e. (X_1, X_2) is a pseudo-Runge pair at bidegree (n,q), $q \geq n-2+1$. The following result then follows immediately from $[\mathrm{O}, \mathrm{Lemma\ 2.2}]$.

Lemma 3.4
For every $N \geq N_o$ there exists an integer k_o and a constant $C > 0$ such that for any $k \geq k_o$

$$\|u\|^2_{N,k} \leq C(\|\overline{\partial}u\|^2_{N,k} + \|\overline{\partial}^*_{N,k}u\|^2_{N,k})$$

for any $u \in L^2_{n,q}(X_2,N,\omega_k)$ satisfying $u_{|X_1} \perp \mathrm{Ker}\overline{\partial} \cap \mathrm{Ker}\overline{\partial}^*_{N,o}$, $q \geq n-2$.

Lemma 3.4 implies the following theorem.

Theorem 3.5
For every $N \geq N_o$ there exists an integer k_o such that for any $k \geq k_o$ the following holds: Let f be a $(0,q)$-form on X_2 with $q \leq 2$,

$$\int_{X_2} |f|^2_{\omega_k} \frac{1}{d_M^N} e^{\varphi_k} dV_{\omega_k} < +\infty$$

and $f_{|X_1} \equiv 0$. Then there exists a $(0,q-1)$-form u on X_2 such that $\overline{\partial} u = f$ in the sense of distributions and

$$\int_{X_2} |u|^2_{\omega_k} \frac{1}{d_M^N} e^{\varphi_k} dV_{\omega_k} < +\infty$$

4 Nonexistence of Levi-flat CR manifolds

Theorem 4.1
Let X be an irreducible compact Hermitian symmetric manifold of complex dimension n whose bisectional curvature is $(s-1)$-nondegenerate. Assume that M is a smooth Levi-flat CR manifold of real codimension $n-s$ and CR dimension $s \geq 2$. Let $f \in \Lambda^1(T^{0,1}M)^*$ satisfy $\overline{\partial} f = 0$ restricted to each leaf of the foliation of M, and let $k \in \mathbb{N}$. Then there exists $u \in \mathcal{C}^k(M)$ satisfying $\overline{\partial} u = f$ on each leaf of the foliation of M.

Proof. Let f be as in the theorem. It follows from [S2, Lemma 5.2] that there exists a smooth $(0,1)$-form $\tilde{f}$ on X which extends f as a class (in the sense that the restriction of $\tilde{f}$ to any leaf of the foliation of M agrees with $f + \overline{\partial} h$ for some smooth function h on M) such that $\overline{\partial}\tilde{f}$ vanishes to infinite order along M.

Keeping the notations of Theorem 3.5, it is no loss of generality to assume that $\overline{\partial}\tilde{f}_{|X_1} \equiv 0$. Then Theorem 3.5 yields a $(0,1)$-form g on $X \setminus M$ satisfying $\overline{\partial} g = \overline{\partial}\tilde{f}$ and

$$\int_{X \setminus M} |g|^2 \frac{1}{d_M^{2m}} < +\infty, \ m \gg 1.$$

Set $F = \tilde{f} - g \in L^2_{0,1}(X)$. It is then easy to prove that we have $\overline{\partial} F = 0$ on X. Let G be the Green operator on X. We set $U = \overline{\partial}^* G F$. Due, for example, to the simple connectedness of X we have $H^{0,1}(X) = 0$. Since $\overline{\partial} F = 0$, this implies that $\overline{\partial} U = F$ (by basic properties of the Green operator G). Set $u = U_{|M}$. Then one can show that u belongs to $\mathcal{C}^k(M)$ (if m is sufficiently big). We claim that we have $\overline{\partial} u = f$ on each leaf of the foliation of M.

Indeed, if $\mathcal{L}_z$ is the holomorphic leaf of the foliation of M passing through $z \in M$ and if $(\overline{\partial}u - f)(z) \neq 0$, there exists a neighborhood V_z of z and a constant $C > 0$ such that $|g|^2 \geq C$ on V_z (recall that $\overline{\partial}U = F = \tilde{f} - g$). But this implies $|g| d_M^{-m} \geq C\, d_M^{-m} \notin L^2(V_z)$ if $2m \geq n - s + 1$, and we obtain a contradiction. $\square$

We can now prove the following conjecture by Siu [S2].

Theorem 4.1
Let X be an irreducible compact Hermitian symmetric manifold of complex dimension n whose bisectional curvature is $(s-1)$-nondegenerate. Then in X there exists no smooth Levi-flat CR manifold M with real codimension $n-s$ and CR dimension $s \geq 2$, such that the determinant $\mathbb{C}$-line bundle of $N^{1,0}_{M,X}$ is smoothly trivial.

Proof. Our arguments follow [S2].
Let $N^{1,0}_{M,X}$ be the holomorphic normal bundle of M and set $L = \det N^{1,0}_{M,X}$. We endow L with the metric induced from the standard Kähler form on X. Let $\theta = i\Theta(L)$ be the $(1,1)$-form on M which is the curvature of that metric of L. Since L is by assumption smoothly trivial over M, there exists a smooth real-valued 1-form α on the holomorphic leaves of the foliation of M (i.e., α is a function on $T^{1,0}M \oplus T^{0,1}M$) such that $\theta = d\alpha$ as functions on $\Lambda^2(T^{1,0}M \oplus T^{0,1}M)$. Let $\alpha = \alpha^{1,0} + \alpha^{0,1}$ so that $\alpha^{1,0}$ is a function on $T^{1,0}M$ and $\alpha^{0,1}$ is a function on $T^{0,1}M$. From type considerations we conclude that $\overline{\partial}\alpha^{0,1} = 0$ on each local holomorphic leaf of the foliation of M. By Theorem 4.1 we can find a smooth function u on M such that $\overline{\partial}u = \alpha^{0,1}$ on each local holomorphic leaf of the foliation of M. Hence $\theta = \overline{\partial}\alpha^{1,0} + \partial\alpha^{1,0} = \partial\overline{\partial}\psi$ on each local holomorphic leaf of the foliation of M, where $\psi = u - \overline{u}$.

Let p_o be the point on M where ψ achieves its maximum value. Let $z_1, \dots, z_s$ be the local holomorphic coordinates at p_o of the holomorphic leaf through p_o of the foliation of M such that $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_s}$ are mutually orthogonal unit tangent vectors of type $(1,0)$ of X at p_o. Furthermore let $e_1, \dots, e_{n-s}$ be mutually orthogonal unit tangent vectors of X of type $(1,0)$ at p_o which are orthogonal to $T^{1,0}M$. Thus, since the bisectional curvature of X is $(s-1)$-nondegenerate (and in particular s-nondegenerate), it follows that

$$\begin{aligned}
\sum_{j=1}^{s} \frac{\partial^2 \psi}{\partial z_j \partial \overline{z}_j} &= \sum_{j=1}^{s} \theta\Big(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}\Big) = \sum_{j=1}^{s}\sum_{\lambda=1}^{n-s} i\Theta(N^{1,0}_{M,X})\Big(\frac{\partial}{\partial z_j} \otimes e_\lambda, \frac{\partial}{\partial \overline{z}_j} \otimes \overline{e}_\lambda\Big) \\
&\geq \sum_{j=1}^{s}\sum_{\lambda=1}^{n-s} i\Theta(T^{1,0}X)\Big(\frac{\partial}{\partial z_j} \otimes e_\lambda, \frac{\partial}{\partial \overline{z}_j} \otimes \overline{e}_\lambda\Big) > 0
\end{aligned}$$

at p_o, which contradicts the maximum principle. $\square$

5 Nonexistence results for CR manifolds with constant Levi-rank

Using the same methods described before, the nonexistence result in Theorem 4.1 may be generalized to non necessarily Levi-flat CR manifolds.

Let M be a smooth CR manifold with real codimension $n-s$ and CR dimension s. The characteristic bundle H^oM is defined to be the annihilator of $\mathrm{Re}(T^{1,0}M \oplus T^{0,1}M)$ in T^*M. Given $\xi \in H^o_xM$, $X \in T^{1,0}_xM$, let us choose $X' \in \Gamma(M, T^{1,0}M)$ such that $X'(x) = X$. In this way we associate to $\xi \in H^o_xM$ the Levi form at ξ:

$$\mathcal{L}(\xi, X) = \xi([X', \overline{X}']).$$

Theorem 5.1
In $\mathbb{CP}^n$ there exists no smooth CR manifold M with real codimension $n-s$ and CR dimension s such that the determinant $\mathbb{C}$-line bundle of $N^{1,0}_{M,X}$ is smoothly trivial and such that there exists $q, \ell \in \mathbb{N}$ with $2q + \ell = s$, $\ell \geq 1$, $q + \ell \geq 2$ so that for every $x \in M$ and each $\xi \in H^o_xM \setminus \{0\}$ the Levi form $\mathcal{L}(\xi, \cdot)$ has q negative, ℓ zero and q positive eigenvalues,

In the hypersurface situation one can show a better result:

Theorem 5.2
In $\mathbb{CP}^n$ there exists no smooth real hypersurface M such that the Levi form of M has constant signature (p^-, p^o, p^+) along M with $p^- + p^o \geq 2$, $p^+ + p^o \geq 2$ and $p^o \geq 1$.

6 The Hartogs phenomenon for CR manifolds with constant Levi-rank

Let X be a Stein manifold of complex dimension n, and let M be a smooth, closed CR submanifold of X with real codimension $n-s$ and CR dimension s. We assume that there exist $q, \ell \in \mathbb{N}$ with $2q + \ell = s$ such that for every $x \in M$ and each $\xi \in H^o_xM \setminus \{0\}$ the Levi form $\mathcal{L}(\xi, \cdot)$ has q negative, ℓ zero and q positive eigenvalues.

Let $D \subset\subset X$ be a smoothly bounded, strictly pseudoconvex domain such that M intersects ∂D transversally. Using the same methods as described before again (see also [B1]) we can show the following:

Let f be a smooth $\overline{\partial}$-closed $(0,2)$-form on X such that supp$f \subset \overline{D}$ and $f_{|M} = 0$. Then there exists a smooth $(0,1)$-form u on X such that $\overline{\partial} u = f$, supp$u \subset \overline{D}$ and $u_{|M} = 0$.
This implies that the tangential Cauchy-Riemann cohomology group with compact support $H_c^{0,1}(M)$ is zero. As a consequence we can prove the following result on the Hartogs phenomenon in CR manifolds:

Theorem 6.1
Let M be as above. Let K be a compact subset of M such that $M \setminus K$ is connected and globally minimal. Then every smooth CR function on $M \setminus K$ extends to a smooth CR function on M.

The assumption of global minimality is needed in order to assure that the weak analytic continuation principle holds for CR function. However, this assumption is always satisfied as long as the Levi form is not identically zero (or if M is of finite bracket type).

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