INVARIANT THEORY OF THE BERGMAN KERNEL OF STRICTLY PSEUDOCONVEX DOMAINS

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In the paper "Parabolic invariant theory in complex analysis [16]," Fefferman proposed a program of studying the geometry and analysis of strictly pseudoconvex domains (with C^∞ boundary). His basic idea is to consider the Bergman kernel of strictly pseudoconvex domains as an analogy of the heat kernel of Riemannian manifolds; the theory of the heat kernel based on invariant theory is well developed and applied to prove index theorems. The goal of this program is to construct "Invariant theory of the Bergman kernel" corresponding to the heat kernel case. The structure group appearing in this new setting is a parabolic subgroup of $SU(1,n)$, and this is the origin of the name "Parabolic invariant theory." In this article, we report recent developments in Fefferman's program.

For a domain $\Omega \subset \mathbb{C}^n$, the Bergman kernel $K_\Omega(z,w)$ of Ω is defined as the reproducing kernel of the Hilbert space of L^2 holomorphic functions. Its restriction to the diagonal $K_\Omega(z) = K_\Omega(z,z)$ is a smooth function on Ω. This function is also called the Bergman kernel, and in the following K_Ω (the variables omitted) will mean this function. Under a weak assumption, e.g. boundedness of Ω, we have $K_\Omega(z) > 0$, and if we assume more, e.g. strictly pseudoconvexity, we get $K_\Omega(z) \to +\infty$ as z tends to the boundary $\partial\Omega$. Here we consider the local geometric information contained in the singularity of K_Ω at the boundary.

The fundamental property that enables us to relate the Bergman kernel with local geometry of the boundary is the transformation law under the biholomorphic map $\Phi\colon \Omega_1 \to \Omega_2$:

$$K_{\Omega_2} \circ \Phi = |\det \Phi'|^{-2} K_{\Omega_1}. \tag{0.1}$$

Here $\det \Phi'$ is the holomorphic Jacobian of Φ. It is well-known that this transformation law implies the biholomorphic invariance of the Bergman metric $-i\partial\overline{\partial}\log K_\Omega$ and this metric enables us to apply the theory of Riemannian geometry to the geometry of complex manifolds. On the other hand, if we regard (0.1) as a relation satisfied by Φ, we can estimate the differentiability of biholomorphic map between strictly pseudoconvex domains through analyzing the boundary behavior of the Bergman kernel [14]. In fact, the work [14] is the origin of [16] and in which the type of the singularity of the Bergman kernel is identified and applied. In general, it is difficult to write the singularity of the kernel explicitly. An elementary and closed form of the Bergman kernel is known only when the domain has large automorphism group — with in the class of strictly pseudoconvex domains, such formula is known only for (the domains biholomorphic to) the ball.

The first object of the program is to give an algorithm of writing down the boundary singularity of the Bergman kernel for a strictly pseudoconvex domain in terms of local geometric invariants of the boundary — its explanation is the main part of the present notes. The key to study this problem is the transformation law of the Bergman kernel (0.1). Generalizing this transformation law, we consider domain functionals $L = \{L_\Omega\}$ (assigning to each strictly pseudoconvex domain Ω a function $L_\Omega \in C^\infty(\Omega)$) satisfying the transformation law

$$L_{\Omega_2} \circ \Phi = |\det \Phi'|^{-2w/(n+1)} L_{\Omega_1}. \tag{0.2}$$

We call such a functional L a *biholomorphic invariant of weight w*. In particular, the Bergman kernel has weight $n+1$. Thus, if we can describe the structure of all biholomorphic invariants, then we can hope to find a naturally way of expressing the singularity of the Bergman kernel in a biholomorphically invariant manner — this is the principle of our problem-solving. To put it more precisely, we only need to consider local biholomorphic invariants L, that is, L_Ω is determined locally at each boundary point. Moreover, we regard L as formal power series at each boundary point and formulate the problem in an algebraic setting.

Local geometry of strictly pseudoconvex domains is the CR geometry of their boundaries. Hence our first task is to relate the local biholomorphic invariants defined on domains to the CR structures of the boundaries. For this purpos, in addition to the domain functionals, we introduce localizable boundary functionals — assignments of functions $L_{\partial\Omega} \in C^\infty(\partial\Omega)$ for each strictly pseudoconvex domain Ω — that satisfy the transformation law (0.2) on the boundaries. We here call such boundary functionals *CR invariants*. To give an explicit relation between CR invariants and local biholomorphic invariants, we employ the ambient metric construction, introduced by Fefferman [16], which based on the analysis of the complex Monge-Ampère operators. For strictly pseudoconvex domains, the Dirichlet boundary value problem of the complex Monge-Ampère equation admits a unique solution u and we can define a complete Einstein-Kähler metric $g^{\mathrm{EK}} = i\partial\overline{\partial}\log u$ (see [7]). Moreover, if we consider the bundle $(z_0, z) \in \mathbb{C}^* \times \Omega$ over Ω, which is regarded as a $(n+1)$st root of the canonical bundle of Ω, then $g[u] = i\partial\overline{\partial}(|z_0|^2 u(z))$ gives a Ricci-flat Lorentz-Kähler metric on the bundle, which we call the *ambient metric*. Since the ambient metric is shown to be biholomorphically invariant, we can hope that the invariants of the metric (restricted to the boundary) give all CR invariants. However, there are two obstructions that appear in this procedure. Firstly, u admits weak singularities at the boundary and we cannot define derivatives of $g[u]$ of high order. Secondary, u is not uniquely determined locally. We need to get over these difficulties to obtain a biholomorphic invariant asymptotic expansion of the Bergman kernel to infinite order, which is the main result of this article.

In §1 we recall analytic results about the asymptotic expansions of the Bergman kernel and the heat kernel, and in §2 we apply invariant theory to give partial expressions of these asymptotic expansions. The algebraic and geometric features of the invariant theory are explained in §3 and §§4–5, respectively. In §6 we show how to overcome the obstruction that appears in the solution of the Monge-Ampère equation, explained above, and then give a complete asymptotic expansion of the Bergman kernel in terms of local biholomorphic invariants.

The results described above are the first step of the program. Our next object is to extract global, geometric and analytic information of the domain out of the asymptotic expansion thus obtained. As a prospect toward this direction, we give, in §7, some remarks on the relation between the Bergman kernel and global invariants of the domain.

Fefferman's program is now generalized, beyond the framework of several complex variables, to parabolic invariant theory [23]. As its example, §8 describes the case of conformal geometry, the structure group of which is a parabolic subgroup of $O(n,1)$ and is a subject of the parabolic invariant theory.

Our treatment of Fefferman's program strongly reflects the author's point of view and only a part of the program is explained. The whole picture of the program can be learnt from Fefferman's lecture notes [4]. The developments after these lectures have been explained in the survey papers, e.g. Bailey [1], Hirachi-Komatsu [28], Gover [20] , and Eastwood [11]. About the circumstances of the study of the Bergman kernel before Fefferman's proposal, there is a compact survey article by Diederich [10].

1. Asymptotic expansion of the Bergman kernel

We start by recalling analytic results on the asymptotic expansion of the Bergman kernel by comparing with its counterpart for the heat kernel. These are all we need for this article and are the premise that we can apply invariant theory for the Bergman kernel.

For a domain $\Omega\subset\mathbb{C}^n$, let $A^2(\Omega)$ be the Hilbert space of L^2 holomorphic functions on Ω. Then for a complete orthonormal system $\{\varphi_j(z)\}_{j=1}^{\infty}$ of $A^2(\Omega)$, the series

$$K_\Omega(z,w)=\sum_{j=1}^{\infty}\varphi_j(z)\overline{\varphi_j(w)}$$

converges uniformly on each compact set of $\Omega\times\Omega$ and defies a holomorphic function of $(z,\overline{w})$. $K_\Omega(z,w)$ is determined independently on the choice of the complete orthonormal system and called the *Bergman kernel of* Ω. It is shown for each $f\in A^2(\Omega)$ that $f(z)=\int_\Omega K_\Omega(z,w)f(w)|dw|^2$, and $K_\Omega(z,w)$ can be characterized as the reproducing kerne of $A^2(\Omega)$. Here $|dw|^2$ denote the standard volume element of $\mathbb{C}^n$. On the other hand, the heat kernel $H_t(x,y)$ of a compact Riemannian manifold (M,g) is defined as the integral kernel that gives the solution to the heat equation $(\partial_t+\Delta_x)u(x,t)=0$ with initial value $u(x,0)=f(x)$. If $\{\varphi_j(z)\}_{j=1}^{\infty}$ is a complete orthonormal system of the eigen functions $\Delta_x\varphi_j=\lambda_j\,\varphi_j$, then $H_t(x,y)$ can be expressed as a series

$$H_t(x,y)=\sum_{j=1}^{\infty}e^{-\lambda_j\,t}\varphi_j(x)\varphi_j(y). \tag{1.1}$$

For the euclidean space $\mathbb{R}^n$, which is the model case of the Riemannian geometry, the heat kernel is given by $H_t(x,y)=\text{const.}\,t^{-n/2}\exp(-\text{dist}(x,y)^2/2t)$ and, in particular, $H_t(x,x)=\text{const.}\,t^{-n/2}$. Here "const." is a universal constant depending only on the dimension n. For general (M,g), $H_t(x,x)$ has the same principal part as $t\to 0$ and admits an asymptotic expansion

$$H_t(x,x)\sim\text{const.}\,t^{-n/2}\Big(1+\sum_{k=1}^{\infty}\gamma_k(x)t^k\Big), \tag{1.2}$$

where $\gamma_k(x)$ are smooth functions of M determined locally by the metric g. Thus, by comparing the integral over the diagonal of (1.1) and (1.2), we get a relation between the spectrum $\{\lambda_j\}$ of (M,g) and the integrals of curvatures of (M,g). Now we return to the case of the Bergman kernel. The model case of strictly pseudoconvex domains is the *Siegel domain*

$$\Omega_0=\{(z',z_n)\in\mathbb{C}^n: z_n+\overline{z}_n-|z'|^2>0\}.$$

Its Bergman kernel is

$$K_{\Omega_0}(z) = c_n\,(z_n + \overline{z}_n - |z'|^2)^{-n-1}, \quad c_n = n!/\pi^n.$$

In this case, as z tends to the boundary, $K_{\Omega_0}(z)$ diverges in the magnitude of $-(n+1)$st power of the defining function. For a general *strictly pseudoconvex domain* — a domain whose boundary can be osculated, at each point, to the second order by a local biholomorphic image of $\partial\Omega_0$ — the Bergman kernel admits the same leading singularity while it also contains a weaker logarithmic singularity.

Theorem 1.1. ([31], [14]) *The Bergman kernel of a strictly pseudoconvex domain Ω has the following singularity at the boundary:*

$$K_\Omega = \varphi^{\mathrm{B}}\,\rho^{-n-1} + \psi^{\mathrm{B}}\log\rho, \tag{1.3}$$

where $\varphi^{\mathrm{B}}, \psi^{\mathrm{B}} \in C^\infty(\overline{\Omega})$ (a function smooth up to the boundary) and $\rho \in C^\infty(\overline{\Omega})$ is a defining function of Ω (that is, ρ satisfies $\Omega = \{\rho > 0\}$ and $d\rho \neq 0$ on the boundary $\partial\Omega$). The boundary value of φ^{B} is give by $c_n\,J[\rho]$, where $J[\,\cdot\,]$ is the complex Monge-Ampère operator

$$J[\rho] = (-1)^n \det \begin{pmatrix} \rho & \partial\rho/\partial z_j \\ \partial\rho/\partial\overline{z}_k & \partial^2\rho/\partial z_j\partial\overline{z}_k \end{pmatrix}_{j,k=1,\dots,n}. \tag{1.4}$$

Moreover, for each boundary point $p \in \partial\Omega$, the Taylor series of φ^{B} mod $O^{n+1}(\partial\Omega)$ and ψ^{B} around p is determined by the Taylor coefficients of ρ around p. Here $f = O^m(\partial\Omega)$ stands for a function such that $f/\rho^m \in C^\infty(\overline{\Omega})$.

The coefficients of the expansions of $\varphi^{\mathrm{B}}, \psi^{\mathrm{B}}$ contain local geometric invariants of Ω. As in the case of the asymptotic expansion of the heat kernel, we further expand $\varphi^{\mathrm{B}}, \psi^{\mathrm{B}}$ in powers of a defining function ρ:

$$\begin{aligned} \varphi^{\mathrm{B}} &= \varphi_0 + \varphi_1\rho + \cdots + \varphi_n\rho^n + O^{n+1}(\partial\Omega), \\ \psi^{\mathrm{B}} &\sim \varphi_{n+1} + \varphi_{n+2}\rho + \cdots + \varphi_{n+k+1}\rho^k + \cdots \end{aligned} \tag{1.5}$$

and try to express the coefficients $\varphi_k(z) \in C^\infty(\overline{\Omega})$ in terms of the curvature of the boundary. Remark that (1.5) are not Taylor expansions; ρ depends on the variable z. Of course, it is possible to take a C^∞ diffeomorphism that transforms $\{0 \le \rho < \epsilon\}$ into a product space $\partial\Omega \times [0, \epsilon)$ and define Taylor expansions of φ^{B} and ψ^{B} with respect to this decomposition; but then the expansions (1.5) lose their biholomorphic invariance. Our plan is to construct a defining function ρ that satisfies the transformation law of weight -1 and then give φ_k as biholomorphic invariant of weight k.

2. Asymptotic expansion of kernel functions using geometric invariants

We first give the expansion of φ^{B}. The method is modeled on the description of the asymptotic expansion of the heat kernel in terms of the Riemannian curvature. The coefficients γ_k of the expansion of the heat kernel (1.2) can be identified by using Weyl' invariant theory for the orthogonal group $O(n)$. As a result, each γ_k is written as a linear combination of complete contraction of the form

$$\operatorname{contr}\bigl(\nabla^{p_1}R \otimes \cdots \otimes \nabla^{p_s}R\bigr). \tag{2.1}$$

Here R is the Riemannian curvature of g, $\nabla^p R$ is its p-th iterated covariant derivatives, and the contraction is taken with respect to the metric g. Moreover, by considering the scaling of the metric and correspondence transformation law of the heat kernel, we see that γ_k contains only the terms satisfying $\sum p_j = 2(k-s)$. For each k, such choice of p_j are finite, and are easily listed. For example, first three terms are:

$$H_t(x,x) \sim t^{-n/2}\Big(c_n^0 + c_n^1 S\, t + (c_n^2\, S^2 + c_n^3\, \|\mathrm{Ric}\|^2 + c_n^4\, \|R\|^2 + c_n^5\, \Delta S)\, t^2 + \cdots\Big),$$

where S is the scalar curvature and c_n^j are numerical constants depending only on the dimension n.

For the case of the Bergman kernel, we need to start from the construction of invariants of the strictly pseudoconvex domains. The construction for the Riemannian geometry above can be also applied to this case. Take a defining function ρ of Ω and set $\rho_\#(z_0, z) = |z_0|^2\rho(z)$, which is a defining function of $\mathbb{C}^* \times \Omega$. The strictly pseudoconvexity of Ω then ensures that

$$g[\rho] = \sum_{j,k=0}^{n} \frac{\partial^2 \rho_\#}{\partial z_j \partial \overline{z}_k}\, dz_j d\overline{z}_k$$

is a Lorentz-Kähler metric in a neighborhood of $\mathbb{C}^* \times \partial\Omega$ in $\mathbb{C}^* \times \overline{\Omega}$. Let $R = R[\rho]$ be the curvature of $g = g[\rho]$ and $R^{(p,q)} = \overline{\nabla}^{q-2}\nabla^{p-2}R$ be its iterated covariant derivatives. Then we define a *Weyl polynomial of weight* $w \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ to be a linear combination of complete contractions of the form

$$W_\# = \mathrm{contr}\big(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\big) \tag{2.2}$$

with $\sum p_j = \sum q_j = w + s$. A Weyl polynomial $W_\#$ assigns for each defining function ρ of Ω a smooth function $W_\#[\rho]$ on $\mathbb{C}^* \times \overline{\Omega}$ (to be more precise, it is defined on a small neighborhood of $\mathbb{C}^* \times \partial\Omega$). This function admits an expression of the form $W_\#[\rho](z_0, z) = |z_0|^{-2w} W[\rho](z)$, and $W[\rho](z)$ gives a smooth function on $\overline{\Omega}$. In the followings, by this correspondence, we identify $W_\#$ with W and also call W a *Weyl polynomial*.

Theorem 2.1. ([16], [3]) *There exist Weyl invariants W_k of weight $k \in \{0, 1, \dots, n\}$ such that*

$$K_\Omega = r^{-n-1} \sum_{k=0}^{n} W_k[r]\, r^k + O(\log r) \tag{2.3}$$

holds for any strictly pseudoconvex domain Ω. Here r is a defining function of Ω satisfying $J[r] = 1 + O^{n+1}(\partial\Omega)$, and $O(\log r)$ denotes a function f such that $|f/\log r|$ is bounded near the boundary.

The first three terms of the expansion are given by (note that $W_1 = 0$):

$$K_\Omega = r^{-n-1}(c_n + c_n'\, \|R\|^2\, r^2 + \cdots),$$

where c_n, c_n' are constants depending only on n (it is clear that $c_n = n!/\pi^n$, while $c_n' = (n-2)!/(24\,\pi^n)$ is shown by a direct computation [29]). This expansion has less terms compared with that of the heat kernel; this results from the fact that $g[r]$ is Ricci-flat (approximately near the boundary).

The defining function r satisfying the condition $J[r] = 1 + O^{n+1}(\partial\Omega)$ imposed in Theorem 2.1 can be constructed by an inductive procedure and is shown to be unique modulo $O^{n+2}(\partial\Omega)$ [15]. We call such a defining function r a *Fefferman's defining function*, and call $g[r]$ an *ambient metric associated with* Ω. Fefferman's defining function satisfies the transformation law of weight -1, which we have imposed in the previous section (here we neglect the ambiguity of $O^{n+2}(\partial\Omega)$ contained in the definition of r). The next lemma shows that the right-hand side of (2.3) is independent of the choice of r and is determined by Ω.

Lemma 2.2. ([16], [3]) *For a Weyl polynomial W of weight w, $W[r]\,r^w$ mod $O^{n+1}(\partial\Omega)$ is independent of the choice of Fefferman's defining function r.*

In case $w \geq n+1$, this lemma becomes empty and we cannot extract geometric information of Ω out of Weyl polynomial. Thus, in order to remove the restriction on weight from Theorem 2.1, we need a new idea, which will be described later in §6. In the following three sections, we outline the proof of Theorem 2.1 by recalling the results of invariant theory and biholomorphic geometry.

3. Invariant theoretic problems

The expression of the Bergman kernel in terms of Weyl polynomials (2.3) appears to be a literal translation of corresponding expression for the heat kernel. But the invariant theories applied to obtain the expressions are quite different. In this section, we describe what sort of invariant theoretic problems arise in the expansion of the Bergman kernel and how these problems are solved (and which parts are still open).

For a semisimple group G and a parabolic subgroup $H \subset G$, we consider several problems of constructing invariant polynomials for these groups. We here list the problems in order of the easiness (in the sense that it is well-studied and understood).

Problem 0. Find all G-invariant polynomials on a G-module V.

This problem appears in the expansion of the heat kernel, in which $G = O(n)$ and V is a tensor space over $\mathbb{R}^n$. In this case, all G-invariant polynomials are shown to be Weyl polynomials (linear combinations of complete contraction of tensors (2.1)) as an application of the representation theory of semisimple groups. The similar results based on the representation theory hold for more general settings of Problem 0 (see [4]).

Problem 1. Find all H-invariant polynomials on a G-module V.

This problem can be reduced to Problem 0, because we can show that all H-invariant polynomials are G-invariant (see [13]).

Problem 2. Let Y be an H-submodule of G-module V, which may not be G-invariant. Find all H-invariant polynomials of Y.

This is the problem that arises in the expansion of the Bergman kernel, in which $G = SU(1,n)$, V is a tensor space over $\mathbb{C}^{n+1}$, Y is the range of curvature tensor (together with its iterated covariant derivatives) when the ambient metric varies over all strictly pseudoconvex domains. Since the ambient metric is homogeneous only in the z_0-direction, Y is not G-invariant. (The curvature tensors subject to nonlinear relations such as Ricci identity and thus Y is not a vector space. Nevertheless, by considering the tangent space of Y at the origin, we may reduce the problem to the case of H-module.) Writing down the relations satisfied by the curvature tensor of the ambient metric, we can prove all H-invariant polynomials on Y are given by restrictions of Weyl polynomials of V to Y (cf. Theorem 3.1 below). In this argument, the shape of Y in V is important. In fact, there are H-submodule of V for which not all H-invariant polynomials come from Weyl polynomials of V (cf. Remark 3.3). This dependence on Y of the structure of H-invariant polynomials is a property that does not appear in the invariant theory of semisimple groups, which have complete reductivity.

Problem 3. Find all H-invariant polynomials of an H-module Y (a G-module containing Y is not given).

The definition of CR invariants is given in this setting (Definition 4.1). In this situation, even making examples of H-invariant polynomials is not easy [21], [30]. We thus construct the ambient metric and reduce this problem to Problem 2. However, so far, the construction of the ambient metric is obstructed at finite order and the reduction is not complete. Thus construction of CR invariants is still one of the main open problem of the parabolic invariant theory — we will further discuss this problem in §5 and §6.

In the rest of this section, we describe the results on Problem 2 that will be used to obtain the expansion of the Bergman kernel (Theorem 2.1). Among these results, Theorem 3.1 below is one of the main (and deep) result of parabolic invariant theory obtained by Fefferman [16] and Bailey-Eastwood-Graham [3].

We embed the Siegel domain Ω_0 into the projective space by $z \mapsto [1 : z_1 : \cdots : z_n] \in \mathbb{P}^n$. Then biholomorphic automorphisms of Ω_0 can be realized as the action on $\mathbb{P}^n$ of the special unitary group $G = SU(1, n)$ for the hermitian form

$$Q(\zeta, \overline{\zeta}) = \zeta_0\overline{\zeta}_n + \zeta_n\overline{\zeta}_0 - \zeta' \cdot \overline{\zeta'}, \quad \zeta = (\zeta_0, \zeta', \zeta_n) \in \mathbb{C}^{n+1}.$$

The isotropy group at the origin $0 \in \Omega_0$ corresponds to the parabolic subgroup of G:

$$H = \{h \in G : h\, e_0 = \lambda\, e_0,\ \lambda \in \mathbb{C}^*\}.$$

Here e_0 is the column vector ${}^t(1, 0, \ldots, 0)$. Note that $\sigma_{p,q}(h) = \lambda^p\overline{\lambda}^q$, $p, q \in \mathbb{Z}$, defines a character of H.

For a general domain Ω, we also consider the same embedding into the projective space and identify the $\mathbb{C}^*$-bundle $\mathbb{C}^* \times \Omega \to \Omega$ with the restriction over Ω of the natural surjection $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$. Then a function on the bundle $\mathbb{C}^* \times \mathbb{C}^n$ of the form $r_\#(z_0, z) = |z_0|^2 r(z)$ can be identified with a function $\varphi(\zeta)$ on $\mathbb{C}^{n+1} - \{0\}$ with homogeneity $\varphi(\lambda\zeta) = |\lambda|^2\varphi(\zeta)$, $\lambda \in \mathbb{C}^*$. Let us denote by $\mathcal{E}(1)$ the totality of the jets at e_0 of real-valued C^∞ functions with this homogeneity.

If we consider the perturbation of Ω_0 with the common boundary point 0, then the corresponding families of Fefferman's defining functions $r_\#$ (as the jets at e_0) move in $\mathcal{E}(1)$; their first variations form the harmonic subspace of $\mathcal{E}(1)$:

$$\mathcal{H}(1) = \{\varphi \in \mathcal{E}(1) : \Delta\varphi = 0\}, \quad \Delta = \partial_0\partial_{\overline{n}} + \partial_n\partial_{\overline{0}} - \sum_{k=1}^{n-1} \partial_j\partial_{\overline{j}},$$

where $\partial_j = \partial/\partial\zeta_j$, $\partial_{\overline{j}} = \partial/\partial\overline{\zeta}_j$. The Lorentz-Laplacian Δ arises as the linearization of the complex Monge-Ampère operator around $r_0 = z_n + \overline{z}_n - |z'|^2$.

Remark 3.1. Strictly speaking, Fefferman's defining function r is not uniquely determined, and accordingly its first variation is defined as φ mod $O(Q^{n+2})$. We will consider this point later in §6.

We now generalize the homogeneity to real number s and define $\mathcal{E}(s)$ to be the totality of the jets at e_0 of C^∞ functions satisfying $\varphi(\lambda\zeta) = |\lambda|^{2s}\varphi(\zeta)$, $\lambda \in \mathbb{C}^*$. We also define $\mathcal{H}(s)$ to be the harmonic subspace of $\mathcal{E}(s)$. Then the action of H on $\mathbb{C}^{n+1}$

induces a structure of H-module on $\mathcal{H}(s)$. If we identify $\varphi \in \mathcal{H}(s)$ with tensors $T = (T_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 0}$, where $T_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}}\varphi(e_0)$, then the action of H to φ agrees with that of the covariant tensorial action of $h \in G$ to T up to scalar multiples by the character $\sigma_{p,q}$. We write this action of $h \in H$ to $T \in \mathcal{H}(s)$ as $h.T$.

Model Problem 1. Find all H-invariant polynomials of $\mathcal{H}(s)$. Here, by an H-invariant polynomial (of weight w) we mean a polynomial $P(T)$ in the components of T that satisfies $P(h.T) = \sigma_{w,w}(h)P(T)$ for all $h \in H$.

According to s, this problem is reduced to either Problem 1 (which has been solved) or Problem 2 (which is still open in general).

Case 1: $s \notin \mathbb{Z}_+$. This case falls under the category of Problem 1. If we modify the action of H on $\mathcal{H}(s)$ by multiplying the character $\sigma_{p,q}$, then we can define G-action on the polynomial $P(T)$ in such a way that $P(T)$ is invariant under the H-action obtained by the restriction. It is shown ([12]) that $P(T)$ is also invariant under G and hence an H-invariant polynomial is a *Weyl polynomial*, that is, $P(T)$ is written as a liner combination of complete contractions with respect to Q of the form

$$\mathrm{contr}\left(T^{(p_1,q_1)} \otimes \cdots \otimes T^{(p_k,q_k)}\right), \quad T^{(p,q)} = (T_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}.$$

Case 2: $s \in \mathbb{Z}_+$. In this case, $\mathcal{H}(s)$ admits a decomposition as H-modules: $\mathcal{H}(s) = \mathcal{H}_s \oplus \mathcal{H}^s$,

$$\mathcal{H}_s = \{T : T^{(p,q)} = 0 \text{ if } \min(p,q) \leq s\}, \quad \mathcal{H}^s = \{T : T^{(p,q)} = 0 \text{ if } p,q > s\}.$$

We consider H-invariant polynomial of each H-module. These problems fall under the category of Problem 2. In fact, for each H-invariant polynomial $P(T)$ of $\mathcal{H}_s$, we may take p,q so that $P(T)$ depends only on $\mathcal{H}_s^{(p,q)} = \{T^{(p,q)} : T \in \mathcal{H}_s\}$ and $\mathcal{H}_s^{(p,q)} \otimes \sigma_{s-p,s-q}$ is an H-submodule of the covariant tensor space on $\mathbb{C}^{n+1}$ of type (p,q) that is not G-invariant. Similar statement also holds for $\mathcal{H}^s$ (see [12]).

Theorem 3.1. ([16], [3]) *All H-invariant polynomials of $\mathcal{H}_s$ are given by Weyl polynomials.*

The case $s = 1$ is what we need for the description of the biholomorphic invariants. $\mathcal{H}_1$ is the space of first variations of the curvatures $(R^{(p,q)}(e_0))_{p,q\geq 2}$ of the ambient metrics with respect to the perturbation of the Siegel domain Ω_0. The proof of this fact in [3] contains two different arguments according to the weights of the polynomials. For high weights, the structure of $\mathcal{H}_s$ as a $(\mathfrak{g}, H)$-module is used essentially. For low weights, Weyl's theory for the subgroup $U(n-1) \subset H$ is applied; in which, very explicit form of the defining equation of $\mathcal{H}_s$ as a subspace of $\mathcal{E}(s)$ is used and the argument is technical. Bailey [1] contains a clear explanation of the proof.

Remark 3.2. The similar results for the groups $G = O(n,1)$, $SL(n)$ and their parabolic subgroups H are obtained in [3], [13] and [18]; they can be applied to the description of the invariants of the conformal structures and the projective structures, respectively.

Remark 3.3. There exit H-invariant polynomials of $\mathcal{H}^s$ that cannot be expressed as Weyl polynomials (see [12]). We do not known how to give general H-invariant polynomials of $\mathcal{H}^s$.

4. Moser's normal form

We next explain some geometric tools that are needed to describe the asymptotic expansion. In the case of the heat kernel, each coefficient γ_k of the asymptotic expansion is expressed, at the center of normal coordinates $x = (x_1, \dots, x_n)$, as a polynomial $P_k(g_{ij,ab\dots c})$ whose variables are the partial derivatives $(g_{ij,ab\dots c}(0))$ of the components of the metric $g = (g_{ij})$ with respect to x. This polynomial is shown to be invariant under the action of $O(n)$, which acts as transformation between normal coordinates and accordingly on $(g_{ij,ab\dots c}(0))$. Thus the invariant theory for $O(n)$ can be applied to determine the explicit formula for γ_k.

As an analogy to this argument, we will define CR invariants as H-invariant polynomials, in which the correspondence of the basic concepts are as follows:

	Riemannian manifold:	Strictly pseudoconvex domain:
Basic example:	Euclidean space $\mathbb{R}^n = E(n)/O(n)$	Siegel domain $\Omega_0 = SU(1,n)/H$
Analytic problems:	$\Delta u = f$	$\overline{\partial} u = \alpha,\ \ \overline{\partial}_b u = \alpha$
Geometric concepts:	Geodesics, Levi-Civita connection, Normal coordinates	Chain, Cartan-Tanaka-Chern connection, Moser's normal form

Let us start by explaining Moser's normal form. Recall that a strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^n$ can be osculated, at each point, to the second order by a local biholomorphic image of the boundary of the Siegel domain $\partial\Omega_0$ (this is equivalent to the definition of strictly pseudoconvexity). Moser [35] (cf. [8]) gave for each point p on a real-analytic strictly pseudoconvex hypersurface M a local biholomorphic map Φ in a neighborhood of 0 such that $\Phi(\partial\Omega_0)$ gives the best approximation at p. Then Φ^{-1} gives a coordinate system about p, which is called *Moser's normal coordinates centered at p*. For each $p \in M$, if we take a Moser's normal coordinates $z = (z', z_n) \in \mathbb{C}^n$ centered at p, then M is given by the equation

$$2\,\mathrm{Re}\, z_n - |z'|^2 - \sum_{|\alpha|,|\beta|\geq 2,\, l\geq 0} A^l_{\alpha\overline{\beta}}\, z'_\alpha \overline{z}'_\beta (\mathrm{Im}\, z_n)^l = 0, \tag{4.1}$$

where ${z'}_\alpha = z_{\alpha_1} \cdots z_{\alpha_q}$ are monomials of z' and $|\alpha| = q$ is the length of multi index $\alpha = \alpha_1 \dots \alpha_q$. The Taylor coefficients $(A^l_{\alpha\overline{\beta}})$ satisfy several normalization conditions, which are linear; these are derived from the equation of chains, the CR analog of geodesics in Riemannian geometry. $A = (A^l_{\alpha\overline{\beta}})$ is called *Moser's coefficients at* (M, p), and the (germ of) surface of the form (4.1) is called *Moser's normal form* and is denoted by $N(A)$. As in the case of the normal coordinates in Riemannian geometry, Moser's normal coordinates have ambiguity that are parametrized by the isotropy group H of the model domain. Using this parametrization, we can define an action of H on the space of all Moser's coefficients in such a way that the orbits give the biholomorphic equivalence class of the surfaces $N(A)$.

We can also formulate this H-action as an H-principal bundle over M. The bundle, introduced by E. Cartan, Tanaka [36], [37], Chern (cf. [8]), is defined as a higher coframe bundle together with a canonical Cartan connection. The curvature of this connection has natural relation to Moser's coefficients A, in which a choice of the frame corresponds to a choice of Moser's normal coordinates.

To obtain invariants of the surfaces that are independent of the choice of normal coordinates (or frame of the bundle), we now consider functions $P(A)$ of A that are invariant under the H-action. Here we restrict our attention to the case where $P(A)$ are polynomials of A, and define CR invariants as follows.

Definition 4.1. A polynomial $P(A)$ of Moser's coefficients $A = (A^l_{\alpha\overline{\beta}})$ is said to be a *CR invariant of weight* w if $P(A)$ satisfies

$$P(\widetilde{A}) = |\det \Phi'(0)|^{-2w/(n+1)} P(A)$$

for any biholomorphic map Φ that maps $N(A)$ to $N(\widetilde{A})$.

The coordinates changes Φ can be parametrized by the isotropy group H and $\det \Phi'(0)$ can be seen as a character of H. So $P(A)$ becomes an H-invariant polynomial (c.f. Model Problem 1). This definition of H-invariant polynomials falls under the category of Problem 3 of the previous section, and it is difficult to find CR invariants based on this definition (moreover, since the action of H to A is non-linear, we encounter another difficulty; see [21], [30]). It is also possible to define CR invariants using the curvature of Cartan connection, which is intrinsic to the CR structures, instead of Moser's coefficients. However, to relate CR invariants with the Bergman kernel, the use of normal coordinates is essential and we employ the extrinsic definition as above.

Remark 4.2. Recently, Kuranishi [32] has been studying the construction of local embedding of CR manifolds into $\mathbb{C}^n$ that based on the Cartan connection. If such an embedding were constructed, it would become possible to directly relate the curvature of Cartan connection with the asymptotic expansion of the Bergman kernel.

5. Construction of CR invariants using the ambient metric

The problem of finding all CR invariants, for weight $\leq n$, can be solved by using the ambient metric defined §2 and the parabolic invariant theory explained in §3. As an application of the result, we get an expansion of the Bergman kernel (2.3). In this section, we explain the mechanism that all CR invariants of low weights are constructed out of the ambient metric.

Recall that the ambient metric on $\mathbb{C}^* \times \overline{\Omega}$ is defined as the Lorentz-Kähler metric $g[r]$, where r is Fefferman's defining function. (To be more precise, r is defined up to the additions of $O^{n+2}(\partial\Omega)$ and accordingly $g[r]$ only has meaning as a finite jets along $\mathbb{C}^* \times \partial\Omega$. Meanwhile, we disregard this point to simplify the explanation.) For the case of the Siegel domain Ω_0, Fefferman's defining function is $r_0 = z_n + \overline{z}_n - |z'|^2$. If we introduce coordinates $\zeta_0 = z_0, \zeta_1 = z_0 z_1, \ldots, \zeta_n = z_0 z_n$ on $\mathbb{C}^* \times \Omega_0$, we can write the ambient metric $g_0 = g[r_0]$ as

$$g_0 = d\zeta_0 d\overline{\zeta}_n + d\zeta_n d\overline{\zeta}_0 - d\zeta' \cdot d\overline{\zeta'},$$

which corresponds to the quadratic form $Q(\zeta, \overline{\zeta})$ used in §3. Each automorphism $\Phi \in \mathrm{Aut}(\Omega_0)$ can be lifted to an isometry of $\mathbb{C}^* \times \Omega_0$,

$$\Phi_{\#}(z_0, z) = \left(z_0 \cdot (\det \Phi'(z))^{-1/(n+1)}, \Phi(z)\right), \tag{5.1}$$

which agrees with a linear map $h \in SU(1,n)$ defined with respect to the coordinates ζ. Here the branch of the fractional power corresponds to the center $\mathbb{Z}_{n+1}$ of $SU(1,n)$ and we have $\mathrm{Aut}(\Omega_0) = SU(1,n)/\mathbb{Z}_{n+1}$.

For a biholomorphic map $\Phi : \Omega_1 \to \Omega_2$, we can also define a bundle map

$$\Phi_\# : \mathbb{C}^* \times \Omega_1 \to \mathbb{C}^* \times \Omega_2$$

by the formula (5.1). Then $\Phi_\#$ preserves the functions $r_\# = |z_0|^2 r$ and gives an isometry for the ambient metrics. Thus we can construct biholomorphic invariants from the metric $g[r]$. For a Weyl polynomial, which is homogeneous in z_0 and admits an expression $W_\#[r](z_0, z) = |z_0|^{-2w} W[r](z)$, the invariance of $W_\#[r]$ under $\Phi_\#$ can be rewritten as

$$W[r_2] \circ \Phi = |\det \Phi'|^{-2w/(n+1)} W[r_1], \tag{5.2}$$

where r_j is Fefferman's defining fucntion of Ω_j. Moreover, if the boundary $\partial\Omega$ is in Moser's normal form $N(A)$ and W has weight $\le n$ (see Lemma 2.2), then the value of a Weyl polynomial at the origin $W[r](0)$ is shown to depend polynomially on A; thus $W[r](0)$ is a CR invariant.

Theorem 5.1. ([16], [3]) *All CR invariants of weight $\le n$ are given by Weyl polynomials.*

If we use this theorem, we can easily prove Theorem 2.1 by induction. Suppose we have found $W_0, \ldots, W_{k-1}$ and define $\varphi_k \in C^\infty(\overline{\Omega})$ by the formula $\varphi^{\mathrm{B}} = \sum_{j=0}^{k-1} W_j[r]\, r^j + \varphi_k r^k$. If $\partial\Omega$ is locally in Moser's normal form $N(A)$, then $\varphi_k(0)$ depends polynomially on A (see Theorem 1.1) and gives a CR invariant of weight k. Thus, using Theorem 5.1, we can find Weyl polynomial W_k of weight k such that $\varphi_k(0) = W_k[r](0)$. Since φ_k and W_k satisfy the transformation law of weight k, we can also see $\varphi_k(p) = W_k[r](p)$ for any boundary point p. Namely, we have $\varphi_k = W_k[r] + O(\partial\Omega)$, and thus $\varphi^{\mathrm{B}} = \sum_{j=0}^{k} W_j[r]\, r^j + O^{k+1}(\partial\Omega)$ as required.

In the following, we explain the method of the proof of Theorem 5.1 using a linearized model. In this model, we neglect the non-linear terms which appear in the action of H to A and also neglect the normalization conditions imposed on A. Even after these simplification, the main problem of the ambient metric construction remains.

We consider the restriction to the quadric $\mathcal{Q} = \{\zeta \in \mathbb{C}^{n+1} : Q(\zeta, \overline{\zeta}) = 0\}$ of the jet space of functions $\mathcal{E}(s)$ introduced in §3 and define an H-module

$$\mathcal{J}(s) = \{f|_{\mathcal{Q}} : f \in \mathcal{E}(s)\}.$$

With respect to the coordinates $(z_0, z', v) = (\zeta_0, \zeta'/\zeta_0, \mathrm{Im}(\zeta_n/\zeta_0))$ of $\mathcal{Q}$, each $f \in \mathcal{J}(s)$ can be written as

$$f = |z_0|^{2s} \sum_{|\alpha|, |\beta|, l \ge 0} A^l_{\alpha\overline{\beta}}\, z'_\alpha \overline{z}'_\beta\, v^l. \tag{5.3}$$

So identifying f with the list of Taylor coefficients $A = (A^l_{\alpha\overline{\beta}})$, we can define an action of H to A. In particular, when $s = 1$, this action agrees with the action of H to Moser's coefficients A up to the terms which are non-linear in A and the correction terms caused by the normalization condition on A (see also Remark 5.1 below). Under these settings, we pose the following problem, which is a linearized model of the problem of the construction of CR invariants.

Model Problem 2. Find all H-invariant polynomials of $\mathcal{J}(s)$.

Remark 5.1. The precise relation between the H-module $\mathcal{J}(1)$ and the space of Moser's coefficients $\mathcal{N}$, on which H-acts, can be described as follows. If we embed

$\mathcal{N}$ as a subspace of $\mathcal{J}(1)$ by (5.3), then we get a vector space decomposition $\mathcal{J}(1) = \mathcal{N} \oplus \mathcal{J}^1$, where $\mathcal{J}^1 = \{\varphi|_{\mathcal{Q}} : \varphi \in \mathcal{H}^1\}$ is a H-submodule of $\mathcal{J}(1)$. The linearization of the action of H on $\mathcal{N}$ at $A = 0$ (that is, the induced H-action on the tangent space $T_0\mathcal{N}$) is then isomorphic to the quotient H-module $\mathcal{J}(1)/\mathcal{J}^1$.

The situation of this problem also varies according to s. The easiest case is $n + 2s \notin \mathbb{N} = \{1, 2, \dots\}$. Then the restriction to $\mathcal{Q}$,

$$\mathcal{H}(s) \to \mathcal{J}(s), \tag{5.4}$$

gives an isomorphism of H-module, and thus Model Problem 2 is reduced to Model Problem 1 (Case 1) of §3. This isomorphism asserts that each function on $\mathcal{Q}$ admits a unique harmonic extension to $\mathbb{C}^{n+1}$ (as formal power series). Note that the initial value problem of the second order partial differential equation $\Delta\varphi = 0$ admits a unique solution because $\varphi(\zeta)$ is assumed to be homogeneous in ζ.

When $m = n+2s \in \mathbb{N}$, the map (5.4) in neither surjective nor injective. However, we can approximately construct harmonic extension so that $\Delta\varphi = O(Q^{m-1})$ holds. (If $s = 1$, then $\Delta\varphi = O(Q^{n+1})$ is the linearization about r_0 of the equation $J[r] = 1 + O^{n+1}(\partial\Omega)$, which is the equation for Fefferman's defining function.) Thus, for the jets of enough low order, (5.4) give an H-equivariant bijection. Using this map, we can reduce Model problem 2 for H-invariant polynomials of weight $\le n + 2s - 2$ to Model Problem 1 (Case 2). If we restrict the map (5.4) to the subspace $\mathcal{N} \subset \mathcal{J}(1)$ consisting of Moser's coefficients A (see Remark 5.1), then we can reduce the problem of finding H-invariant polynomials of weight $\le n$ to the analogous problem for $\mathcal{H}_1$. Therefore, after linearization, Theorem 5.1 is reduced to Theorem 3.1.

6. Refinement of the ambient metric

In this section, we explain the refinement of the ambient metric given in [26]. In virtue of this refinement, we can remove the restriction on weight in Theorems 2.1 and 5.1.

We start by studying the complex Monge-Ampère equation, which is fundamental to the construction of the ambient metric. It is shown by Cheng-Yau [7] that the boundary value problem of the complex Monge-Ampère equation

$$J[u] = 1 \text{ and } u > 0 \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega \tag{6.1}$$

admits a unique solution. This solution u^{CY} has weak singularity at the boundary, and admits the following asymptotic expansion [33]:

$$u^{\mathrm{CY}} \sim r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k. \tag{6.2}$$

Here r is Fefferman's defining function and $\eta_k \in C^\infty(\overline{\Omega})$. Part of this expansion is locally determined by the boundary (while u^{CY} itself is determined globally). In fact, the equation $J[u] = 1$ admits a formal solution of the form $r \sum_{k=0}^\infty \eta_k \cdot (r^{n+1} \log r)^k$, the ambiguity of which is a choice of a function on the boundary $a \in C^\infty(\partial\Omega)$ (see [22]). In (6.2), the function a is determined globally, and if we specify a, then the formal solution can be constructed uniquely at each boundary point. There are (at least) two ways of constructing invariants of the boundary out of this formal solution:

A) Extract the parts of the formal solutions that are independent of the parameter a, and use them to construct CR invariants.

Graham ([21], [22]) used this method. He showed that $\eta_k \mod O^{n+1}(\partial\Omega)$ are independent of a and defined Weyl polynomials that contain η_k. These new Weyl polynomials enable us to improve Theorem 5.1, but they still do not give all CR invariants (see also [30]).

B) Define ambient metrics that contain the parameter function a and construct Weyl polynomials that depend on a.

In this setting, we can describe the full asymptotic expansion of the Bergman kernel in terms of Weyl polynomials, and can also remove the restriction on weight from Theorem 5.1 after weakening the statement (Theorem 6.2). In the rest of this section, we explain this method.

Keeping the transformation law in mind, we construct formal solutions on the $\mathbb{C}^*$-bundle. Using the coordinates ζ of $\mathbb{C}^* \times \Omega$, we define the complex Monge-Ampère operator by

$$J_\#[U] = (-1)^n \det \left(\frac{\partial^2 U}{\partial \zeta_j \partial \overline{\zeta}_k} \right)_{j,k=0,\dots,n}$$

and consider the formal solutions of the equation $J_\#[U] = 1$ of the form:

$$U = r_\# + r_\# \sum_{k=1}^{\infty} \eta_k (r^{n+1} \log r_\#)^k, \quad \eta_k \in C^\infty(\overline{\Omega}),$$

where $r \in C^\infty(\overline{\Omega})$ is a defining function of Ω, and $r_\# = |z_0|^2 r$ is its lift to $\mathbb{C}^* \times \overline{\Omega}$. As in the case of the equation on Ω, such formal solutions exist and are parametrized by $C^\infty(\partial\Omega)$. Collecting the defining functions $r = r[U]$ that appear as the smooth part of formal solutions U, we now set

$$\mathcal{F}_{\partial\Omega} = \{ r[U] : U \text{ is a formal solution along } \mathbb{C}^* \times \partial\Omega \}.$$

This family of defining functions satisfies the transformation law of weight -1 in the following sense: if we define the weighted pull-back $\Phi^*(r) = |\det \Phi'|^{-2/(n+1)} r \circ \Phi$ for each biholomorphic map $\Phi : \Omega_1 \to \Omega_2$, then we have $\Phi^*(\mathcal{F}_{\partial\Omega_2}) = \mathcal{F}_{\partial\Omega_1}$. This results from the fact that U has log term $\log r_\#$ (instead of $\log r$). Note that we cannot get such a transformation law if we define $\mathcal{F}_{\partial\Omega}$ with respect to the formal solutions of the form (6.2).

Using this family of defining functions, we now generalize Theorem 2.1 to the following theorem.

Theorem 6.1. ([26]) *There exist Weyl polynomials W_k of weight k for $k = 0, 1, 2, \dots$ such that the asymptotic expansion*

$$K_\Omega \sim r^{-n-1} \sum_{k=0}^{n} W_k[r]\, r^k + r^{-n-1} \sum_{k=n+1}^{\infty} W_k[r]\, r^k \log r \tag{6.3}$$

holds for any strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and for any $r \in \mathcal{F}_{\partial\Omega}$.

When $n = 2$, the expansion of the log term coefficient ψ^{B} starts as follows:

$$\psi^{\mathrm{B}} = c_1\, \Delta^2 S + c_2 \|R^{(2,4)}\|^2\, r + (c_3 \|R^{(2,5)}\|^2 + c_4 \|R^{(3,4)}\|^2) r^2 + O^3(\partial\Omega), \tag{6.4}$$

where S and Δ are the scalar curvature and the Laplacian of $g[r]$, respectively, and $\|R^{(p,q)}\|^2$ is a complete contraction of the form $\mathrm{contr}(R^{(p,q)} \otimes R^{(p,q)})$.

The first $n+1$ terms of the expansion (6.3) agree with those given in Theorem 2.1. In fact, each $r \in \mathcal{F}_{\partial\Omega}$ is a Fefferman's defining function (i.e. $J[r] = 1 + O^{n+1}(\partial\Omega)$ holds) and hence the boundary values of the coefficients $W_k[r]|_{\partial\Omega}$, $k \le n$, become CR invariants. For $k > n$, $W_k[r]|_{\partial\Omega}$ may depend on r and may not give a CR invariant. However, in the series (6.3), the dependence on r of $W_k[r]$ is cancelled out and (6.3) gives an asymptotic series determined by the domain. To be more precise, for each $m \ge n+1$, the partial sum $\sum_{k=n+1}^{m} W_k[r]\, r^k \bmod O^{m+1}(\partial\Omega)$ is independent on the choice of $r \in \mathcal{F}_{\partial\Omega}$.

We next describe the dependence of Weyl polynomial $W[r]$ on the defining function r. Recall that the formal solution U is uniquely specified by a choice of a function on the boundary. Thus $\mathcal{F}_{\partial\Omega}$ is also parametrized by the same function. If we take a real vector field T on $\mathbb{C}^n$ that is transversal to $\partial\Omega$, then the parametrization is given by $T^{n+2} r|_{\partial\Omega} = a \in C^\infty(\partial\Omega)$; we denote $r \in \mathcal{F}_{\partial\Omega}$ satisfying this condition by r_a. We now assume that the boundary is locally in Moser's normal form $N(A)$, and consider the value at the origin of $W[r_a]$ for each $r_a \in \mathcal{F}_{N(A)}$. Since this value depends on A and the Taylor coefficients of a at the origin, we may write $P_W(A, a) = W[r_a](0)$. If $P_W(A, a)$ is independent of the parameter a, then we call W an *a-independent Weyl polynomial.*

Theorem 6.2. ([26]) *For an a-independent Weyl polynomial W, the polynomial $P(A) = P_W(A, a)$ is a CR invariant, and all CR invariants are given in this way.*

This theorem is a generalization of Theorem 5.1; the condition on weight is removed. Since Weyl polynomials of weight $\le n+2$ are shown to be a-independent, Theorem 6.2 implies Theorem 5.1. So far, we know no practical criterion for determining which Weyl polynomial is a-independent. Thus Theorem 6.2 does not give a method of constructing CR invariants.

Remark 6.3. In a project with R. Gover, the author is trying to find a method of generating all a-independent Weyl polynomials [20]. This can be seen as a CR analog of the construction of conformal invariants by Gover [18] (see also [19]).

In the following, we explain the main idea of the proof of Theorem 6.2 by using the linearized model $\mathcal{J}(1)$ introduced in the previous section. We linearize the problem by considering the first variation of the formal solution under a perturbation of the Siegel domain Ω_0. The first variation satisfies the Laplace equation for the metric g_0. Since the formal solution has log terms, the first variation also contains a log term. The solution space of the Laplace equation with log term is given by

Proposition 6.3. *For each $f \in \mathcal{J}(1)$, there exists a pair $(\varphi, \eta) \in \mathcal{E}(1) \oplus \mathcal{E}(-n-1)$ such that*

$$\Delta(\varphi + \eta\, Q^{n+2} \log Q) = 0, \qquad \varphi|_{\mathcal{Q}} = f$$

hold. If $(\widetilde{\varphi}, \widetilde{\eta})$ satisfies the same condition, then there exists a $\psi \in \mathcal{H}(-n-1)$ such that

$$(\varphi, \eta) - (\widetilde{\varphi}, \widetilde{\eta}) = (Q^{n+2}\psi, 0)$$

(Thus ψ describes the ambiguity of the solution).

Let $\widetilde{\mathcal{H}}(1)$ be the image of the projection of $\big\{(\varphi, \eta) \in \mathcal{E}(1) \oplus \mathcal{E}(-n-1) : \Delta(\varphi + \eta\, Q^{n+2} \log Q) = 0\big\}$ to the $\mathcal{E}(1)$ component (this projection corresponds to the map

$r \mapsto r[U]$ and $\widetilde{\mathcal{H}}(1)$ gives the space of the first variations of r at r_0). Then the proposition above can be written as an exact sequence of H-modules:

$$0 \to \mathcal{H}(-n-1) \to \widetilde{\mathcal{H}}(1) \to \mathcal{J}(1) \to 0. \tag{6.5}$$

This is obtained by extending the domain of (5.4).

Using (6.5), we pull back the H-invariant polynomials of $\mathcal{J}(1)$ to the H-invariant polynomials of $\widetilde{\mathcal{H}}(1)$. Hence denoting by $I(\widetilde{\mathcal{H}}(1))$ and $I(\mathcal{J}(1))$ the spaces of H-invariant polynomials of each module, we may regard $I(\mathcal{J}(1)) \subset I(\widetilde{\mathcal{H}}(1))$. Introduction of this larger space $I(\widetilde{\mathcal{H}}(1))$ enables us to avoid the restriction on weight in the arguments of the previous section.

By restriction, (6.5) gives $0 \to 0 \to \mathcal{H}^1 \to \mathcal{J}^1 \to 0$, where $\mathcal{H}^1$ and $\mathcal{J}^1$ are defined in §3 and Remark 5.1, respectively. Thus the quotient of (6.5) by this sequence gives

$$0 \to \mathcal{H}(-n-1) \to \widetilde{\mathcal{H}}_1 \to \mathcal{N} \to 0, \tag{6.6}$$

where $\widetilde{\mathcal{H}}_1 = \widetilde{\mathcal{H}}(1)/\mathcal{H}^1$ and $\mathcal{N} = \mathcal{J}(1)/\mathcal{J}^1$, which can be identified with the space of Moser's coefficients (see Remark 5.1). Using this sequence, we can reduce the problem of finding all CR invariants (H-invariant polynomials of $\mathcal{N}$) to the problem of finding H-invariant polynomials of $\widetilde{\mathcal{H}}_1$ that vanish on the image of $\mathcal{H}(-n-1)$ (after the process of linearization of the H-actions). Since we can generalize Theorem 3.1 to the module $\widetilde{\mathcal{H}}_1$, we obtain the Weyl polynomial explession of CR invariants as in Theorem 6.2.

7. Asymptotic expansion of the Bergman kernel and global invariants

The result about the invariant theoretic description of the Bergman kernel, explained so far, is just a first step of Fefferman's program. In the heat kernel case, by considering the heat equations on the differential forms, we can obtain a relation between the asymptotic expansion of the heat kernels and the characteristic classes; this relation can be applied to prove index theorems in various settings. It is hoped that the invariant theory for the Bergman kernel will bear such fruits. However, so far, little is known about the link between the Bergman kernel and index theorems. Here we explain some related examples, which are hopefully clues of future research.

In view of the analogy between the time variable and the defining function, we consider the following method of constructing global invariants of strictly pseudoconvex domains in complex manifolds. Fixing a defining function ρ, we set $\Omega_\epsilon = \{\rho > \epsilon\}$ for $\epsilon > 0$. Let $\mathrm{Vol}(\Omega_\epsilon)$ be the volume of Ω_ϵ with respect to the Bergman volume form $K_\Omega dV$, where dV is the smooth volume form on $\overline{\Omega}$ used to define the Hilbert space $A^2(\Omega)$. $\mathrm{Vol}(\Omega_\epsilon)$ diverges as ϵ tends to 0 and admits an expansion

$$\mathrm{Vol}(\Omega_\epsilon) \sim v_0\epsilon^{-n} + v_1\epsilon^{-n+1} + \cdots + v_{n-1}\epsilon^{-1} + v_n \log \epsilon + \cdots \tag{7.1}$$

If we take a canonical defining function ρ, e.g. Fefferman's defining function, then some of the coefficients v_j give biholomorphic invariants of the domain. In particular, if $(L, h) \to M$ is a positive line bundle over a compact complex manifold and $\Omega = \{v \in L^* : |v| < 1\} \subset L^*$ is the unit tube in the dual bundle of L, then the asymptotic series (7.1) for the defining function $\rho = -\log |v|^2$ is shown to be the

Laplace transform of the Hilbert polynomial $P(m) = \dim H^0(M, L^m)$ of L:

$$\mathrm{Vol}(\Omega_\epsilon) \sim \text{const.} \int_0^\infty e^{-\epsilon t} P(t) dt,$$

where the Bergman kernel is defined with respect to the volume $dV = (i\,\partial\overline{\partial}\rho)^{n+1}$ (see [27], where the result is stated for the Szegö kernel but it can be modified to the statement above). This suggests a link between (7.1) and an index theorem.

The construction of global invariants using integrals over the subdomains Ω_ϵ has been done by Burns-Epstein [6], where they used, instead of the Bergman kernel, the complete Einstein-Kähler metric constructed by Cheng-Yau [7] and its characteristic classes. In this case, the coefficients of the expansion contains the Euler characteristic of the domain and the Burns-Epstein invariant of the boundary, a Chern-Simons type invariant for CR structures.

In the setting of conformal geometry, the expansion (7.1) appears in the framework of AdS/CFT correspondence in quantum gravity, in which Ω is a complete Einstein metric g and the boundary admits a conformal structure defined by the boundary value of $\rho^2 g$. The coefficient of $\log \epsilon$ is shown to be a conformal invariant and the coefficient of ϵ^0 (the constant term) is called the renormalized volume of (Ω, g). Note that the Einstein metric g has intimate relation with the ambient metric for conformal structure as we explain in the next section. For more details of this subject see Graham [24], which is a short survey for mathematicians.

8. Relation to conformal geometry

Fefferman's program has been generalized, beyond CR geometry, to parabolic invariant theory or parabolic geometry, [23], [11], [9]. The objects of which are the geometric structures modeled on $G/H =$ (semisimple group)/(parabolic subgroup). This framework merges with the theory of Cartan connection developed by Tanaka [36], [37], [38], Morimoto [34]. In this section we briefly explain the case of conformal geometry, which has the closest connection with CR geometry.

The relation between CR geometry and conformal geometry can be described in terms of the ambient metric. The restriction of the ambient metric on the $\mathbb{C}^*$-bundle to $S^1 \times \partial\Omega$ gives a real Lorentz metric, which is called the *Fefferman metric*. While this definition of the Fefferman metric depends on the embedding of $\partial\Omega$ into $\mathbb{C}^n$, its conformal class is determined locally by the CR structure of the boundary [15]. Note that the construction of the Fefferman metric can be generalized to the abstract CR structures and, moreover, it is shown that this "map" from CR structures to Lorentz conformal structures is injective [5].

The ambient metric for the conformal structures has been constructed in Fefferman-Graham [17]. For a conformal manifold $(M, [g])$, we denote by $\mathcal{G} \subset S^2T^*M$ the metric bundle (the $\mathbb{R}^+$-bundle consisting of the metrics in $[g]$). Then the ambient metric is defined as a Ricci-flat Lorentz metric on $\widetilde{\mathcal{G}} = \mathcal{G} \times (-\epsilon, \epsilon)$. If M has odd dimensions, the ambient metric is determined formally along $\mathcal{G} \times \{0\}$ to infinite order, and the Weyl polynomials of the ambient metric give all local scalar conformal invariants [3]. However, if M has even dimensions, the ambient metric contains logarithmic singularity along $\mathcal{G} \times \{0\}$ and the uniqueness does not hold. Thus we need further work as in the CR case, but the details are not known. We here just note that the CR and the conformal ambient metrics are compatible: if $\widetilde{\Omega}$ is a collar neighborhood of $\partial\Omega$, then $\mathbb{C}^* \times \widetilde{\Omega}$ can be identified with the ambient space $\widetilde{\mathcal{G}}$ for the conformal manifold $S^1 \times \partial\Omega$, and the ambient metric on $\mathbb{C}^* \times \widetilde{\Omega}$ also gives an ambient metric for the conformal structure on $S^1 \times \partial\Omega$.

On the other hand, Bailey-Eastwood-Gover [2] introduced another way of constructing conformal invariants, which is called *Tractor Calculus*. This gives a method of defining Weyl polynomial by using the connection on the vector bundle associated with the Cartan bundle over conformal manifolds. Gover [19] proved that Tractor Calculus produces all conformal invariants for odd dimensional manifolds, while for even dimensions, there are finite number of exceptions which has low weights. An important feature of this method is that it is effective for the construction of the invariants of high weights (relative to the dimension). This is complementary to the ambient metric construction which give all invariants of low weight relative to the dimension. The application of Tractor Calculus to the CR geometry is also in progress [20].

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