ON LINEAR INDEPENDENCE OF CLASSES IN $H^*(\mathrm{WU}_q)$

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Abstract. It is a classical question if the characteristic mapping is injective or surjective. In this article, the injectivity for transversely holomorphic foliations is discussed. More precisely, a family of rigid, linearly independent classes in $H^*(\mathrm{WU}_q)$ will be introduced. The family is closely related to the one obtained by Baker in the real case, and different from the one studied by Hurder in [8]. This article is based on a talk given at Foliations 2012, and also on a paper 'On independent rigid classes in $H^*(\mathrm{WU}_q)$' which is to appear in Illinois Journal of Mathematics [3].

1. Introduction

Secondary characteristic classes are defined for transversely holomorphic foliations in a parallel way to the usual ones for smooth foliations, and called complex secondary characteristic classes. As in the real case, injectivity and surjectivity of the characteristic mappings are classical questions. If complex normal bundles are assumed to be trivial, then it concerns the mapping $H^*(\mathrm{W}_q^{\mathbb{C}}) \to H^*(\overline{B\Gamma_q^{\mathbb{C}}})$. The mapping is studied by Hurder in [8], and a family of rigid, linearly independent classes of $H^*(\mathrm{W}_q^{\mathbb{C}})$ is obtained, where 'rigid' means they are rigid under deformations of foliations (otherwise the class is said to be variable). In this article, we will discuss the case where complex normal bundles are not necessarily trivial. The relevant mapping is $H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathbb{C}})$. By studying a complexification of an example of Baker [4], a family of linearly independent classes of $H^*(\mathrm{WU}_q)$ is obtained.

Date: January 15, 2013.

2010 *Mathematics Subject Classification.* Primary 58H10; Secondary 53C12, 58H15, 57R32.

Key words and phrases. characteristic classes, holomorphic foliations, rigid classes.

The author is supported in part by Grant for Basic Science Research Projects from the Sumitomo Foundation.

Some of the classes in the family are rigid classes and form a family different from Hurder's one.

This article is based on the talk 'On independent rigid classes in $H^*(\mathrm{WU}_q)$' which I gave at Foliations 2012 held at Wydział Matematyki i Informatyki, Uniwersytetu Łódzkiego, Łódź, and also on a paper in the same title which is to appear in Illinois Journal of Mathematics [3].

2. Preliminaries

Throughout the paper, the coefficients of cohomology groups are chosen in $\mathbb{C}$. First we recall some basic notions of characteristic classes of transversely holomorphic foliations. A fundamental reference is [6].

Let $B\Gamma_q^{\mathbb{C}}$ be the classifying space of transversely holomorphic foliations of complex codimension q, and let $\overline{B\Gamma_q^{\mathbb{C}}}$ be the classifying space of transversely holomorphic foliations of complex codimension q with trivialized complex normal bundles. Characteristic classes of transversely holomorphic foliations are elements of $H^*(B\Gamma_q^{\mathbb{C}})$ and $H^*(\overline{B\Gamma_q^{\mathbb{C}}})$. Among them, secondary characteristic classes are given by means of the following differential graded algebras (dga's for short).

Definition 2.1. Let $\mathbb{C}[v_1,\dots,v_q]$ be the polynomial ring generated by $v_1,\dots,v_q$, where the degree of v_j is set to be $2j$. We denote by I_q the ideal generated by monomials of degree greater than $2q$, and set $\mathbb{C}_q[v_1,\dots,v_q] = \mathbb{C}[v_1,\dots,v_q]/I_q$. We also define $\mathbb{C}_q[\bar{v}_1,\dots,\bar{v}_q]$ by replacing v_j by $\bar{v}_j$. We set

$$\begin{aligned}
\mathrm{WU}_q &= \textstyle\bigwedge[\widetilde{u}_1,\dots,\widetilde{u}_q]\otimes\mathbb{C}_q[v_1,\dots,v_q]\otimes\mathbb{C}_q[\bar{v}_1,\dots,\bar{v}_q],\\
\mathrm{W}_q &= \textstyle\bigwedge[u_1,\dots,u_q]\otimes\mathbb{C}_q[v_1,\dots,v_q],\\
\overline{\mathrm{W}_q} &= \textstyle\bigwedge[\bar{u}_1,\dots,\bar{u}_q]\otimes\mathbb{C}_q[\bar{v}_1,\dots,\bar{v}_q],\\
\mathrm{W}_q^{\mathbb{C}} &= \textstyle\bigwedge[u_1,\dots,u_q,\bar{u}_1,\dots,\bar{u}_q]\otimes\mathbb{C}_q[v_1,\dots,v_q]\otimes\mathbb{C}_q[\bar{v}_1,\dots,\bar{v}_q].
\end{aligned}$$

These algebras are equipped with derivations such that $d\widetilde{u}_i = v_i - \bar{v}_i$, $du_i = v_i$, $d\bar{u}_i = \bar{v}_i$ and $dv_i = d\bar{v}_i = 0$. The degree of $\widetilde{u}_i$, u_i and $\bar{u}_i$ are set to be $2i-1$.

Cohomology of these dga's are related as follows. First, we have $\mathrm{W}_q^{\mathbb{C}} = \mathrm{W}_q \wedge \overline{\mathrm{W}_q}$. Therefore, $H^*(\mathrm{W}_q^{\mathbb{C}})$ is isomorphic to $H^*(\mathrm{W}_q)\otimes H^*(\overline{\mathrm{W}_q})$ and there is a natural inclusion from $H^*(\mathrm{W}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$. There is also a natural mapping from $H^*(\mathrm{WU}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ which corresponds to the natural mapping from $\overline{B\Gamma_q^{\mathbb{C}}}$ to $B\Gamma_q^{\mathbb{C}}$. It is induced by the mapping from WU_q to $\mathrm{W}_q^{\mathbb{C}}$ which maps $\widetilde{u}_i$ to $u_i - \bar{u}_i$, v_i to v_i and $\bar{v}_i$ to $\bar{v}_i$, respectively. Note that the mapping from $\overline{B\Gamma_q^{\mathbb{C}}}$ to $B\Gamma_q^{\mathbb{C}}$ is a part of the homotopy fibration $\overline{B\Gamma_q^{\mathbb{C}}} \to B\Gamma_q^{\mathbb{C}} \to \mathrm{BGL}(q;\mathbb{C})$ and also is the classifying map of the $\Gamma_q^{\mathbb{C}}$-structure of $\overline{B\Gamma_q^{\mathbb{C}}}$, namely, the map which forgets the triviality of the complex normal bundle. In what follows, elements of $H^*(\mathrm{WU}_q)$, etc., are denoted by their representatives by abuse of notations.

The following result is classical.

Theorem 2.2. *There is a well-defined homomorphism $H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathbb{C}})$ and $H^*(\mathrm{W}_q^{\mathbb{C}}) \to H^*(\overline{B\Gamma_q^{\mathbb{C}}})$.*

The homomorphisms are called the *universal characteristic homomorphisms*. If a transversely holomorphic foliation of a manifold, say M, is given, then the classifying map induces a mapping from $H^*(\mathrm{WU}_q)$ to $H^*(M)$, or from $H^*(\mathrm{W}_q^{\mathbb{C}})$ to $H^*(M)$. These mappings are called *characteristic homomorphisms*.

Definition 2.3. The classes in $H^*(\mathrm{WU}_q)$, $H^*(\mathrm{W}_q^{\mathbb{C}})$ or its image by (the universal) characteristic mappings which involve $\widetilde{u}_i$, u_i or $\bar{u}_i$ are called *secondary characteristic classes*. If $\mathcal{F}$ is a transversely holomorphic foliation, then, the image of a class, say ω, of $H^*(\mathrm{WU}_q)$ or $H^*(\mathrm{W}_q^{\mathbb{C}})$ under the characteristic mapping associated with $\mathcal{F}$ is represented by $\omega(\mathcal{F})$.

Remark 2.4. The structure of $H^*(\mathrm{W}_q)$ and $H^*(\mathrm{WO}_q)$, where WO_q is a certain dga which plays the role of WU_q for real foliations, are completely understood [7]. Therefore, the structure of $H^*(\mathrm{W}_q^{\mathbb{C}})$ is also understood. However, the structure of $H^*(\mathrm{WU}_q)$ is quite unknown if $q > 3$ (see [2] for the case where $q \leq 3$).

There are some significant classes.

Definition 2.5. 1) The class $u_1 v_1^q$ in $H^{2q+1}(\mathrm{W}_q)$ is called the Bott class and denoted by Bott_q.

2) The class $\sqrt{-1}\widetilde{u}_1(v_1^q + v_1^{q-1}\bar{v}_1 + \cdots + \bar{v}_1^q)$ in $H^{2q+1}(\mathrm{WU}_q)$ is called the *imaginary part of the Bott class* and denoted by ξ_q.
3) The class $\sqrt{-1}\widetilde{u}_1 v_1^q \bar{v}_1^q$ in $H^{4q+1}(\mathrm{WU}_q)$ is called the *Godbillon-Vey class* and denoted by GV_{2q}.
4) A class in $H^*(\mathrm{WU}_q)$ or $H^*(\mathrm{W}_q^{\mathbb{C}})$ is called a *variable class* if it admits a continuous deformation. Classes which are not variable are said to be *rigid*.

The images of above classes under the natural mappings from $H^*(\mathrm{WU}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ and from $H^*(\mathrm{W}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$, or characteristic mappings are also named in the same way.

The following is known.

Theorem 2.6. 1) *Elements of $H^{2q+1}(\mathrm{WU}_q)$ and $H^{2q+1}(\mathrm{W}_q^{\mathbb{C}})$ are variable.*

2) *Let $\rho\colon \mathrm{WU}_{q+1} \to \mathrm{WU}_q$ be the homomorphism which satisfies*

$$\rho(\widetilde{u}_i) = \begin{cases} \widetilde{u}_i, & i \neq q+1, \\ 0, & i = q+1 \end{cases},$$

$$\rho(v_j) = \begin{cases} v_j, & j \neq q+1, \\ 0, & j = q+1 \end{cases},$$

$$\rho(\bar{v}_k) = \begin{cases} \bar{v}_k, & k \neq q+1, \\ 0, & k = q+1 \end{cases}.$$

The classes in the image of $H^*(\mathrm{WU}_q)$ under $\rho_*\colon H^*(\mathrm{WU}_{q+1}) \to H^*(\mathrm{WU}_q)$ are rigid. A similar mapping from $H^*(\mathrm{W}_{q+1}^{\mathbb{C}})$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ is also defined. The classes in the image are also rigid.

Remark 2.7. 1) Actually, the Bott class is well-defined if the q-th exterior product of the complex normal bundle (called the *anticanonical bundle*) is trivial.

2) The Godbillon-Vey class in the sense of Definition 2.5 coincides with the usual one if we regard transversely holomorphic foliations as real foliations by forgetting transverse complex structures.
3) It is known that the image of ξ_q under the natural mapping from $H^*(\mathrm{WU}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ is equal to $\sqrt{-1}(\mathrm{Bott}_q - \overline{\mathrm{Bott}_q})$.

4) The Bott class and its imaginary part are variable class. The Godbillon-Vey class is a rigid class, although it is variable in the category of real foliations. Indeed, if we set $\gamma = \sqrt{-1}\widetilde{u}_1(v_1^{q+1}\bar{v}^{q-1} + v_1^q\bar{v}_1^q + v_1^{q-1}\bar{v}_1^{q+1})$, then $\gamma \in H^{4q+1}(\mathrm{WU}_{q+1})$ and $\rho_*\gamma = \mathrm{GV}_{2q}$.

Theorem 2.8 ([1]). *We have* $H^{2q+1}(\mathrm{WU}_q) \cong H^{2q+1}(\mathrm{W}_q)$.

A basis for $H^{2q+1}(\mathrm{W}_q)$, indeed for $H^*(\mathrm{W}_q)$, is given by Vey [7]. Hence we can find a basis for $H^{2q+1}(\mathrm{WU}_q)$.

A priori there can be a difference between $H^*(B\Gamma_q^{\mathbb{C}})$ and the secondary characteristic classes. There are old open problems on this point.

Question 2.9. 1) *Is the universal characteristic mapping from* $H^*(\mathrm{WU}_q)$ *to* $H^*(B\Gamma_q^{\mathbb{C}})$ *injective?*

2) *Is the universal characteristic mapping from* $H^*(\mathrm{WU}_q)$ *to* $H^*(B\Gamma_q^{\mathbb{C}})$ *surjective?*

3) *How about it if we replace* WU_q *and* $B\Gamma_q^{\mathbb{C}}$ *by* $\mathrm{W}_q^{\mathbb{C}}$ *and* $\overline{B\Gamma_q^{\mathbb{C}}}$, *respectively?*

Note that the kernel of the mapping $H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathbb{C}})$ corresponds to nothing but linear relations (dependence) of characteristic classes.

3. Linear independence of secondary characteristic classes

The linear independence of variable classes can be seen by using a classical example of Bott [5].

Example 3.1. Let $\lambda_0, \ldots, \lambda_q$ be non-zero complex numbers. Let X be a holomorphic vector field on $\mathbb{C}^{q+1}$ defined by $X(z) = \sum_{i=0}^{q} \lambda_i z^i \frac{\partial}{\partial z^i}$, where $z = (z^0, \ldots, z^q)$ is the standard coordinates. The integral curves of X define a foliation of $\mathbb{C}^{q+1}$ which is invariant under homothecies. Hence, if c is a non-zero complex number and if we denote by M_c the quotient of $\mathbb{C}^{q+1} \setminus \{0\}$ by the multiplication by c, then we have a foliation of M_c which we denote by $\mathcal{F}_\lambda$. Note that M_c is diffeomorphic to $S^1 \times S^{2q+1}$ and that the anticanonical bundle of $\mathcal{F}_\lambda$ is trivial. Let $J = (j_1, \ldots, j_q)$ be a q-tuple of non-negative integers. We set $v_J = v_1^{j_1} \cdots v_q^{j_q}$ and $c_J = c_1^{j_1} \cdots c_q^{j_q}$, where c_i denotes the

i-th Chern polynomial. Let $u_i v_J \in H^*(\mathrm{W}_q^{\mathbb{C}})$. Under normalizing the volume, we have

$$u_i v_J(\mathcal{F}_\lambda) = \frac{c_i c_J(\lambda_0, \ldots, \lambda_q)}{\lambda_0 \cdots \lambda_q} \mathrm{vol}_{S^{2q+1}},$$

where $\mathrm{vol}_{S^{2q+1}}$ denotes the normalized volume form of the image of the standard $(2q+1)$-sphere in $\mathbb{C}^{q+1}$ in M_c.

Example 3.1 implies that classes in $H^{2q+1}(\mathrm{W}_q)$ admit continuous deformations and are linearly independent. Indeed, if we have a relation of the form $a_1\alpha_1 + \cdots + a_k\alpha_k = 0$, where $\{\alpha_1, \ldots, \alpha_k\}$ is a basis for $H^{2q+1}(\mathrm{W}_q)$, then it should be invariant when $\lambda_0, \ldots, \lambda_q$ vary. This implies that $a_1 = \cdots = a_k = 0$. Some additional arguments show that classes in $H^{2q+1}(\mathrm{WU}_q)$ and $H^{2q+1}(\mathrm{W}_q^{\mathbb{C}})$ are also linearly independent.

There is a complexification of an example of Baker [4], which is useful in studying rigid classes (see also [2], where an example of the same kind is used for studying the Godbillon-Vey class).

Example 3.2. Let $G = \mathrm{SL}(k+n;\mathbb{C})$, where $n > k > 0$ or $n = k = 1$, $H = \{(a_{ij}) \in G \,|\, a_{ij} = 0 \text{ if } i > k \text{ and } j \leq k\}$, $K = \mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n))$ and $T = T^{k+n-1}$ the maximal torus in G realized as diagonal matrices. Then, the left cosets of H induce transversely holomorphic foliations of complex codimension kn on $\Gamma\backslash G/T$ and $\Gamma\backslash G/K$, where Γ is a discrete subgroup of G such that $\Gamma\backslash G/K$ is a closed manifold.

Example 3.2 is studied in [3], and we have a complexification of [4, Theorem 5.3] as follows. The proof of the following theorems and the corollary is given in [3].

If $I = \{i_1, \ldots, i_l\}$ then we set $\widetilde{u}_I = \widetilde{u}_{i_1} \cdots \widetilde{u}_{i_l}$. We denote $\widetilde{u}_I$ also by $u_{i_1,\ldots,i_l}$. If $I = \varnothing$ then we set $\widetilde{u}_I = 1$ and regard $i_1 = +\infty$. We define h_I in a similar way.

Theorem 3.3. *Let $I = \{i_1, \ldots, i_l\}$ and suppose that $k < i_1 < \cdots < i_l \leq n$. The classes of the form $\widetilde{u}_{1,\ldots,k}\widetilde{u}_I v_1^{kn} \bar{v}_1^{kn}$ are non-trivial and linearly independent in $H^*(\Gamma\backslash\mathrm{SL}(k+n;\mathbb{C})/T)$. These classes are rigid classes, namely, rigid under deformations of foliations if $k = 1$ and $i_1 > 2$.*

There is also a complexification of [9, Theorem 5.37] as follows.

Theorem 3.4. *Let $k = 1$ in Example 3.2 and $I = \{i_1, \dots, i_l\}$, and let $q(v_1, \dots, v_n)$ and $r(\bar{v}_1, \dots, \bar{v}_n)$ be monomials of degree $2i$ and $2n$, respectively. Then, we have the following.*

1) *We fix $r(\bar{v}_1, \dots, \bar{v}_n)$ and denote it by $\bar{r}$. Then, the classes of the form $\widetilde{u}_{n-i+1}\widetilde{u}_I q(v_1, \dots, v_n)\bar{r}$, $n - i + 1 < i_1 < \cdots < i_l \le n$, are linearly independent in $H^*(\Gamma\backslash\mathrm{SL}(n+1;\mathbb{C})/T)$. These classes are multiplies of the classes $\widetilde{u}_1\widetilde{u}_I v_1^n \bar{v}_1^n$ in $H^*(\Gamma\backslash\mathrm{SL}(n+1;\mathbb{C})/T)$.*
2) *The classes of the form $\widetilde{u}_I q(v_1, \dots, v_n) r(\bar{v}_1, \dots, \bar{v}_n)$, where $n - i + 1 < i_1 < \cdots < i_l \le n$, are trivial in $H^*(\Gamma\backslash\mathrm{SL}(n+1;\mathbb{C})/T)$.*

Note that the classes in 2) are rigid ones. Some of the classes in 1), e.g. the Godbillon-Vey class, are also rigid.

Rigid classes are firstly studied by Hurder [8]. Let $q = 2q' - 2$ with $q' > 1$. Let $\mathscr{R}$ be a subset of $H^*(\mathrm{W}_q) \subset H^*(\mathrm{W}_q^{\mathbb{C}})$ defined by $\mathscr{R} = \{u_{2,i_2,\dots,i_s} v_2^{q'-1} | 2 < i_2 < \cdots < i_s \le q\} \cup \{u_{q',i_2,\dots,i_s} v_{q'} | q' < i_2 < \cdots < i_s \le q\}$. The family $\mathscr{R}$ is a set of rigid secondary classes studied in [8]. Indeed, $\mathscr{R}$ consists of linearly independent secondary classes. The families in Theorems 3.3 and 3.4 differ from $\mathscr{R}$. At present we know only the following classes α_q and β_q as elements of WU_q which belongs to the subspace of $H^*(\mathrm{W}_q^{\mathbb{C}})$ spanned by $\mathscr{R}$ and $\overline{\mathscr{R}}$. Let $\alpha_{q'} = \widetilde{u}_2(v_2^{q'-1} + \cdots + \bar{v}_2^{q'-1})$ and $\beta_{q'} = \widetilde{u}_{q'}(v_{q'} + \bar{v}_{q'})$. Note that $\alpha_2 = \beta_2$. These classes are rigid classes also in $H^*(\mathrm{WU}_q)$. We have the following.

Corollary 3.5. *α_2 is non-trivial in $H^7(\Gamma\backslash\mathrm{SL}(3;\mathbb{C})/T;\mathbb{C})$. If $q' > 2$, then $\alpha_{q'}$ and $\beta_{q'}$ are linearly independent in $H^{4q'-1}(\Gamma\backslash\mathrm{SL}(q+1;\mathbb{C})/T;\mathbb{C})$.*

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