On foliations with singularities

“smooth foliations with Morse singularities”

A mini-course at The University of Tokyo

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Introduction

These are the notes of a series of lectures delivered by the author at the Graduate School of Mathematical Sciences of the University of Tokyo, during the month of November 2024. They are meant to be as self-contained as possible, taking into account time and space. The basic idea was to introduce the concepts and some of the basic results in the theory of smooth Morse foliations, with a focus on the characterization of the spherical case in terms of the numbers of centers and saddles.

I wish to express my gratitude to Professor Taro Asuke for his personal effort in making this project possible and for his warm hospitality. I want to thank all those at the Graduate School of Mathematical Sciences of the University of Tokyo for their support and for making my stay in Tokyo such a pleasant and fruitful period.

Preliminaries

1 Morse singularities

We recall the classical notion of Morse singularity for functions.

Given a real-valued C^2 function $f : M^m \to \mathbb{R}$ on a manifold M^m of dimension m, we say that a point $p \in M$ is a Morse type singularity of f if:

(i) p is a critical point, i.e., $df(p) = 0$.

(ii) p is non-degenerate as a critical point: this means that the bilinear form defined by the second derivative of f at p is non-singular. In local coordinates $(x_1, \ldots, x_m)$ centered at p, we may express this condition as:

$$\det \left[\frac{\partial^2 f(x_1, \ldots, x_m)}{\partial x_i \partial x_j} \right]_{(x_1, \ldots, x_m)=0} \neq 0 \qquad (*)$$

Since f has a critical point at p, the fact that $(*)$ is true does not depend on the local system of coordinates $(x_1, \ldots, x_m)$ valid and centered at p.

For this class of singularities we have the classical Morse Lemma (see, for instance, [Milnor]).

Lemma 1.1 [Morse Lemma]. *Let p be a Morse type singularity of f. There is a local system of coordinates $(x_1, \ldots, x_m)$ valid and centered at p, such that*

$$f(x_1, \ldots, x_m) = f(p) - x_1^2 - \cdots - x_\ell^2 + x_{\ell+1}^2 + \cdots + x_m^2$$

where $\ell \in \{0, \ldots, m\}$. In other words, $f(x_1, \ldots, x_m) - f(p)$ is a quadratic form with eigenvalues ± 1.

The number $\ell \in \{0, \ldots, m\}$ does not depend on the chart $(x_1, \ldots, x_m)$ and is called the Morse index of f at p. A geometric consequence of the Morse Lemma is that the levels of f, in a neighborhood of p in M, are diffeomorphic to the level surfaces of $Q(x_1, \ldots, x_m)$ in a neighborhood of the origin $0 \in \mathbb{R}^m$, where $Q(x_1, \ldots, x_m)$ is the quadratic form

$$Q(x_1, \ldots, x_m) = -x_1^2 - \cdots - x_\ell^2 + x_{\ell+1}^2 + \cdots + x_m^2.$$

Example 1.2. $m = 2$.

$\ell = 0$ or 2: $x_1^2 + x_2^2 = \text{constant}$.

$\ell = 1$: $x_1^2 - x_2^2 = \text{const}$.

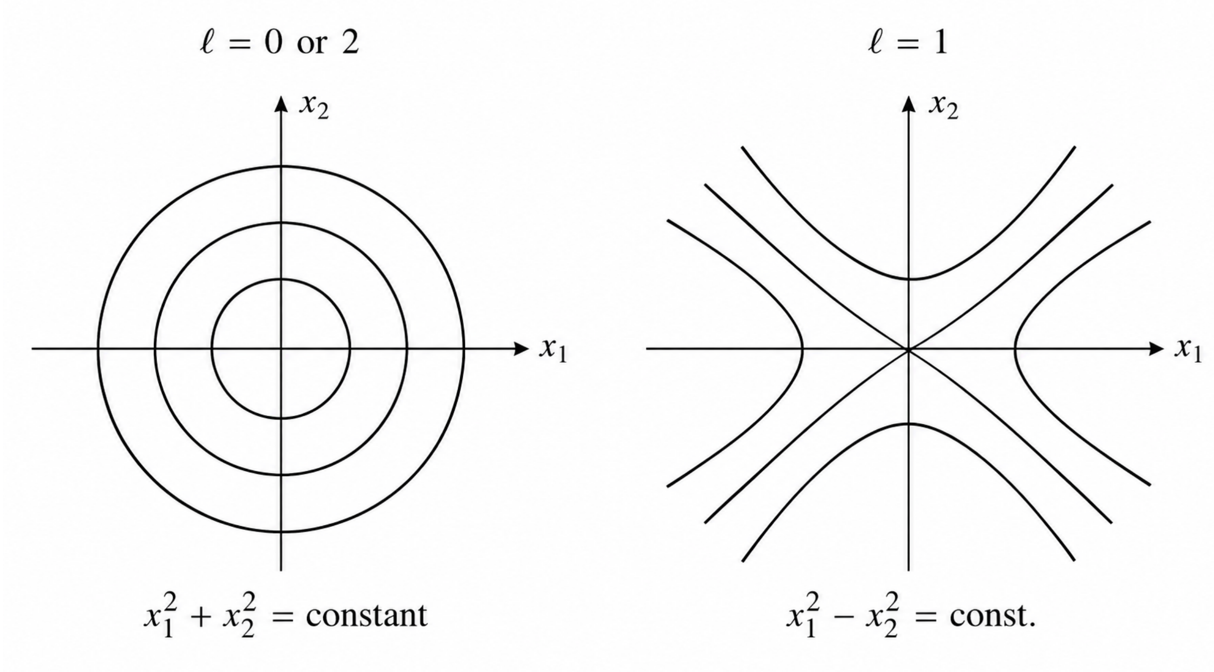

Figure 1: Two-dimensional local models for Morse singularities.

A Morse type singularity will be called a <u>center</u> if the Morse index ℓ is 0 or m (minimum or maximum), in which case the nearby non-singular levels of f are diffeomorphic to $(m-1)$-spheres S^{m-1}.

If $\ell \notin \{0, m\}$, i.e., if p is not a center, then p will be called a <u>saddle</u>. In this case there are leaves which accumulate at the singularity; we call such leaves <u>separatrices</u>. Locally they are given by expressions such as

$$x_1^2 + \cdots + x_\ell^2 = x_{\ell+1}^2 + \cdots + x_m^2.$$

An important case is when $\ell \in \{1, m-1\}$. In this case we have the separatrix given by

$$x_1^2 = x_2^2 + \cdots + x_m^2$$

which is a cone.

The complement of the separatrix has three connected regions R_1, R_2, R_3 depicted above.

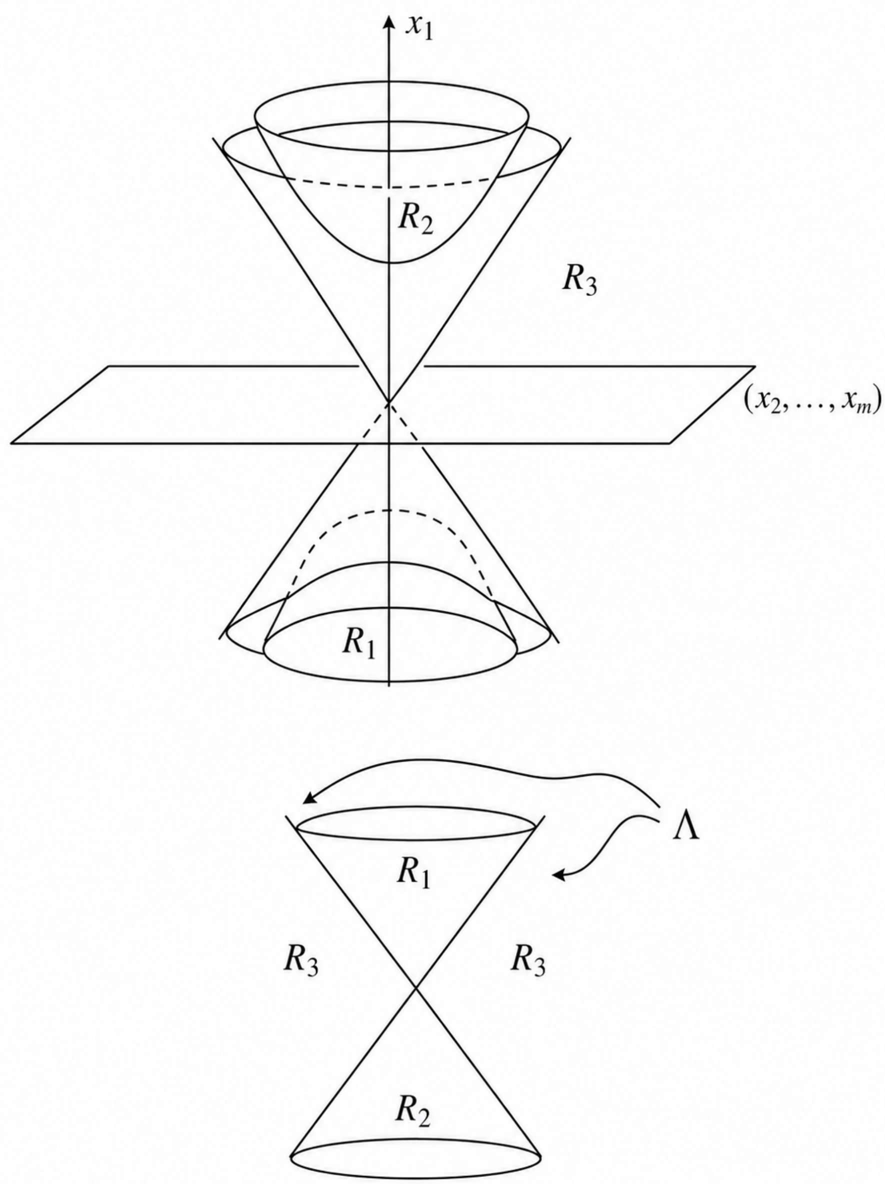

Figure 2: Cone separatrix and the three connected regions R_1, R_2, R_3.

2 Foliations on manifolds

Next we recall the concept of a real smooth foliation on a manifold. Let M be a real C^r manifold, $r \geq 1$, of dimension $m \geq 1$. Denote by B^k the unit ball centered at the origin of $\mathbb{R}^k$. For standard background on foliations, see [C-LN, G].

Definition 2.1. A C^r foliation $\mathcal{F}$ of codimension k on M can be defined in the following three equivalent ways:

Fol. I: A maximal foliated atlas

$$\mathcal{F} = \{(U_j, \varphi_j)\}_{j \in J}$$

of M satisfying the following properties:

(I.1). $\varphi_j(U_j) = B^{m-k} \times B^k$.

(I.2). Whenever $U_i \cap U_j \neq \varnothing$, the transition map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \subset \mathbb{R}^m \longrightarrow \varphi_j(U_i \cap U_j) \subset \mathbb{R}^m$$

is a C^r diffeomorphism of the form

$$\varphi_j \circ \varphi_i^{-1}(x,y) = \big(f_{ij}(x,y), g_{ij}(y)\big)$$

preserving therefore the "horizontal foliation"

in $\mathbb{R}^m = \mathbb{R}^{m-k} \times \mathbb{R}^k$.

In this description $m-k$ is the dimension of $\mathcal{F}$, a plaque of $\mathcal{F}$ is a set $P = \varphi^{-1}(\{y = c\})$ for some $c \in \mathbb{R}^k$. Two points $x, y \in M$ are said to be $\mathcal{F}$-equivalent if there is a finite collection $P_1, \ldots, P_\ell$ of plaques such that $x \in P_1$, $y \in P_\ell$ and $P_i \cap P_{i+1} \neq \varnothing$.

This equivalence relation splits (decomposes) M into disjoint classes called the leaves of $\mathcal{F}$ in M. The leaf of $\mathcal{F}$ through $x \in M$ is denoted by $L_x = \mathcal{F}_x = [x]$ (equivalence class of x).

Fol II: $\mathcal{F}$ is given by a maximal covering by open sets $\{U_j, j \in J\}$ with submersions

$$g_j : U_j \longrightarrow B^k$$

such that $\forall i, j$, with $U_i \cap U_j \neq \varnothing$ there is a diffeomorphism

$$g_{ij} : B^k \longrightarrow B^k$$

satisfying the cocycle relations

$$\begin{cases} g_j = g_{ij} \circ g_i, \\ g_{ii} = \mathrm{Id}\,. \end{cases}$$

In this case the plaques of $\mathcal{F}$ may be defined as the sets $P = \{g_i = c\}$.

Fol. III: $\mathcal{F}$ is a partition of M consisting of immersed C^r submanifolds $F \subset M$, pairwise disjoint, such that every $x \in M$ admits a neighborhood U and a C^r diffeomorphism

$$\varphi : U \longrightarrow B^{m-k} \times B^k$$

such that $\forall y \in B^k$

$$\exists F \in \mathcal{F} \quad \text{(a submanifold of the collection } \mathcal{F}) \quad \text{such that} \quad \varphi^{-1}(B^{m-k} \times y) \subset F.$$

3 Foliations with Morse Singularities

3.1 Singular foliations

Let M be a C^r manifold of dimension m. A foliation with singularities $\mathcal{F}$ on M consists of a pair $\mathcal{F} = (\mathcal{F}_0, S)$ where S is a submanifold of M of dimension $\dim(S) < \ell$ and $\mathcal{F}_0$ is a smooth C^r foliation of dimension ℓ on the open subset $M \setminus S$. By a leaf of $\mathcal{F}$ we shall mean a leaf of $\mathcal{F}_0$ in $M \setminus S$. We call S the singular set of $\mathcal{F}$ and write $S = \mathrm{Sing}(\mathcal{F})$.

Foliations with Morse type singularities

In what follows we will consider singular foliations of codimension one. For our purposes it is enough to consider the case $r = \infty$. For the classical theory of integrable Pfaff forms with non-degenerate singularities, see [W].

Definition 3.1. Let $\mathcal{F}$ be a C^∞ codimension one foliation on a manifold M and let $p \in M$ be an isolated singularity of $\mathcal{F}$.

We say that p is a Morse type singularity if there is a neighborhood $p \in U \subset M$ and a C^∞ function $f : U \to \mathbb{R}$ such that:

(i) $\mathcal{F}|_U$ is given by $df = 0$: this means that in U the leaves of $\mathcal{F}$ are given by the level surfaces of f.

(ii) p is a Morse type singularity of f.

We may assume that p is the only singularity of f in U.

Definition 3.2. A codimension one C^∞ foliation with singularities $\mathcal{F}$ on M will be called a Morse foliation if each singularity in $\mathrm{Sing}(\mathcal{F})$

is of Morse type and has no saddle-connections in the following sense:

given a Morse type singularity $q \in \mathrm{Sing}(\mathcal{F})$ if the Morse index $\ell \notin \{0, m\}$ then we have some leaves locally given by an expression like

$$x_1^2 + \cdots + x_\ell^2 = x_{\ell+1}^2 + \cdots + x_m^2.$$

Those leaves are the local separatrices of $\mathcal{F}$ through q.

A saddle-connection for $\mathcal{F}$ is a leaf L of $\mathcal{F}$ that contains separatrices of two distinct saddle singularities of $\mathcal{F}$.

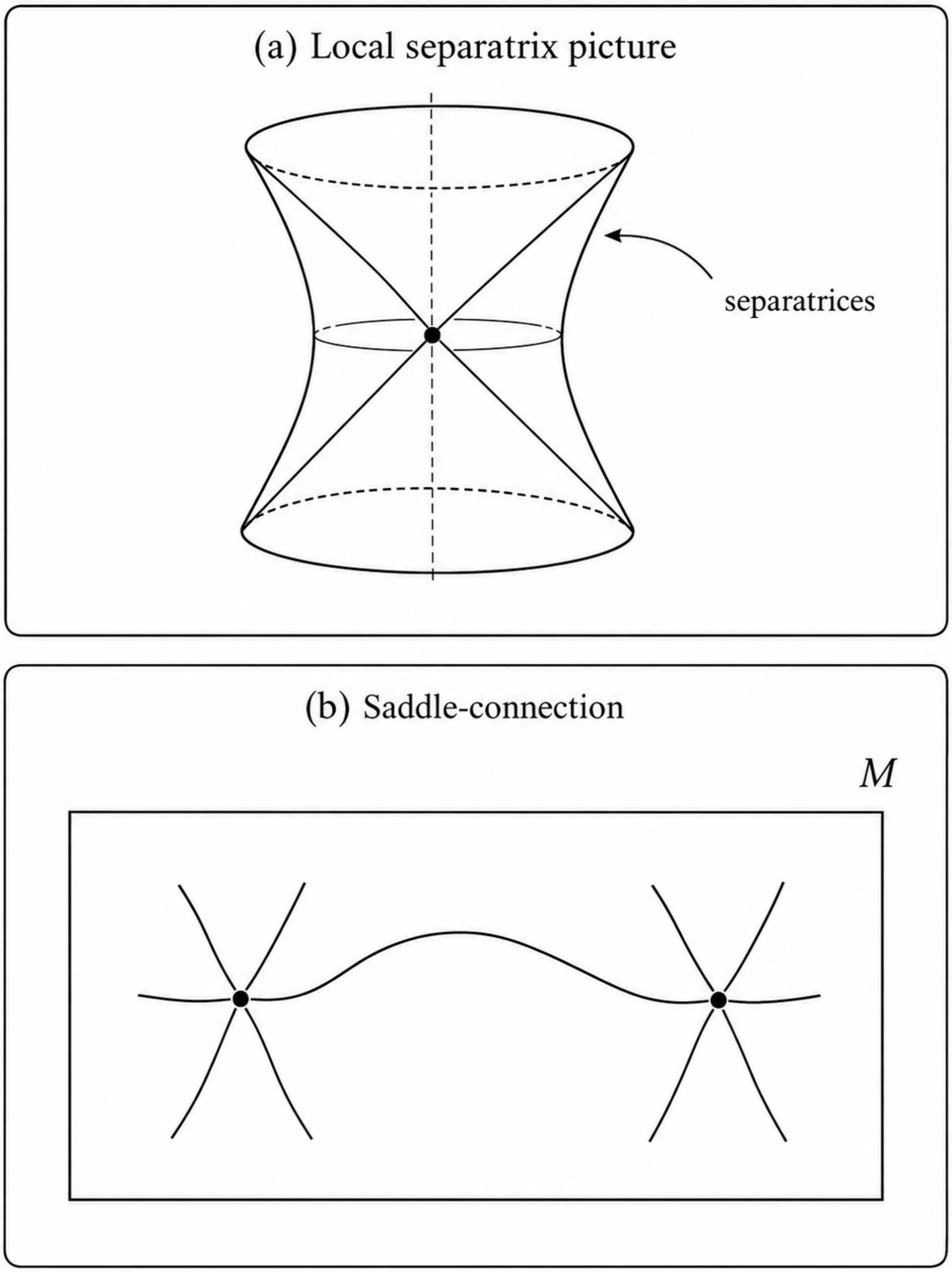

Figure 3: Local separatrix picture and a saddle-connection.

The very first example of a Morse foliation with non-empty singular set on a manifold is given by the levels of a Morse function:

Recall that a Morse function on a manifold M is a C^∞ function $f : M \to \mathbb{R}$ all of whose critical points are of Morse type. In the example considered here we additionally assume that distinct critical points have distinct critical values. Under this additional assumption the level foliation has no saddle-connections.

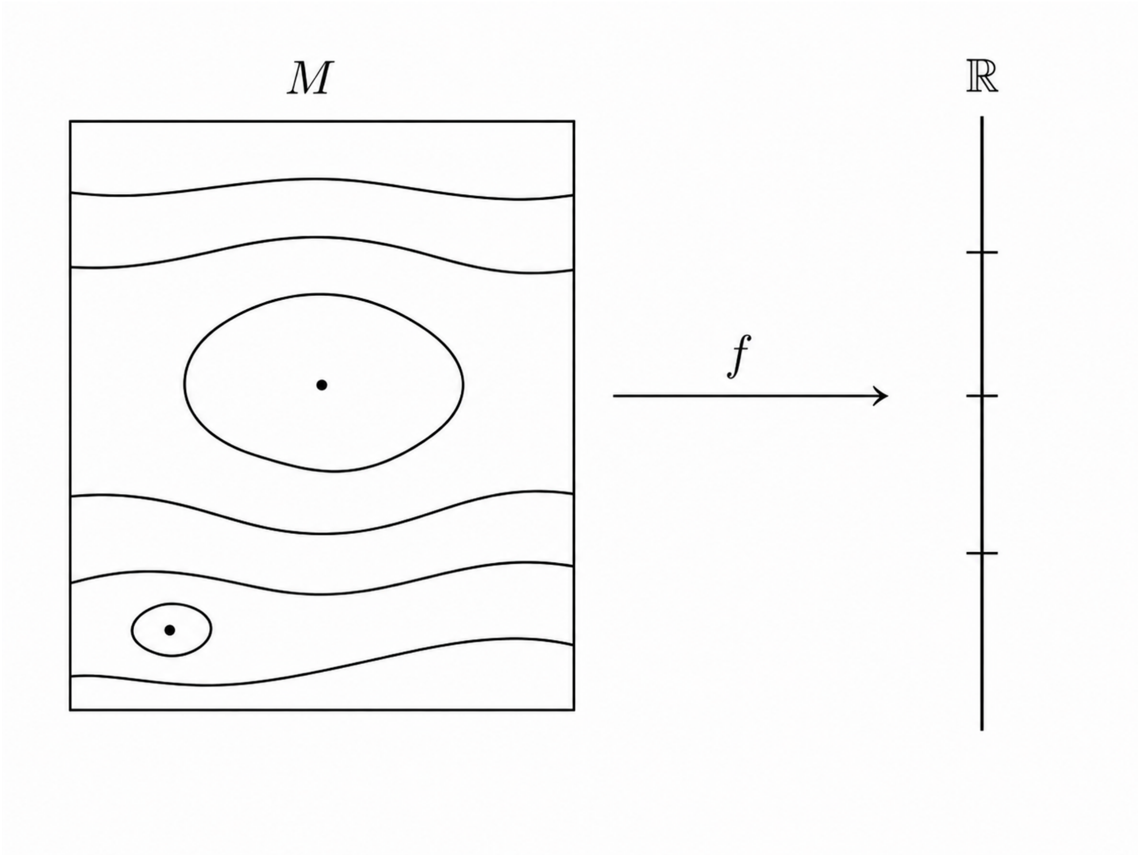

Figure 4: A Morse function on M and its critical values in $\mathbb{R}$.

3.2 Orientability

A foliation with isolated singularities, of codimension one and class C^∞, $\mathcal{F}$ on a manifold M will be called transversely orientable (or simply orientable, in the terminology used below) if there exists a C^∞ one-form Ω satisfying the integrability condition $\Omega \wedge d\Omega = 0$ and such that $\mathcal{F}$ outside the singular set $\mathrm{Sing}(\Omega) = \mathrm{Sing}(\mathcal{F})$ coincides with the distribution $\Omega = 0$.

A foliation is locally integrable if each point $p \in M$ admits a neighborhood where $\mathcal{F}$ is transversely orientable, i.e., given by a local one-form Ω_p as above.

Remark 3.3. (i) A foliation with Morse type singularities is locally transversely orientable.

(ii) A locally transversely orientable foliation on a simply connected manifold is transversely orientable.

3.3 The Index theorem

We recall that if M is a C^∞ oriented, compact manifold and X is a C^1 vector field with a finite number of singularities on M then the Euler-Poincaré-Hopf Index theorem asserts that the sum

$$\sum_{p\in \mathrm{Sing}(X)} I(X,p) = \chi(M)$$

(called the Euler characteristic) is a topological invariant of M, does not depend on X. By $I(X,p)$ we mean the Index of X at p and can be viewed as the degree of the map

$$\left.\frac{X}{\|X\|}\right|_{\partial B(p)} : \partial B(p) \longrightarrow S^{m-1}$$

where $B(p)$ is a "ball" centered at $p \in M$ and $\partial B(p)$ its spherical boundary in M; $\|X\|$ is the norm of X in any Riemannian metric in M.

The singularity $p \in \mathrm{Sing}(X)$ is simple if $DX(p)$ is non-singular, in which case

$$I(X,p) = \pm 1,$$

according to whether $\dfrac{X}{\|X\|}$ preserves $(+1)$ or inverts (-1) the orientation.

If f is a Morse function on M then we may compute this topological invariant $\chi(M)$ as follows: Take a Riemannian metric on M. Then

$$\chi(M) = \sum_{p\in \mathrm{Sing}(\mathrm{grad}\, f)} I(\mathrm{grad}\, f, p).$$

Recall that, in suitable coordinates we have

$$\mathrm{grad}\, f(x_1,\dots,x_m) = \mathrm{grad}\big(f(p) + x_{\ell+1}^2 + \cdots + x_m^2 - x_1^2 - \cdots - x_\ell^2\big)$$

for some $\ell_p \in \{0,\dots,m\}$.

Therefore,

$$I(\mathrm{grad}\, f, p) = (-1)^{\ell_p}$$

so that

$$\sum_{p\in \mathrm{Sing}(\mathrm{grad}\, f)} I(\mathrm{grad}\, f, p) = \sum_{p\in \mathrm{Sing}(\mathrm{grad}\, f)} (-1)^{\ell_p}.$$

For every $\ell \in \{0,1,\dots,m\}$ we define the number

$$M(f,\ell) := \#\{p \in \mathrm{Sing}(\mathrm{grad}\, f);\ \ell_p = \ell\}.$$

$M(f,\ell)$ is the number of singularities of f with Morse index equal to ℓ.

Then

$$\sum_{p\in \mathrm{Sing}(\mathrm{grad}\, f)} I(\mathrm{grad}\, f, p) = \sum_{\ell=0}^{m} (-1)^\ell M(f,\ell).$$

Therefore

$$\chi(M) = \sum_{\ell=0}^{m} (-1)^\ell M(f,\ell).$$

Let us now extend this to our framework:

We shall consider the following situation:

M is a C^∞ manifold, $S \subset M$ is a submanifold. Given a smooth differential one-form Ω in M we shall say that $\ker(\Omega)$ (or $\Omega = 0$) is transverse to S if $\forall p \in S$ we have $\Omega(p) \neq 0$ and $\ker \Omega(p) + T_pS = T_pM$ as real vector spaces.

Let now $p \in \operatorname{Sing}(\Omega)$ be an isolated singularity

$$\big(\operatorname{Sing}(\Omega) = \{q \in M;\ \Omega(q) = 0\}\big).$$

We assume that M is oriented and choose a positive local chart $\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^m$ with $p \in U$ and $\varphi(p) = 0$.

We also assume that $\overline{U} \cap \operatorname{Sing}(\Omega) = \{p\}$ and put

$$\omega = \varphi_*(\Omega) = (\varphi^{-1})^*(\Omega) \in \Lambda^1(\varphi(U)).$$

Write

$$\omega = \sum_{j=1}^{m} f_j \, dx_j$$

with $f_j \in C^\infty(\varphi(U), \mathbb{R}),\ f_j(0) = 0,\ j = 1, \dots, m$. The gradient vector field of ω is defined as

$$\operatorname{grad}(\omega) := \sum_{j=1}^{m} f_j \frac{\partial}{\partial x_j}.$$

Definition 3.4. The *index of* Ω *at* p is defined by

$$I(\Omega; p) = I(\operatorname{grad}(\omega), 0),$$

which is the ordinary Poincaré-Hopf index defined above for the vector field $\operatorname{grad}(\omega)$ at 0.

This definition does not depend on the positive chart $\varphi : U \to \varphi(U)$ chosen above.

All this combined gives us the following

Theorem 3.5 [Index Theorem]. *Let M be an oriented manifold and $D \subset\subset M$ a bounded domain with connected regular boundary of class C^2. Given a C^∞ one-form Ω in M such that* $\operatorname{Sing}(\Omega)$ *is discrete and the distribution* $\ker(\Omega)$ *is either everywhere transverse to ∂D or everywhere tangent to ∂D. Then we have*

$$\chi(D) = \sum_{p \in \operatorname{Sing}(\Omega) \cap D} I(\Omega; p).$$

4 Holonomy and Reeb Stability

4.1 Holonomy of a leaf

The notion of holonomy of a leaf (of a foliation without singularities) is an important tool in the study of the dynamics of foliations. We recall that this is obtained by following plaques through lifting paths on a fixed leaf.

If $\mathcal{F}$ is a C^∞ codimension k foliation on M and L is a leaf of $\mathcal{F}$, given a point $p \in L$ and a transverse section $\Sigma \hookrightarrow M$, $p \in \Sigma$, $\Sigma \pitchfork \mathcal{F}$, the holonomy group of L with respect to Σ is a group

$$\operatorname{Hol}(\mathcal{F}, L, p) \subset \operatorname{Diff}(\Sigma, p)$$

of germs of C^∞ diffeomorphisms of Σ fixing p.

Identifying (Σ, p) with $(\mathbb{R}^k, 0)$, via some diffeomorphism we may identify $\operatorname{Hol}(\mathcal{F}, L, p)$ with a subgroup of $\operatorname{Diff}(\mathbb{R}^k, 0)$, of germs of diffeomorphism fixing the origin $0 \in \mathbb{R}^k$.

One may think of the Poincaré first return map of a closed trajectory, with γ playing the role of the closed trajectory, in a solid torus

$$T \simeq \big(\gamma[0,1]\big) \times B^k,$$

equipped with a real flow transverse to the disks $\gamma(t) \times B^k$.

(recall that $\gamma(0) = \gamma(1)$).

The holonomy group is the image of the holonomy representation

$$\phi : \pi_1(L, p) \longrightarrow \operatorname{Diff}(\Sigma, p)$$

that associates to each homotopy class $[\gamma] \in \pi_1(L, p)$ given by closed paths

$$\gamma : [0,1] \longrightarrow L, \qquad \gamma(0) = \gamma(1) = p,$$

the holonomy diffeomorphism

$$h_\gamma := \phi([\gamma])$$

defined as follows:

By definition, given a point $z \in \Sigma$, $h_\gamma(z)$ is the final point of the lift $\widetilde{\gamma}_z$ of the path γ to the leaf L_z of $\mathcal{F}$, starting at z, using a local fibration by disks transverse to $\mathcal{F}$ along γ.

4.2 Reeb Stability theorems

The notion of holonomy has special interest when the leaf is compact and has a finite fundamental group:

Theorem 4.1 [Reeb local stability theorem]. *Let $\mathcal{F}$ be a C^1 codimension k foliation of a manifold M and L a compact leaf with finite holonomy group. Then there exists a fundamental system of saturated neighborhoods U of L in M in which all leaves are compact with finite holonomy groups. Indeed, we have a retraction $\pi : U \to L$ such that, for every leaf $L' \subset U$, the restriction*

$$\pi|_{L'} : L' \longrightarrow L$$

is a finite covering map and for each $x \in L$ the fiber $\pi^{-1}(x)$ is transverse to $\mathcal{F}$ and diffeomorphic to the open ball $B^k \subset \mathbb{R}^k$.

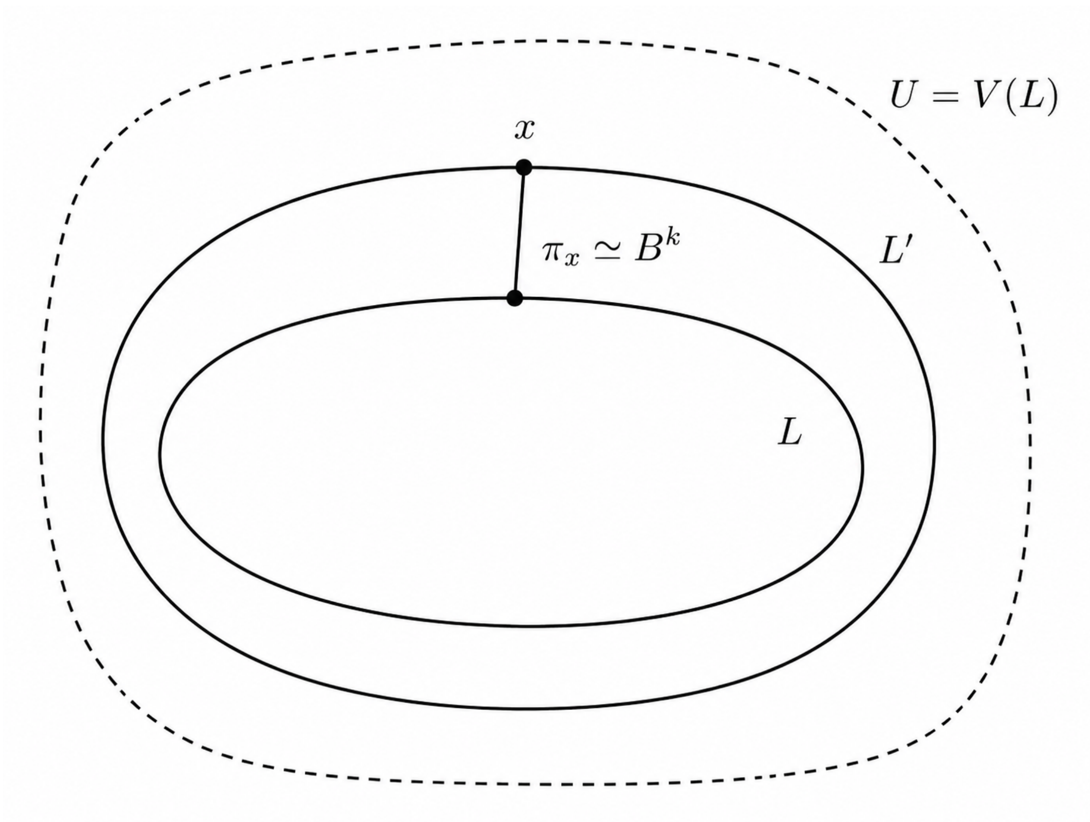

Figure 5: A Reeb neighborhood of a compact leaf.

An important consequence is:

Corollary 4.2. *Let $\mathcal{F}$ be a codimension k, C^1 foliation of a manifold M and L a compact leaf with finite fundamental group. Then there exists an arbitrarily small invariant neighborhood U of L in which all leaves are compact with finite fundamental groups.*

Indeed, a finite covering of a compact manifold with finite fundamental group also has a finite fundamental group.

A more technical but equally useful consequence of the local stability theorem is:

Lemma 4.3 [Trivialization lemma]. *Let $\mathcal{F}$ be a transversely oriented codimension one C^r foliation, $r \geq 1$. Let L be a compact leaf with finite holonomy. Then L has trivial holonomy $\mathrm{Hol}(\mathcal{F}, L) = \{\mathrm{Id}\}$. Moreover, there is an open neighborhood $V(L)$ of L in M, saturated by $\mathcal{F}$, and a C^r diffeomorphism*

$$h : (-1,1) \times L \longrightarrow V(L)$$

such that the leaves of $\mathcal{F}$ in $V(L)$ are the sets

$$h(\{t\} \times L), \qquad t \in (-1,1),$$

with $L = h(\{0\} \times L)$.

In particular, $V(L) \setminus L$ has two connected components.

Remark 4.4. A finite subgroup $H \subset \mathrm{Diff}^r(\mathbb{R}, 0)$, $r \geq 0$, is either trivial or has order two. If the maps in H are orientation preserving then H is trivial.

Lemma 4.5 [Spherical product region]. *Let $\mathcal{F}$ be a transversely oriented codimension-one C^2 foliation on an m-manifold M, $m \geq 3$, and let U be a connected saturated open subset of the regular part of $\mathcal{F}$. Assume that every leaf of $\mathcal{F}|_U$ is diffeomorphic to S^{m-1}. Then the leaf space*

$$U/\mathcal{F}$$

is a connected one-dimensional manifold and the quotient map

$$\pi : U \longrightarrow U/\mathcal{F}$$

is a locally trivial bundle with fiber S^{m-1}. If U is a proper subset of the connected ambient manifold M, then

$$U/\mathcal{F} \simeq (0, 1)$$

and consequently

$$U \simeq S^{m-1} \times (0, 1)$$

as a foliated manifold.

Proof. Every leaf $L \subset U$ is compact and, since $m \geq 3$, simply connected. Hence the holonomy group of L is trivial. By Reeb local stability, each leaf has a saturated neighborhood foliated-diffeomorphic to

$$S^{m-1} \times (-\varepsilon, \varepsilon),$$

with the leaves corresponding to the slices $S^{m-1} \times \{t\}$. Thus the quotient $U/\mathcal{F}$ is locally an interval. Moreover, the compact leaves have stable saturated product neighborhoods; the usual separation argument with these neighborhoods shows that distinct leaves determine distinct points which can be separated in the quotient. Hence $U/\mathcal{F}$ is Hausdorff. Since U is connected, its leaf space is therefore a connected one-dimensional manifold without boundary.

Consequently, $U/\mathcal{F}$ is diffeomorphic either to $\mathbb{R}$ or to S^1, and the local Reeb product charts show that $\pi : U \to U/\mathcal{F}$ is a locally trivial S^{m-1}-bundle. If $U/\mathcal{F} \simeq S^1$, then U is compact because both the base and the fiber are compact. Since U is also open in the connected manifold M, it is then both open and closed, so $U = M$. Therefore, if U is proper, the circular case cannot occur and

$$U/\mathcal{F} \simeq \mathbb{R} \simeq (0, 1).$$

Finally, every smooth fiber bundle over an interval is trivial. Hence

$$U \simeq S^{m-1} \times (0, 1),$$

and the trivialization can be chosen so that the leaves of $\mathcal{F}|_U$ are precisely the slices $S^{m-1} \times \{t\}$. $\square$

Following the ideas above we have the following

Theorem 4.6 [Reeb global stability theorem]. *"Let $\mathcal{F}$ be a C^1 codimension one foliation of a closed manifold M. If $\mathcal{F}$ has a compact leaf L with finite fundamental group then all leaves of $\mathcal{F}$ are compact with finite fundamental group. If $\mathcal{F}$ is orientable then every leaf of $\mathcal{F}$ is diffeomorphic to L; M is the total space of a fibration $f : M \to S^1$ over the circle S^1 with fiber L and $\mathcal{F}$ is the foliation $\{f^{-1}(\theta); \theta \in S^1\}$."*

Remark 4.7. In the terminology used in these notes, for a regular foliation of codimension one the following are equivalent:

(i) $\mathcal{F}$ is transversely orientable.

(ii) The normal line bundle of $\mathcal{F}$ is orientable.

(iii) $\exists\, \Omega \in \Lambda^1(M)$ with $\mathrm{Sing}(\Omega) = \varnothing$ and $\mathcal{F}$ given by $\Omega = 0$, $\Omega \wedge d\Omega = 0$.

(iv) There exists a vector field $X \in \mathfrak{X}^1(M)$ such that $X \pitchfork \mathcal{F}$ everywhere.

Now we mention a couple of important basic results connected to Reeb global stability and its consequences:

Lemma 4.8 [Lemma A]. *Let $\mathcal{F}$ be a codimension one smooth foliation and L a compact leaf of $\mathcal{F}$. If $\{L_\nu\}_{\nu \in \mathbb{N}}$ is a sequence of compact leaves of $\mathcal{F}$ accumulating to some point in L then, given a neighborhood W of L in M, one has $L_\nu \subset W$, $\forall \nu \gg 1$ (for all ν large enough).*

Remark 4.9. The important facts here are:

(1) $\mathcal{F}$ is of codimension one and (2) L_ν and L are compact.

Another underlying fact we are using is

Lemma 4.10 [Lemma B]. *Let $\{L_\nu\}_{\nu \in \mathbb{N}}$ be a sequence of leaves of a foliation $\mathcal{F}$ of M. If a point $x \in M$ satisfies*

$$x \in \overline{\bigcup_{\nu \in \mathbb{N}} L_\nu},$$

then the leaf L_x also satisfies

$$L_x \subset \overline{\bigcup_{\nu \in \mathbb{N}} L_\nu}.$$

Proof. The set $\bigcup_\nu L_\nu$ is saturated. In a foliated box, the closure of a saturated set is again saturated, because a plaque meeting the closure is obtained as a limit of plaques belonging to the saturated set. Propagating this observation along a finite chain of foliated boxes in the leaf L_x shows that every point of L_x belongs to $\overline{\bigcup_\nu L_\nu}$. $\square$

4.3 Holonomy of an invariant subset of codimension one

We shall now extend the notion of holonomy to a connected invariant subset of codimension one. The set Λ can be either a leaf or the union of a saddle singularity with some of its separatrices (recall that we are avoiding saddle connections).

We consider the case where Λ is the union of a saddle singularity $q \in \mathrm{Sing}(\mathcal{F})$ and some of its separatrices.

We first assume that $\Lambda = \gamma \cup \{q\}$ where γ is either a cone leaf or a union of two cone leaves accumulating at q.

In a small neighborhood of q, γ can consist of two components γ_1 and γ_2. In this case Λ locally splits the manifold into three components. One of them, say R_3, is the union of (regular) leaves which are transversely hyperboloids of one sheet.

The other components, say R_1 and R_2, are the union of one of the connected components of hyperboloids of two sheets.

Let $\gamma : [0,1] \longrightarrow \Lambda$ be a path on Λ which passes through the singularity q (from γ_1 to γ_2). In this case, the holonomy "map" associated to γ can be defined in the usual manner (lifting paths to leaves) in the region R_3. Nevertheless, there is no canonical extension of this holonomy to the other side in general.

Thus we will adopt the following notion of holonomy: Fix a neighborhood U of $q \in \mathrm{Sing}(\mathcal{F})$ where $\mathcal{F}$ is given by a Morse function

$$f(x_1, \dots, x_m) = f(q) - x_1^2 - \cdots - x_\ell^2 + x_{\ell+1}^2 + \cdots + x_m^2.$$

Let $\gamma : [0,1] \longrightarrow \Lambda \cap \overline{U}$ be any piecewise smooth path and let Σ_0 and Σ_1 be local transverse sections to $\mathcal{F}$ at $\gamma(0)$ and $\gamma(1)$ respectively. The holonomy along γ will be the mapping which assigns to $t \in \Sigma_0$ the point $f^{-1}(f(t)) \cap \Sigma_1 \in \Sigma_1$.

This holonomy map is then well-defined, even if γ is not contained in $U \setminus \{q\}$.

5 Dead branches, pairings and elimination of pairs of singularities

We shall now show how to perform modifications (surgery) on foliations, under suitable conditions, in order to eliminate certain arrangements of singularities.

5.1 Trivial center–saddle pairing

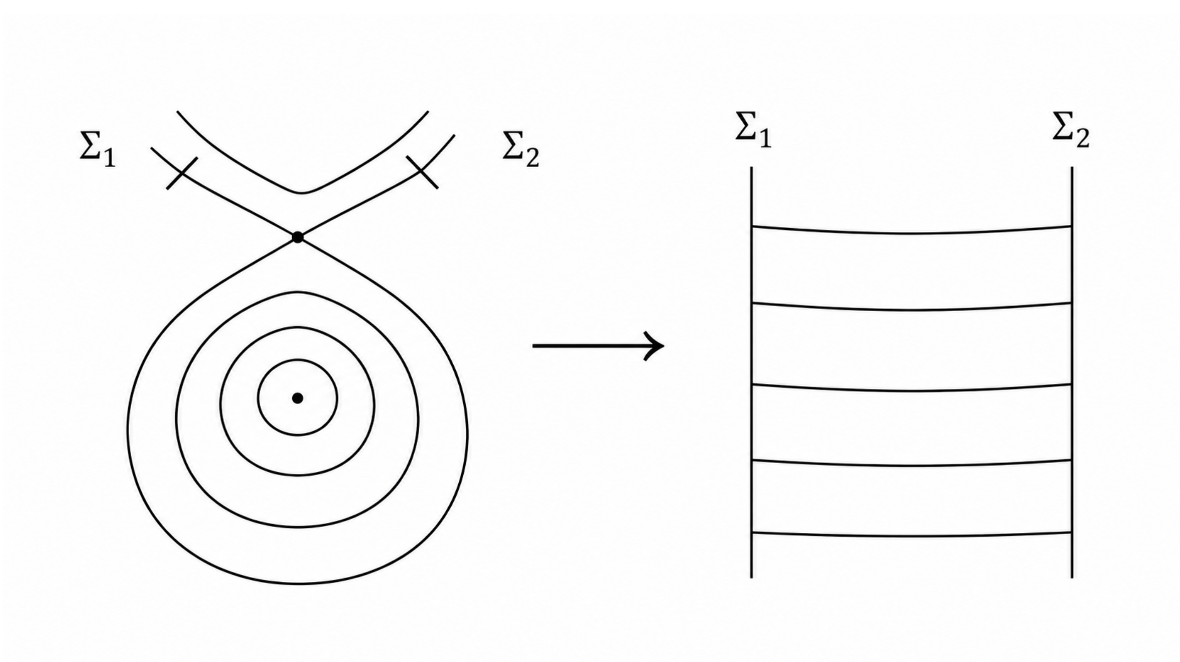

Figure 6: A trivial center–saddle pairing and the resulting trivial foliation.

We have a center-saddle pair which is replaced by a trivial foliation, without affecting the holonomy map $\Sigma_1 \to \Sigma_2$.

In dimension $m \geq 3$ we have the same construction, which can be obtained from the two-dimensional case by rotation. We obtain an isolated center

$$\sum_{j=1}^{m} x_j^2 = \text{constant}$$

and an isolated saddle singularity of type

$$x_m^2 - \sum_{j=1}^{m-1} x_j^2 = \text{constant}.$$

(we shall refer to a saddle

$$-\sum_{j=1}^{\ell} x_j^2 + \sum_{j=\ell+1}^{m} x_j^2 = \text{constant}$$

as a type $(\ell, m-\ell)$-saddle.)

Other center-saddle (non-trivial) pairings are depicted below.

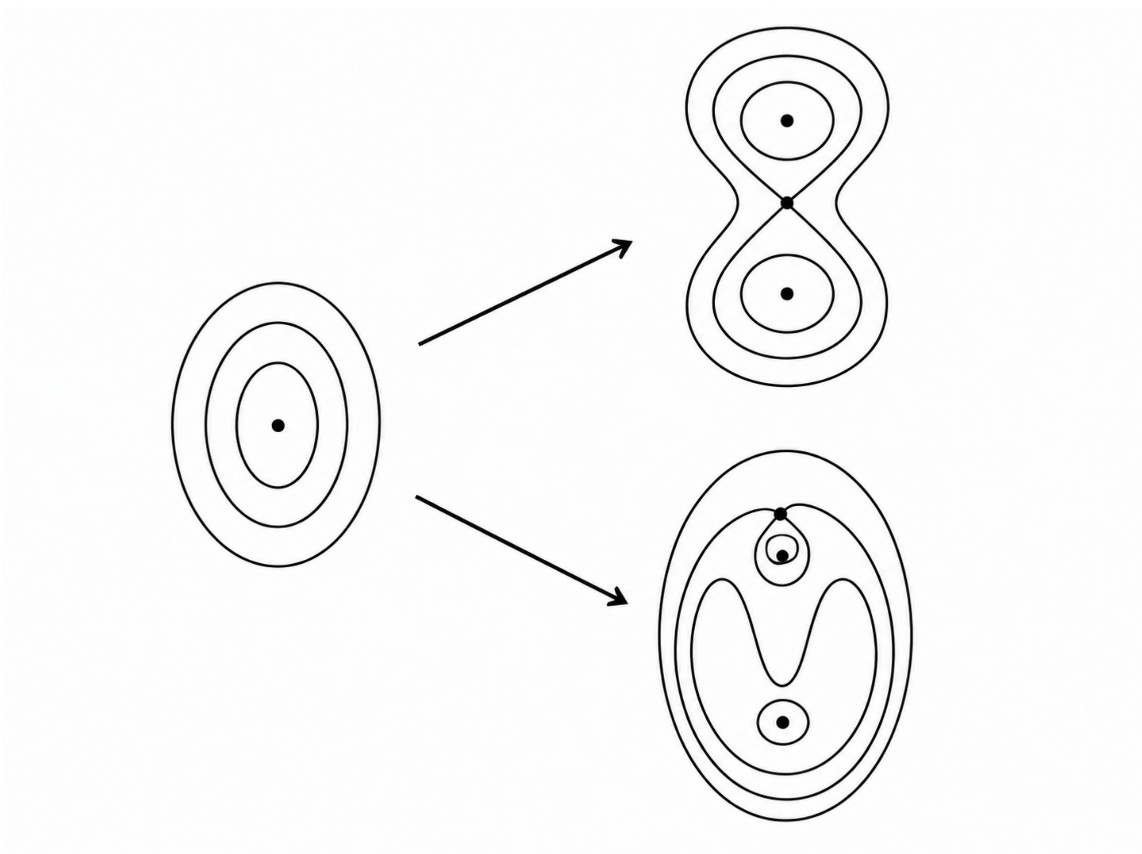

Figure 7: Examples of non-trivial center–saddle pairings.

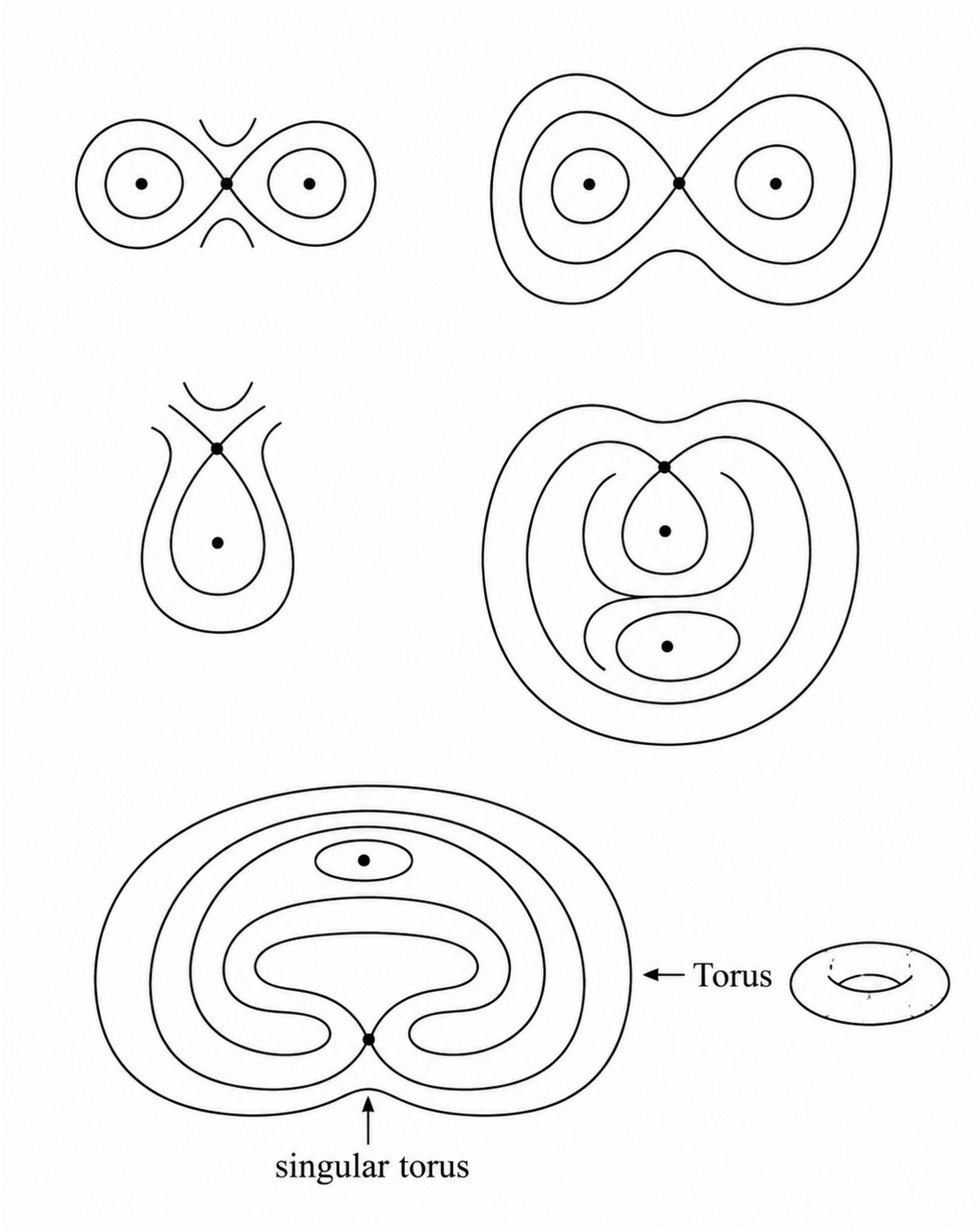

Figure 8: Further non-trivial pairings, including a torus and a singular torus.

5.2 Trivial pairings

Given $n > 0$ we write $D(n) \subset \mathbb{R}^{m-1}$ for the closed disk of radius $n > 0$ and $I(n) \subset \mathbb{R}$ centered at zero. We also write

$$B(n) := D(n) \times I(n) \subset \mathbb{R}^{m-1} \times \mathbb{R}.$$

We denote by $\mathcal{H}$ the foliation on $B(n)$ given by the submersion $(x, t) \mapsto t$.

Then, we consider a center $p \in \operatorname{Sing}(\mathcal{F})$ and a saddle $q \in \operatorname{Sing}(\mathcal{F})$. We shall say that p and q are paired if $q \in \partial C_p(\mathcal{F})$.

Definition 5.1. We say that p and q are in a *trivial pairing* if they are paired and there are open neighborhoods $V \supset V' \supset \overline{C_p(\mathcal{F})}$, $p, q \in V'$ and a diffeomorphism $\varphi : V \to B(1)$, such that $\varphi(V') = B(1/2)$ and

$$\mathcal{F}|_{V \setminus V'} = \varphi^*(\mathcal{H}).$$

Definition 5.2 [Dead branch]. Let $\mathcal{F}$ be a codimension one foliation, with isolated singularities on a manifold M^m. By a dead branch of $\mathcal{F}$ we mean a region $R \subset M$, diffeomorphic to the product $\overline{B}^{m-1} \times [0,1]$ of the closed unit ball $\overline{B}^{m-1} \subset \mathbb{R}^{m-1}$ by the interval $[0,1]$, as a manifold with boundary and corners, such that:

$$\partial R \cong \overline{B}^{m-1} \times \partial[0,1] \ \cup \ \partial B^{m-1} \times I$$

is the union of two connected invariant components (pieces of leaves of $\mathcal{F}$) and on $\partial B^{m-1} \times I$, $\mathcal{F}$ is a trivial foliation transverse to the fibers $\{q\} \times I$.

5.3 Basin of a center singularity

Let $\mathcal{F}$ be a Morse foliation in M, manifold of dimension $m \geq 3$. A singularity $p \in \operatorname{Sing}(\mathcal{F})$ is a center if there is a Morse function f, defining $\mathcal{F}$ in a neighborhood of p, which is of the form

$$f(x_1, \ldots, x_m) = f(p) \pm \sum_{j=1}^{m} x_j^2.$$

In this case, there is a fundamental system of invariant neighborhoods W of p in M, such

that $\mathcal{F}|_W$ is a foliation by $(m-1)$-spheres centered at p.

If p is not a center we shall call it a saddle singularity.

Given a center singularity $p \in \operatorname{Sing}(\mathcal{F})$ each nearby leaf L is diffeomorphic to a $(m-1)$-sphere and is therefore compact and simply connected ($m \geq 3 \Rightarrow m-1 \geq 2$). By theorem 4.1 this implies that there exists a neighborhood of L where all leaves are diffeomorphic to L. We introduce therefore the sets $C(\mathcal{F})$ as the union of centers in $\operatorname{Sing}(\mathcal{F})$ and all leaves diffeomorphic to S^{m-1}; and $C_p(\mathcal{F})$ as the connected component of $C(\mathcal{F})$ that contains p.

$C_p(\mathcal{F})$ will be called the basin of attraction of the center $p \in \operatorname{Sing}(\mathcal{F})$. We have:

Lemma 5.3 [Lemma C]. *Given centers $p_1 \neq p_2 \in \operatorname{Sing}(\mathcal{F})$ the sets $C_{p_1}(\mathcal{F})$ and $C_{p_2}(\mathcal{F})$ are open in M and $C_{p_1}(\mathcal{F}) \cap C_{p_2}(\mathcal{F}) \neq \varnothing$ if and only if, $C_{p_1}(\mathcal{F}) = C_{p_2}(\mathcal{F})$. Moreover $C_p(\mathcal{F}) = M$ if and only if $\partial C_p(\mathcal{F}) = \varnothing$, in which case M is homeomorphic to S^m, provided that M is orientable.*

Proof. The proof is quite straightforward. First observe that $C(\mathcal{F})$ is open. Indeed, this follows from the Morse local model in a neighborhood of a center and, at a regular spherical leaf, from Reeb local stability, since S^{m-1} is compact and simply connected for $m \geq 3$. Hence each component $C_p(\mathcal{F})$ is open. The fact that $C_{p_1}(\mathcal{F}) \cap C_{p_2}(\mathcal{F}) \neq \varnothing \Rightarrow C_{p_1}(\mathcal{F}) = C_{p_2}(\mathcal{F})$ is evident. If $\partial C_p(\mathcal{F}) = \varnothing$ then $C_p(\mathcal{F})$ is open and closed in M and therefore $C_p(\mathcal{F}) = M$.

In this case $\mathcal{F}$ is a foliation of Morse type having only centers and leaves diffeomorphic to S^{m-1}. A classical theorem of Reeb [Reeb] then implies that M is homeomorphic to the sphere S^m.

□

Next we study further the possible intersections for boundaries of two basins.

Lemma 5.4 [Lemma D]. *Let $p_1, p_2 \in \operatorname{Sing}(\mathcal{F})$ be distinct centers such that*

$$\partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F}) \neq \varnothing.$$

Then we have the following (mutually exclusive) possibilities:

(i) $\partial C_{p_1}(\mathcal{F}) = \partial C_{p_2}(\mathcal{F})$ and

$$M = \overline{C_{p_1}(\mathcal{F})} \cup \overline{C_{p_2}(\mathcal{F})}.$$

(ii) $\partial C_{p_1}(\mathcal{F}) \neq \partial C_{p_2}(\mathcal{F})$ and there is a saddle point

$$q \in \partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F})$$

with Morse index 1 or $m-1$, without self-connection.

Proof. Since $\partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F}) \neq \varnothing$ there must be a saddle point

$$q \in \partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F}).$$

Indeed, a center cannot belong to the boundary of a basin, because the Morse local model gives a whole neighborhood contained in $C(\mathcal{F})$. On the other hand, a regular common-boundary leaf cannot constitute a boundary component by itself: the spherical leaves of the two basins accumulate on it and Reeb local stability would continue the spherical foliation across such a regular boundary. Hence the common boundary component must meet $\operatorname{Sing}(\mathcal{F})$, and the singularity so obtained is necessarily a saddle.

We have two distinct cases:

<u>1st case</u>: q has Morse index $r \notin \{1, m-1\}$. In suitable coordinates $(x_1, \ldots, x_m)$ we have $q = (0, \ldots, 0)$ and $\mathcal{F}$ given by

$$-\left(x_1^2 + \cdots + x_r^2\right) + \left(x_{r+1}^2 + \cdots + x_m^2\right) = \text{constant}.$$

The separatrix C_q through q is given by

$$x_1^2 + \cdots + x_r^2 = x_{r+1}^2 + \cdots + x_m^2.$$

In particular, it is connected.

Thus, if C is the global separatrix of $\mathcal{F}$ at q we have

$$\partial C_{p_1}(\mathcal{F}) = \partial C_{p_2}(\mathcal{F}) = \overline{C} = C \cup \{q\}.$$

Here the equality $\overline{C} = C \cup \{q\}$ follows from the absence of saddle-connections: the closure of C cannot contain another saddle, and a regular extra accumulation leaf would contradict Reeb local stability by the same transverse-intersection argument used below in Lemma 5.5 [Lemma E].

It remains only to note that the two basin closures fill M. Put

$$A = \overline{C_{p_1}(\mathcal{F})} \cup \overline{C_{p_2}(\mathcal{F})}.$$

The set A is closed. It is also open. At a regular point of $\overline{C}$ a small foliated box is divided by the plaque of C into two half-boxes; since the point belongs to the boundary of both basins, one half-box lies in $C_{p_1}(\mathcal{F})$ and the other in $C_{p_2}(\mathcal{F})$. At q, since $2 \leq r \leq m-2$, the two regions

$$-\left(x_1^2 + \cdots + x_r^2\right) + \left(x_{r+1}^2 + \cdots + x_m^2\right) < 0$$

and

$$-\left(x_1^2 + \cdots + x_r^2\right) + \left(x_{r+1}^2 + \cdots + x_m^2\right) > 0$$

are connected and are occupied by the two basins. Thus a neighborhood of every point of $\overline{C}$ is contained in A. Since M is connected, $A = M$, and we are in case (i).

<u>2nd case</u>: q has Morse index 1 or $m-1$.

If C as above is a self-connection then

$$\partial C_{p_1}(\mathcal{F}) = \partial C_{p_2}(\mathcal{F}) = \overline{C}.$$

As in the first case, the complement of this common boundary has exactly the two components which contain the two center basins: locally the regions determined by the two branches of the cone are joined according to the self-connection, and every regular point of $\overline{C}$ has the two basins on its two sides. Hence

$$M = \overline{C_{p_1}(\mathcal{F})} \cup \overline{C_{p_2}(\mathcal{F})},$$

and we are again in case (i). Let us now assume that q has no self-connection. There are local coordinates $(x_1, \ldots, x_m)$ such that $q = (0, \ldots, 0)$ and $\mathcal{F}$ is given by the levels of

$$f = -x_1^2 + x_2^2 + \cdots + x_m^2.$$

Then $(f = 0)$ bounds the regions

$$R_1 : x_1 < 0, \quad x_1^2 > x_2^2 + \cdots + x_m^2;$$

$$R_2 : x_1 > 0, \quad x_1^2 > x_2^2 + \cdots + x_m^2$$

and

$$R_3 : x_1^2 < x_2^2 + \cdots + x_m^2.$$

Then, $C_{p_j}(\mathcal{F}) \cap R_3 = \varnothing$, otherwise we would have a saddle-connection at q, $j = 1, 2$. Moreover,

$$C_{p_1}(\mathcal{F}) \cap R_1 \neq \varnothing \Rightarrow C_{p_1}(\mathcal{F}) \cap R_2 = \varnothing$$

for the same reason.

Therefore $\partial C_{p_1}(\mathcal{F}) \neq \partial C_{p_2}(\mathcal{F})$ proving (ii).

□

Our basic elimination result is the following:

Lemma 5.5 [Lemma E – Elimination lemma]. *Let $p_1, p_2 \in \operatorname{Sing}(\mathcal{F})$ be two different centers such that*

$$\partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F}) \neq \varnothing$$

and let

$$q \in \partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F})$$

be a saddle of index 1 or $m-1$ and no self-connection. Then either p_1, q or p_2, q form a trivial pairing.

Proof. In a suitable local system of coordinates we have $q = (0, \ldots, 0)$ and $\mathcal{F}$ given by the levels of

$$f = -x_1^2 + x_2^2 + \cdots + x_m^2.$$

The cone separatrix

$$x_1^2 = x_2^2 + \cdots + x_m^2$$

divides the neighborhood into three regions

$$R_1 : x_1 < 0, \quad x_1^2 > x_2^2 + \cdots + x_m^2,$$

$$R_2 : x_1 > 0, \quad x_1^2 > x_2^2 + \cdots + x_m^2,$$

$$R_3 : x_1^2 < x_2^2 + \cdots + x_m^2.$$

The cone leaves of the local foliation are γ_1 and γ_2, bounding R_1 and R_2 respectively.

Denote by L_j the global leaf of $\mathcal{F}$ that contains γ_j, $j = 1, 2$. Then $L_1 \neq L_2$ because q is not self-connected. There are two basins $C_{p_j}(\mathcal{F})$, $j = 1, 2$, and three regions R_1, R_2, R_3. Therefore some basin $C_{p_j}(\mathcal{F})$ will intersect R_1 or R_2.

Let us suppose that

$$C_{p_1}(\mathcal{F}) \cap R_1 \neq \varnothing.$$

Then $C_{p_1}(\mathcal{F}) \cap R_2 = \varnothing$ and $C_{p_1}(\mathcal{F}) \cap R_3 = \varnothing$. Indeed, R_2 and R_3 have the leaf L_2 in their boundary and if either $C_{p_1}(\mathcal{F}) \cap R_2 \neq \varnothing$ or $C_{p_1}(\mathcal{F}) \cap R_3 \neq \varnothing$ this would imply a self saddle-connection at q.

Thus

$$\partial C_{p_1}(\mathcal{F}) = \overline{L}_1.$$

<u>Claim</u>: $\overline{L}_1 = L_1 \cup \{q\}$.

<u>proof of the claim</u>: Indeed, if there is a regular point $s \in \overline{L}_1 \setminus L_1$ then we take a small transverse section Σ_s centered at s.

Since $s \in \overline{L}_1 \setminus L_1$ then number of intersection points of L_1 with Σ_s is infinite.

On the other hand, given any leaf $L \subset C_{p_1}(\mathcal{F})$ and a local transverse section Σ to $\mathcal{F}$, the number $n(L, \Sigma)$ of intersection points of L with Σ is locally constant with respect to $L \subset C_{p_1}(\mathcal{F})$; this is a consequence of Reeb local stability theorem.

Since L_1 is accumulated by leaves in $C_{p_1}(\mathcal{F})$ we would obtain a sequence of leaves

$$L_\nu \subset C_{p_1}(\mathcal{F}), \qquad \nu \in \mathbb{N},$$

with

$$n(L_\nu, \Sigma_s) \longrightarrow \infty$$

as $\nu \to \infty$; contradiction. Therefore

$$\partial C_{p_1}(\mathcal{F}) = L_1 \cup \{q\}.$$

This proves the claim.

Let us now take a neighborhood U of q small enough so that $\overline{L}_1 \cap U = \overline{\gamma}_1$. Given a leaf $L \subset C_{p_1}(\mathcal{F})$ close enough to $\overline{L}_1$, $L \cap U$ is connected and $f|_{L \cap U} = \delta < 0$ a constant. Let us write $L = L_\delta$ ($\delta = f|_L$ in the above notation).

Observe that as $\delta \to 0$, $L_\delta \setminus U$ approaches $L_1 \setminus \gamma_1$ and this implies that $\overline{L}_1 \setminus \gamma_1$ is homeomorphic to $L_\delta \setminus U$ and therefore to an $(m-1)$-disk. Therefore $\overline{L}_1$ is homeomorphic to the sphere S^{m-1}.

Furthermore, for $\varepsilon > 0$ small enough, each leaf $f^{-1}(\delta)$ of $\mathcal{F}|_{R_1 \cup R_3}$, $-2\varepsilon \leq \delta \leq 2\varepsilon$, bounds an $(m-1)$-disk D_δ close to $L_1 \setminus \gamma_1$, with

$$D_0 = L_1 \setminus \gamma_1.$$

The union

$$T_{2\varepsilon} = \bigcup_{-2\varepsilon \leq \delta \leq 2\varepsilon} \overline{D}_\delta$$

is a trivially foliated neighborhood of $L_1 \setminus \gamma_1$. We can extend the function f to the union $U \cup T_{2\varepsilon}$ by writing $f|_{D_\delta} = \delta$. We then define a saturated neighborhood V_0 of $\overline{L}_1 \cup \gamma_2$ by setting

$$V_0 := f^{-1}([-\varepsilon, \varepsilon]) \cup C_{p_1}(\mathcal{F})$$

and define

$$g = f|_{V_0 \setminus C_{p_1}(\mathcal{F})}.$$

Take now a Riemannian metric on M and a normal vector field $\vec{N}$ to $\mathcal{F}$ that in U takes the form

$$\vec{N} = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_m \frac{\partial}{\partial x_m}.$$

For $a > 0$ small enough, the submanifold

$$E := \{x_1 = a\} \cap \gamma_2$$

is well-defined and diffeomorphic to S^{m-2}.

Let Σ' be a cross section to $\mathcal{F}$ over E, i.e., $\Sigma' \cap \gamma_2 = E$, Σ' contained in $\overline{V}_0$ and invariant by $\vec{N}$. Take Σ' diffeomorphic to $E \times [-\varepsilon, \varepsilon]$ by means of a map that takes each product $E \times s$, $-\varepsilon \leq \delta \leq \varepsilon$, to the leaf $(f = \delta) \cap \Sigma'$ of $\mathcal{F}|_{\Sigma'}$.

Consider the region $V \subset \overline{V}_0$ bounded by $(g = -\varepsilon) \cup (g = \varepsilon)$ and Σ' and define a neighborhood W such that $\partial V \subset W \subset \overline{V}$ as

$$W := (-\varepsilon \leq g \leq -\varepsilon/2) \cup (\varepsilon/2 \leq g \leq \varepsilon) \cup N$$

where N is a neighborhood of $\Sigma' \subset \overline{V}$ invariant by $\vec{N}$.

The foliation $\mathcal{F}|_W$ is trivial in the sense that:

- on $(-\varepsilon \leq g \leq -\varepsilon/2) \cup (\varepsilon/2 \leq g \leq \varepsilon)$ the leaves of $\mathcal{F}$ are levels of g, diffeomorphic to the disk $D(1)$.

- in N the leaves of $\mathcal{F}$ are levels of g $(g = \delta)$, $-\varepsilon/2 \leq \delta \leq \varepsilon/2$, diffeomorphic to $D(1) \setminus D(1/2)$.

- In W the foliation $\mathcal{F}$ and the trajectories of $\vec{N}$ are transverse everywhere.

Therefore, a diffeomorphism

$$\varphi : W \longrightarrow D(1) \setminus D(1/2)$$

can be constructed by sending leaves of $\mathcal{F}|_W$ at the level $(g = \alpha\varepsilon)$ onto leaves of $\mathcal{H}|_{D(1)\setminus D(1/2)}$ at the level $(x = \alpha)$ and orbits of $\vec{N}|_W$ to orbits of $\frac{\partial}{\partial t}$. Then φ is extended to the neighborhood V.

□

Second proof of Lemma 5.5 [Lemma E], using the basin boundary. We keep the notation above and suppose, as before, that

$$C_{p_1}(\mathcal{F}) \cap R_1 \neq \varnothing.$$

The first part of the preceding proof gives

$$\partial C_{p_1}(\mathcal{F}) = \overline{L}_1 = L_1 \cup \{q\}.$$

We now use the global structure of the center basin. Every regular leaf of $C_{p_1}(\mathcal{F})$ is a sphere S^{m-1} and $C_{p_1}(\mathcal{F}) \setminus \{p_1\}$ is a proper connected saturated open subset of the regular part of $\mathcal{F}$. Therefore, by lemma 4.5,

$$C_{p_1}(\mathcal{F}) \setminus \{p_1\} \simeq S^{m-1} \times (0, 1)$$

as a foliated manifold. A spherical leaf sufficiently close to the boundary meets a small Morse neighborhood of q in the standard cap. Its complementary part is an $(m-1)$-disk and tends, as the leaf approaches the boundary, to L_1 outside that neighborhood. Consequently

$$\overline{L}_1 = L_1 \cup \{q\} \simeq S^{m-1},$$

and the closure of the center basin is a topological m-ball with the standard Morse center at p_1 and the standard saddle attachment at q.

Choose a slightly larger neighborhood V of $\overline{C_{p_1}(\mathcal{F})}$. Outside the singular boundary the leaves are regular spheres with trivial holonomy, so Reeb local stability gives a foliated collar of the boundary. Taking a smaller neighborhood $V' \Subset V$ containing p_1 and q, we obtain

$$V \setminus V' \simeq S^{m-1} \times [0, 1]$$

with the product foliation. The local Morse models at p_1 and q show that V' and V are m-balls. Hence there is a diffeomorphism

$$\varphi : V \longrightarrow B(1), \qquad \varphi(V') = B(1/2),$$

which carries the foliation on $V \setminus V'$ to the standard product foliation on $B(1) \setminus B(1/2)$. Thus p_1 and q form a trivial pairing. The other possibility is obtained by interchanging p_1 and p_2.

□

6 Classification of certain Morse foliated manifolds

Let us now present some of our results on Morse foliations with a certain configuration of singularities. Let M be a compact connected oriented manifold, and let $\mathcal{F}$ be a transversely oriented codimension one C^∞ Morse foliation on M. We denote by k the number of centers in $\mathrm{Sing}(\mathcal{F})$ and by ℓ the number of saddles.

Our first result reads as follows (see [C-Sc2]):

Theorem 6.1 [Theorem A]. *Let $\mathcal{F}$ be a Morse foliation on the closed manifold M of dimension $m \geq 3$. If $k \geq \ell + 2$ then $k = \ell + 2$ and M is homeomorphic to the m-sphere S^m.*

In the course of the proof we shall prove that there is a foliation $\mathcal{F}'$ on M which is a Morse foliation with singular set consisting of exactly two centers and $\mathcal{F}'$ being given by a Morse function $f : M \to \mathbb{R}$.

The foliation $\mathcal{F}'$ will be obtained from $\mathcal{F}$ by "elimination" of certain center–saddle pairs if $\ell > 0$.

Lemma 6.2 [Lemma F – Reeb]. *Let $\mathcal{F}$ be a Morse foliation on a closed manifold M of dimension $m \geq 3$. Assume that* $\mathrm{Sing}(\mathcal{F})$ *consists only of centers. Then $\#\,\mathrm{Sing}(\mathcal{F}) = 2$, $\mathcal{F}$ is given by a Morse function and M is homeomorphic to the m-sphere S^m.*

Proof. Since $\mathrm{Sing}(\mathcal{F}) \neq \varnothing$, there is a center singularity $p \in \mathrm{Sing}(\mathcal{F})$. Then $C_p(\mathcal{F})$ is open, non-empty and connected in M. Let us prove that $\partial C_p(\mathcal{F}) = \varnothing$. If the boundary were non-empty, the same Reeb-stability argument used in Lemma 5.3 [Lemma C] would imply that it contains a singularity. But every singularity is a center, and a center has a whole Morse neighborhood contained in $C(\mathcal{F})$; hence a center cannot lie in the boundary of a connected component of $C(\mathcal{F})$. This contradiction proves that $\partial C_p(\mathcal{F}) = \varnothing$, and therefore $C_p(\mathcal{F}) = M$. The classical Reeb sphere theorem [Reeb] now gives $\#\,\mathrm{Sing}(\mathcal{F}) = 2$, shows that $\mathcal{F}$ is defined by a Morse function with two critical points, and implies that M is homeomorphic to S^m.

□

Proof of Theorem 6.1 [Theorem A] . We will proceed by induction on the number ℓ of saddle singularities in $\mathrm{Sing}(\mathcal{F})$.

1st case: $\ell = 0$.

If $\ell = 0$ then $\mathcal{F}$ has only centers in M. In this case the theorem follows from Lemma 6.2 [Lemma F].

2nd case: Assume now that $\ell \geq 1$ and that the result has been proven for foliations with at most $\ell - 1$ saddle type singularities. We are assuming $k \geq \ell + 2$. Thus, there are two centers $p_1, p_2 \in \mathrm{Sing}(\mathcal{F})$ such that there is a saddle q with

$$q \in \partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F}).$$

Indeed, given any center $p \in \mathrm{Sing}(\mathcal{F})$ if $\partial C_p(\mathcal{F}) = \varnothing$ then we can conclude that $\ell = 0$ and $M \simeq S^m$. Thus, we will assume that for each center p there is some saddle singularity $q(p) \in \partial C_p(\mathcal{F})$. Since $k \geq \ell + 2$, there exist $p_1 \neq p_2$ centers such that $q(p_1) = q(p_2)$.

By Lemma 5.4 [Lemma D] we have either

$$M = \overline{C_{p_1}(\mathcal{F})} \cup \overline{C_{p_2}(\mathcal{F})}$$

or q has index 1 or $m - 1$ and is not self-connected.

If

$$M = \overline{C_{p_1}(\mathcal{F})} \cup \overline{C_{p_2}(\mathcal{F})},$$

then the common boundary is the singular separatrix through q and, since there are no saddle-connections, it contains no other saddle. Hence

$$\mathrm{Sing}(\mathcal{F}) = \{p_1, p_2, q\},$$

so that $k = 2$ and $\ell = 1$, contradiction with $k \geq \ell + 2$.

Thus q must have index 1 or $m - 1$ and no self-intersection. By Lemma 5.5 [Lemma E] we have a trivial pairing $p_j - q$ for some $j \in \{1, 2\}$ and therefore we can eliminate this pair obtaining a Morse foliation $\mathcal{F}'$ on M whose number of centers is $k' = k - 1$ and saddles is $\ell' = \ell - 1$. The modification is performed inside the trivial-pairing neighborhood and is unchanged near its boundary; therefore it creates no new singularities and no new saddle-connections, and $\mathcal{F}'$ is again a Morse foliation of the same type. Then from $k \geq \ell + 2$ we get

$$k' = k - 1 \geq \ell + 1 = (\ell - 1) + 2 = \ell' + 2,$$

and by the induction hypothesis we may conclude that $M \simeq S^m$ which proves the theorem.

$\square$

In dimension 3 every saddle has index 1 or 2, so that we are always in the index-$1/m-1$ situation. This allows us to strengthen Theorem 6.1 [Theorem A] as follows (see [C-Sc1]):

Theorem 6.3 [Theorem A′]. *Let $\mathcal{F}$ be an oriented Morse foliation on a closed, oriented 3 manifold M^3. Suppose that the number k of centers and the number ℓ of saddles satisfy $k > \ell$. Then M is diffeomorphic to S^3.*

7 Eells–Kuiper manifolds

A classical result due to Reeb states that a compact manifold admitting a Morse function with only two critical points is homeomorphic to the sphere of the same dimension. This also follows from Theorem 6.1 [Theorem A]. The next interesting case is that of a Morse function with exactly three critical points, two centers and one saddle. Such manifolds were studied and classified by Eells and Kuiper [E-K1, E-K2].

Theorem 7.1 [Eells–Kuiper]. *Let M be a connected closed manifold (not necessarily orientable) of class C^∞ and dimension m. Suppose M admits a Morse function with exactly three singular points. Then:*

(i) $m \in \{2, 4, 8, 16\}$.

(ii) M is topologically a compactification of $\mathbb{R}^m$ by an $m/2$-sphere.

(iii) If $m = 2$ then M is diffeomorphic to $\mathbb{R}P(2)$.

If $m \geq 4$ then M is simply connected and has the integral cohomology ring of

$$\begin{cases} \mathbb{C}P^2 & \text{if } m = 4 \quad \text{(complex projective plane)}, \\ \text{Quaternionic projective plane} & \text{if } m = 8, \\ \text{Cayley projective plane} & \text{if } m = 16. \end{cases}$$

We will call these manifolds Eells-Kuiper manifolds.

In this case the number of centers k and the number of saddles ℓ are related by $k = \ell + 1$.

As an extension of Theorem 6.1 [Theorem A] and Eells-Kuiper above we have:

Theorem 7.2 [Theorem B]. *Let M be a compact connected manifold and $\mathcal{F}$ a Morse foliation on M such that the number k of centers and the number ℓ of saddles in $\mathrm{Sing}(\mathcal{F})$ satisfy $k \geq \ell + 1$ (i.e., $k > \ell$). If $k = \ell + 1$ then M is an Eells–Kuiper manifold. The case $k \geq \ell + 2$ is dealt with by Theorem 6.1 [Theorem A].*

Theorem 7.2 [Theorem B] is proved in a similar way to Theorem 6.1 [Theorem A], i.e., an induction argument and a basic case with minimum k and ℓ. For Theorem 6.1 [Theorem A] this is the case $k = 2$ and $\ell = 0$ which is covered by Reeb's theorem. For Theorem 7.2 [Theorem B] the minimum case is $k = 2$, $\ell = 1$ which will be covered by the following

Lemma 7.3 [Lemma G]. *Let $\mathcal{F}$ be a Morse foliation on a closed connected manifold M of dimension $m \geq 3$. Assume that $k = 2$ and $\ell = 1$. Then $\mathcal{F}$ is given by a Morse function $f : M \to \mathbb{R}$. In particular, M is an Eells–Kuiper manifold.*

The proof of Lemma 7.3 [Lemma G] is contained in [C-Sc2], but we shall give the main steps below.

Proof. Let p_1, p_2 be the two centers and let q be the unique saddle. Since the boundary of a proper center basin contains a saddle, necessarily

$$q \in \partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F}).$$

Apply Lemma 5.4 [Lemma D]. If case (ii) occurs, Lemma 5.5 [Lemma E] gives a trivial pairing between q and one of the centers. Eliminating that pair would produce a Morse foliation on the closed manifold M with one center and no saddles, which is impossible by Lemma 6.2 [Lemma F]. Therefore case (i) of Lemma 5.4 [Lemma D] occurs, and

$$\partial C_{p_1}(\mathcal{F}) = \partial C_{p_2}(\mathcal{F}), \qquad M = \overline{C_{p_1}(\mathcal{F})} \cup \overline{C_{p_2}(\mathcal{F})}.$$

Thus the leaf space is an interval: each $C_{p_j}(\mathcal{F}) \setminus \{p_j\}$ is a product by spherical leaves, one end of the interval corresponds to p_1, the other to p_2, and the common singular level contains q.

We now choose a coordinate on this interval. Near p_1 and p_2 we take the usual Morse first integrals for a center, and near q we take the Morse first integral

$$-\big(x_1^2 + \cdots + x_r^2\big) + \big(x_{r+1}^2 + \cdots + x_m^2\big).$$

On the regular parts two transverse first integrals defining the same oriented foliation differ by a smooth increasing function of one variable. Reparametrizing the interval coordinate on the corresponding subintervals, these local first integrals fit together to give a global smooth function

$$f : M \longrightarrow \mathbb{R}$$

whose levels are the leaves of $\mathcal{F}$ and whose critical points are exactly p_1, q, p_2. Hence f is a Morse function with exactly three critical points. The Eells-Kuiper theorem then implies that M is an Eells-Kuiper manifold.

□

We can now complete the induction for Theorem 7.2 [Theorem B]. The case $k \geq \ell + 2$ is Theorem 6.1 [Theorem A]. Assume $k = \ell + 1$. The case $k = 2$, $\ell = 1$ is Lemma 7.3 [Lemma G]. If $\ell \geq 2$, the same counting argument used in the proof of Theorem 6.1 [Theorem A] gives two centers whose basin boundaries contain the same saddle q. Case (i) of Lemma 5.4 [Lemma D] would force $k = 2$, $\ell = 1$, so case (ii) occurs and Lemma 5.5 [Lemma E] gives a trivial center-saddle pair. After eliminating it we have

$$k' = k - 1, \qquad \ell' = \ell - 1, \qquad k' = \ell' + 1.$$

As in Theorem 6.1 [Theorem A], the elimination preserves the Morse character and creates no saddle-connections. Induction therefore reduces the foliation to the basic case of Lemma 7.3 [Lemma G], proving Theorem 7.2 [Theorem B].

8 Haefliger's Theorem

One of the most important classical results in the geometric theory of codimension one smooth foliations is the theorem of Haefliger below [Haefliger]:

Theorem 8.1 [Haefliger]. *Let $\mathcal{F}$ be a C^2 foliation, of codimension one on a manifold M. Assume that $\mathcal{F}$ admits a null homotopic compact transverse section. Then there is a leaf L of $\mathcal{F}$ and a closed path δ in L such that the holonomy map corresponding to δ is of the form*

$$h : (-\varepsilon, \varepsilon) \longrightarrow (\mathbb{R}, 0), \qquad h|_{(-\varepsilon,0]} = \mathrm{Id}$$

and $h|_{(0,\varepsilon)}$ is not the identity.

We shall refer to the situation in the conclusion as $\mathcal{F}$ has a leaf with some one-sided holonomy map. In particular, $\mathcal{F}$ is not real analytic.

By a null-homotopic compact transverse section we mean a closed transverse map

$$\gamma : S^1 \longrightarrow M$$

which is null-homotopic in M, i.e., whose homotopy class represents the identity element of $\pi_1(M)$.

One important fact, connected to the above hypothesis is:

Lemma 8.2 [Lemma H]. *Let $\mathcal{F}$ be a codimension one C^2 foliation on a manifold M. Given a leaf L of $\mathcal{F}$ if L is not closed then there is a compact C^2 transverse section $\gamma : S^1 \to M$ to $\mathcal{F}$ which intersects L.*

Proof. Given a point $p \in \overline{L} \setminus L$ we consider a transverse interval Σ_0 to $\mathcal{F}$ passing by p depicted below:

Choosing points $q_1, q_2 \in L \cap \Sigma_0$ we have a figure like

Using now a fibration by transverse segments based on $\alpha : [0,1] \to L$, $\alpha(0) = q_1$, $\alpha(1) = q_2$, we may cut and complete Σ_0 into a closed curve transverse to $\mathcal{F}$.

In particular, we conclude:

□

Corollary 8.3. *If a C^2 codimension one foliation on a compact manifold M has some non-compact leaf then it admits a C^2 compact closed transverse section.*

Combining this with theorem 8.1 we obtain the next corollary. Indeed, if $\gamma : S^1 \to M$ is the closed transversal supplied by Lemma 8.2 [Lemma H] and $\pi_1(M)$ is finite, then the class $[\gamma]$ has finite order. Hence, for some $N \geq 1$, the N-fold iterate γ^N is null-homotopic. Since it is obtained by traversing the same transverse loop N times, γ^N is still a compact transverse section, and theorem 8.1 applies.

Corollary 8.4. *Let M be a closed manifold with finite fundamental group. A foliation $\mathcal{F}$ of class C^2 and codimension one on M satisfies one of the following:*

(i) $\mathcal{F}$ has all leaves compact.

(ii) $\mathcal{F}$ has a leaf with a one-sided holonomy map.

In particular, if $\mathcal{F}$ is real analytic then $\mathcal{F}$ is a foliation with all leaves compact.

For instance we obtain:

Corollary 8.5. *There is no real analytic codimension one foliation of the sphere S^m if $m \geq 2$.*

Proof. There are several ways of proving this. By corollary 8.4, all leaves are compact, since a real analytic holonomy germ cannot be the identity on one side and non-trivial on the other. For a codimension-one compact foliation the holonomy groups of the compact leaves are finite. Since S^m is simply connected, the foliation is transversely orientable, and the finite holonomy groups are orientation preserving; by remark 4.4 they are therefore trivial. Reeb local stability then makes the leaf space a compact one-dimensional manifold, so $\mathcal{F}$ is given by a submersion

$$f : S^m \longrightarrow S^1.$$

Now take the angle element 1-form θ in S^1 (in coordinates $(x, y) \in \mathbb{R}^2$, $S^1 : x^2 + y^2 = 1$) and

$$\theta = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Then the pull-back one form

$$\omega = f^*(\theta)$$

is closed and non-singular in S^m. Since $m \geq 2$, S^m is simply connected. Therefore ω is exact, say $\omega = dg$ for some C^1 function

$$g : S^m \longrightarrow \mathbb{R}.$$

Since S^m is compact, g has a critical point, which would be a zero of ω, a contradiction.

□

8.1 Vector fields transverse to the boundary of the disk in the plane

Let us now consider the following situation. We have a C^2 vector field $\vec{X}$ defined in a neighborhood W of the closed unit disk

$$\overline{D}^2(1) \subset \mathbb{R}^2$$

centered at the origin $0 \in \mathbb{R}^2$.

We will also assume that:

(i) $\vec{X}$ is transverse to the boundary $S^1 = \partial D^2(1)$.

(ii) the singularities of $\vec{X}$ in $D^2(1)$ are of Morse type: $\forall p \in \mathrm{Sing}(\vec{X}) \cap D^2(1) \Rightarrow \exists$ local coordinates $(x_1, x_2) \in U_p$ with $p \in U_p \subset D^2(1)$, $x_1(p) = x_2(p) = 0$ and such that

$$X(x_1, x_2) = g_p(x_1, x_2) \cdot \vec{H}_p(x_1, x_2)$$

where $g_p : U_p \to \mathbb{R}$ is a non-vanishing C^2 function and $\vec{H}_p(x_1, x_2)$ is one of the two hamiltonian vector fields

$$\text{(a)} \quad \vec{H}_p(x_1, x_2) = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$$

or

$$\text{(b)} \quad \vec{H}_p(x_1, x_2) = x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}.$$

(iii) $\vec{X}$ has no saddle-connections i.e., if $q_1, q_2 \in \mathrm{Sing}(\vec{X}) \cap D^2(1)$ are saddle (type (b)) singularities then there is no separatrix of q_1 that is also a separatrix of q_2.

We shall say that $\vec{X}$ is a Morse type vector field in (a neighborhood of) the disk $D^2(1)$.

Let us now study the dynamics of such an object. By reversing the orientation we may assume that $\vec{X}$ points inward along $\partial D^2(1)$.

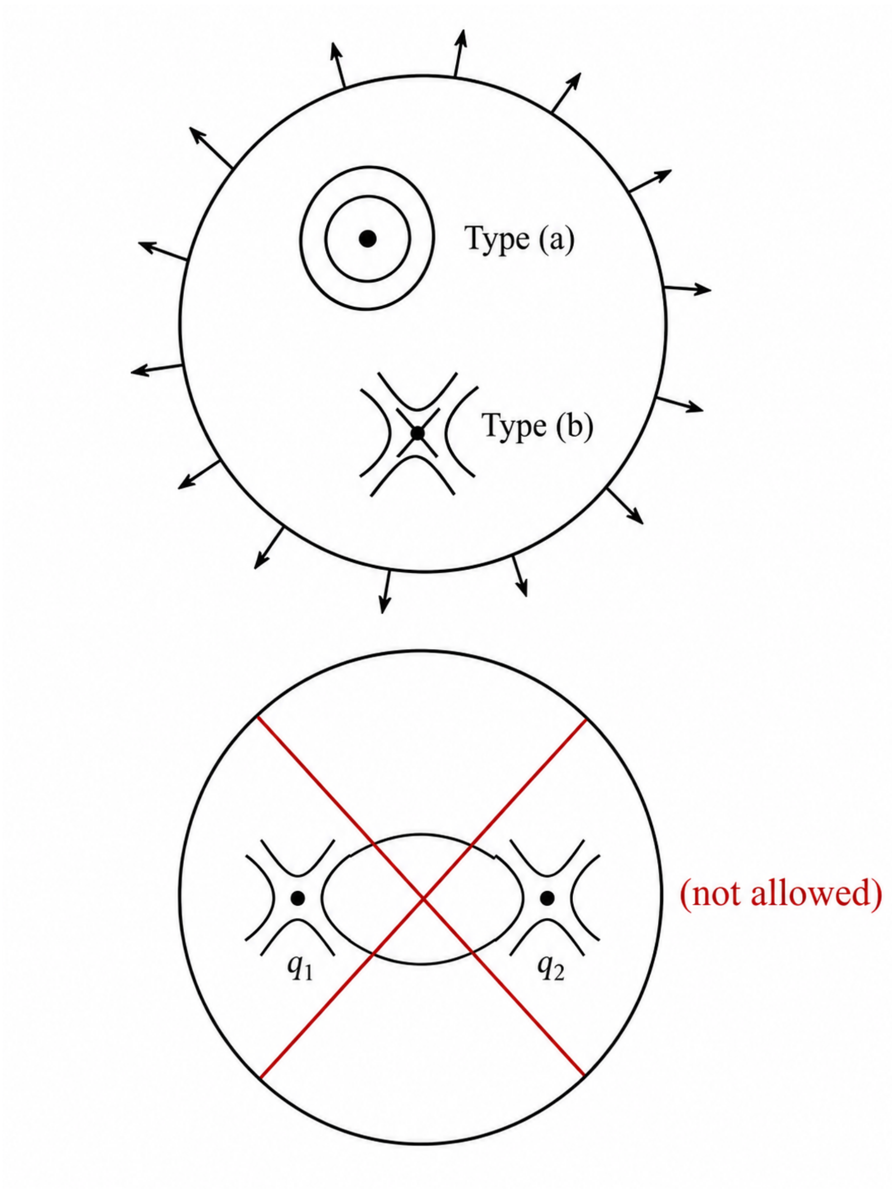

Figure 9: Allowed center/saddle configurations in the disk and a forbidden saddle-connection.

Given a singularity $p \in \mathrm{Sing}(\vec{X}) \subset D^2(1)$, if p is of type (a) (center) then the index of $\vec{X}$ at p is given by

$$I(\vec{X};p) = (+1)^2 = +1.$$

If p is a saddle (type (b)) then $I(\vec{X};p) = -1$.

By Euler-Poincaré-Hopf theorem we have

$$
\begin{aligned}
1 = \chi(D^2(1)) = &\sum_{p \in \mathrm{Sing}(\vec{X})} I(\vec{X}; p) \\
&= \#(\text{centers}) - \#(\text{saddles}) \\
&= k - \ell.
\end{aligned}
$$

Therefore we have $k = \ell + 1$ for $k = \#\text{centers}$ and $\ell = \#\text{saddles}$ in $\mathrm{Sing}(\vec{X})$.

Let us proceed with this information.

1st case: $\ell = 0$. In this case $k = 1$.

Since there is just one singularity, which is a center, and $\vec{X}|_{\partial D^2(1)}$ points inward, consider the maximal annulus of periodic orbits surrounding this center. It cannot reach $\partial D^2(1)$, because the vector field is transverse there. Its outer boundary is therefore a compact invariant subset of the interior of the disk and, since there is no other singularity, the Poincaré-Bendixson theorem gives a limit cycle $\Gamma \subset D^2(1)$ on this boundary.

On the side of Γ belonging to the period annulus the Poincaré map is the identity. On the other side it cannot also be the identity on a whole interval, otherwise the annulus of periodic orbits would extend beyond Γ, contradicting maximality. Thus the Poincaré map of Γ is a non-trivial one-sided holonomy map.

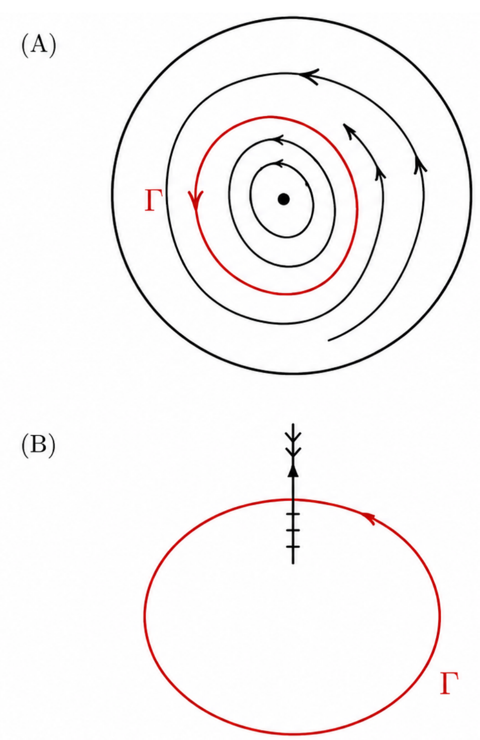

Figure 10: A limit cycle and its one-sided holonomy map.

Assume now that the following claim is valid for Morse type vector fields in $D^2(1)$ having at most

$\ell - 1$ saddle singularities. Here an invariant graph means a compact invariant set made of finitely many saddle singularities and separatrix arcs; a periodic orbit is allowed as the nonsingular case.

Claim 8.6. *The vector field $\vec{X}$ has an invariant graph with non-trivial one-sided holonomy.*

2nd case: $\ell \geq 1$.

We consider any center $p \in \mathrm{Sing}(\vec{X})$.

We introduce the basin of p, $B_p(\vec{X})$ as the union of p and all periodic orbits γ of $\vec{X}$ that bound a region $R(\gamma)$ containing only p and periodic orbits of $\vec{X}$.

If the boundary $\partial B_p(\vec{X})$ does not contain any singularity of $\vec{X}$, then it must give us a one-sided holonomy limit cycle. If a singularity belongs to $\partial B_p(\vec{X})$ then it must be a saddle.

Thus we may assume that for each center $p \in \mathrm{Sing}(\vec{X})$ there is a saddle

$$q_p \in \partial B_p(\vec{X}) \cap \mathrm{Sing}(\vec{X}).$$

The map

$$p \longmapsto q_p$$

cannot be injective because $k > \ell$. Thus there must be two distinct centers p_1, p_2 such that

$$\partial B_{p_1}(\vec{X}) \cap \partial B_{p_2}(\vec{X})$$

contains a saddle singularity.

(Recall that the saddle $q_p \in \partial B_p(\vec{X})$ is unique because $\vec{X}$ has no saddle-connections). Indeed, $\partial B_p(\vec{X})$ is a connected invariant boundary. If it contained two distinct saddles, a nonsingular piece of this boundary joining consecutive saddles would be a separatrix common to the two, i.e., a saddle-connection, contrary to (iii).

Then from Poincaré-Bendixson theorem we conclude that there exists a pairing $p_{j_0} - q$ which is trivial. More explicitly, the separatrix branches of q which bound the two center basins form the corresponding homoclinic lobes; at least one such lobe bounds a disk whose interior contains the corresponding center p_{j_0} and no other singularity, and this disk is filled by the periodic orbits of $B_{p_{j_0}}(\vec{X})$. This is precisely the trivial center-saddle configuration.

This allows us to modify $\vec{X}$ and obtain a new vector field $\vec{X}'$ which is of Morse type, coincides with $\vec{X}$ outside a neighborhood containing only p_{j_0} and q, has the same holonomy maps as $\vec{X}$, and satisfies

$$\mathrm{Sing}(\vec{X}') = \mathrm{Sing}(\vec{X}) \setminus \{p_{j_0}, q\}.$$

The modification is made inside the trivial-pairing disk and is unchanged near its boundary. Hence the remaining singularities keep their Morse type, no new saddle-connection is created, and the transversality to $\partial D^2(1)$ is unchanged. In particular, $\vec{X}'$ falls into the induction hypothesis above and we may conclude that it admits some invariant graph with a one-sided holonomy. Since $\vec{X}' = \vec{X}$ outside the cancellation region, the same holonomy graph is also present for $\vec{X}$.

8.2 Sketch of the proof of Haefliger's Theorem

Now we are ready to give the sketch of the proof of Haefliger's theorem.

First we observe that there is a C^2 closed transverse map

$$A_0 : S^1 \longrightarrow M^m$$

which is transverse to $\mathcal{F}$, with $A_0(S^1)$ homotopic to zero in M.

Using this we can obtain a C^2 immersion

$$A : \overline{D}^2(1) \longrightarrow M$$

of (a neighborhood of) the closed disk $D^2(1)$ in M such that $A|_{\partial D^2(1)} = A_0$ and

(i) $A|_{\partial D^2(1)}$ is transverse to $\mathcal{F}$.

(ii) The tangent points of $\mathcal{F}$ and A in $D^2(1)$ are all non-degenerate.

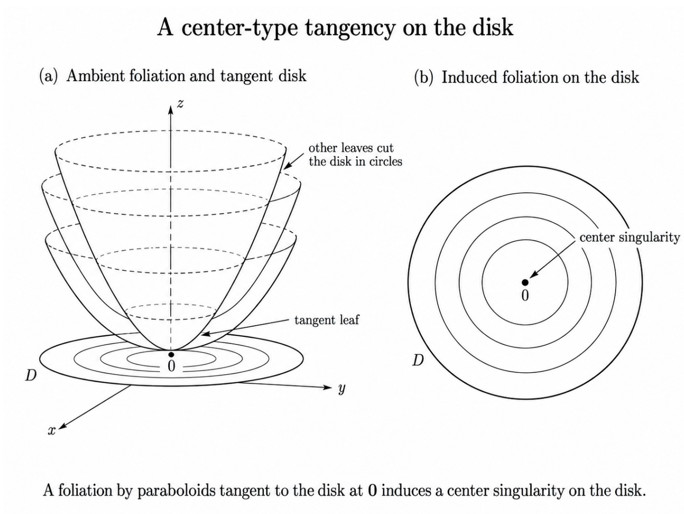

Figure 11: A local center-type tangency model. One ambient leaf is tangent to the disk at a single point, while the nearby leaves cut the disk in concentric circles. The induced foliation on the disk therefore has a center singularity.

(iii) The pull-back foliation $A^*(\mathcal{F})$ is a Morse foliation in the disk. After an arbitrarily small perturbation of A relative to the boundary, we may also assume that $A^*(\mathcal{F})$ has no saddle-connections.

Since the disk is simply connected, the tangent line field of $A^*(\mathcal{F})$ is orientable. We may therefore choose a C^1 vector field tangent to $A^*(\mathcal{F})$, with the same singularities, and reverse its orientation if necessary so that it points inwards along $\partial D^2(1)$. Then we are in the situation addressed above, of a Morse vector field in the disk $D^2(1)$ transverse to the boundary $\partial D^2(1)$. Using the above claim we conclude that $A^*(\mathcal{F})$, and therefore $\mathcal{F}$, has some non-trivial one-sided holonomy map in some leaf.

9 The graph of a Morse foliation

The material in this section was not part of the original Tokyo minicourse. It is about additional data, suggested by the elimination arguments above. We preserve the same framework from the former sections.

Let $\mathcal{F}$ be a transversely oriented Morse foliation on a closed connected manifold M of dimension $m \geq 3$, without saddle-connections. We denote by k the number of centers and by ℓ the number of saddles of $\mathcal{F}$.

Definition [Graph of a Morse foliation]. Let

$$\mathcal{S}(\mathcal{F}) = \mathrm{Sing}(\mathcal{F}) \cup \{\text{separatrix leaves of the saddles of } \mathcal{F}\}$$

be the singular-separatrix skeleton. A connected component U of $M \setminus \mathcal{S}(\mathcal{F})$ which is foliated by spheres will be called a *spherical region.* By lemma 4.5, whenever U is proper it is a spherical product region,

$$U \simeq S^{m-1} \times (0,1).$$

Whenever the two ends of each spherical region approach well-defined singularities, we associate to U an open edge whose endpoints are those singularities. The vertices are the singularities of $\mathcal{F}$. The resulting graph, possibly with loops or multiple edges, is denoted by

$$\mathcal{G}(\mathcal{F})$$

and is called the *graph of the Morse foliation* $\mathcal{F}$.

In the case $k > \ell$ considered below, the elimination argument proves in particular that this construction is globally well-defined and gives a finite graph.

The local Morse models describe the possible vertices. A center has one incident spherical sector. If q is a saddle of index 1 or $m-1$, the local cone separates a neighborhood of q into three connected regions. If q has index r, $2 \leq r \leq m-2$, the two regions $Q < 0$ and $Q > 0$ in the local model

$$Q = -x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_m^2$$

are connected.

Theorem 9.1 [Spherical tree theorem]. *Let $\mathcal{F}$ be a transversely oriented Morse foliation on a closed connected m-manifold, $m \geq 3$, without saddle-connections. Let k be the number of centers and ℓ the number of saddles. Assume that*

$$k > \ell.$$

Then the graph $\mathcal{G}(\mathcal{F})$ of $\mathcal{F}$ is a tree. Moreover, the terminal vertices are precisely the centers; every saddle of index 1 or $m-1$ is trivalent; and there is at most one saddle of intermediate index. When such a saddle occurs, it is bivalent.

Proof. We use exactly the elimination procedure of theorems 6.1 and 7.2. Since $k > \ell$, unless we are already in a basic configuration, the counting argument gives two centers whose basin boundaries contain a common saddle. By lemmas 5.4 and 5.5, this gives a trivial center–saddle pairing, and this pair can be eliminated. Repeating the procedure, we arrive at one of the two basic configurations:

$$(k,\ell) = (2,0)$$

or

$$(k,\ell) = (2,1).$$

In the first case Lemma 6.2 [Lemma F] gives the standard two-center foliation of the sphere. Its spherical graph is a single edge joining the two centers. In the second case Lemma 7.3 [Lemma G] gives a global Morse function with two centers and one saddle. The two center basins are spherical product regions meeting the singular level through the saddle, and the corresponding graph is a path with three vertices.

We now read the eliminations backwards. Reintroducing a trivial center-saddle pair is a local operation inside the ball of the trivial pairing. Outside this ball the foliation is unchanged. Inside it, the product spherical region is replaced by the standard three-sector configuration of an index 1 or $m-1$ saddle, one of the three sectors terminating at the newly introduced center. Consequently, at the level of the graph, an edge

$$u \;---\; v$$

is replaced by

$$u \;---\; q \;---\; v, \qquad q \;---\; p,$$

where p is the new center and q the new saddle. In particular, the reverse operation adds two vertices and two edges and cannot create a cycle. Starting from either of the two basic trees, every reverse elimination therefore produces again a finite tree. Since the reverse sequence reconstructs the original foliation, the graph $\mathcal{G}(\mathcal{F})$ is a finite tree with the local degrees described above.

Finally, every saddle introduced by reversing an elimination has index 1 or $m-1$. Hence an intermediate-index saddle can only be the saddle of the terminal $(2,1)$ configuration. There is therefore at most one such saddle.

□

Let

$$a = \#\{q \text{ saddle} : \mathrm{Ind}(q) = 1 \text{ or } m-1\}$$

and

$$b = \#\{q \text{ saddle} : 2 \le \mathrm{Ind}(q) \le m-2\}.$$

Thus $\ell = a + b$. Since $\mathcal{G}(\mathcal{F})$ is a tree, the degree formula gives

$$2E = k + 3a + 2b, \qquad E = (k + a + b) - 1.$$

It follows that

$$\boxed{k = a + 2}.$$

Since $k > \ell = a + b$, we obtain $b < 2$. Consequently:

$$\boxed{b = 0 \implies k = \ell + 2,}$$

while

$$\boxed{b = 1 \implies k = \ell + 1.}$$

Thus the two numerical alternatives in Theorems A and B have a graph-theoretic interpretation: in the first case the tree has no bivalent saddle vertex, whereas in the second case it has exactly one.

The elimination of a trivial center-saddle pair also has a simple interpretation in $\mathcal{G}(\mathcal{F})$: it is a pruning operation. One removes a terminal center together with its adjacent trivalent saddle and joins the two remaining incident edges. The induction arguments of Theorems A and B may therefore be viewed, without changing their original proofs, as successive pruning of the spherical tree.

The two terminal configurations of the elimination process may be represented schematically as follows. Figure 12 shows the sphere case, where the foliation has two centers and no saddles. Figure 13 shows the Eells–Kuiper case, where the foliation has two centers and one unique intermediate-index saddle. The drawings are intended only as schematic representations of the foliation, the leaf-space picture, and the associated spherical graph; they are not meant to depict an actual Euclidean embedding of the manifold.

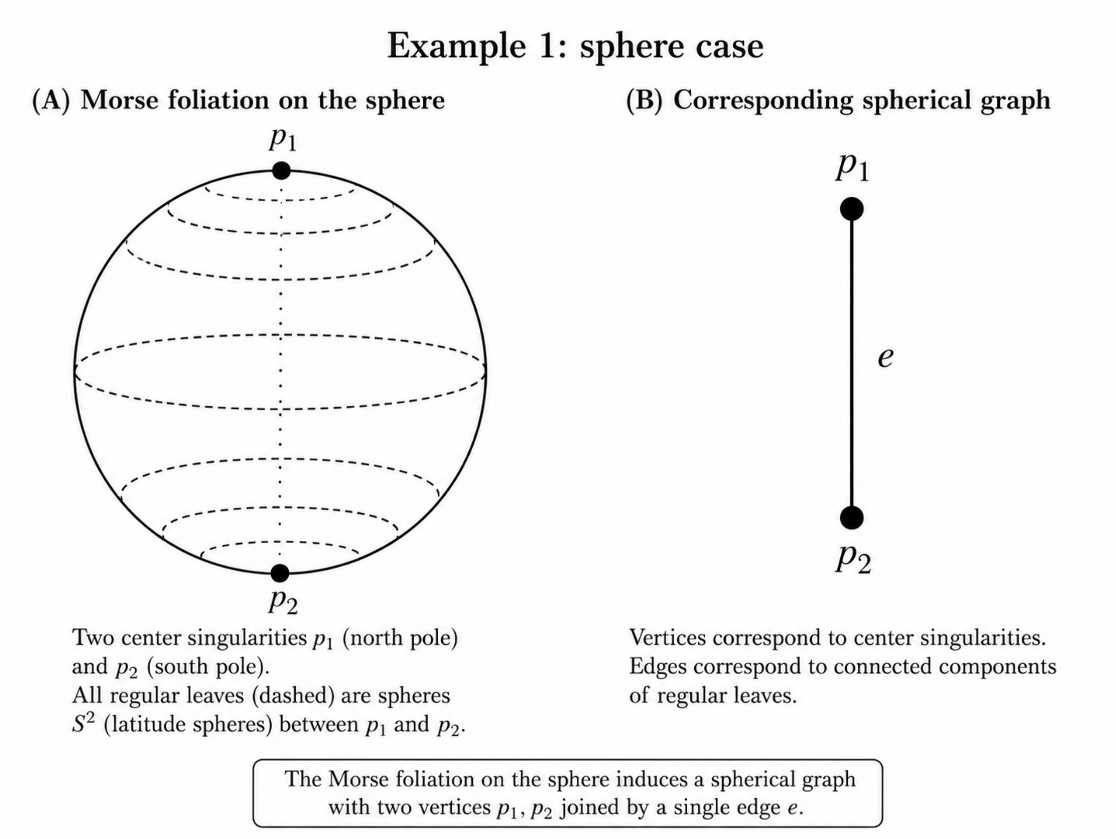

Figure 12: The basic sphere case. The foliation has exactly two center singularities and no saddle. The associated spherical graph consists of a single edge joining the two center vertices.

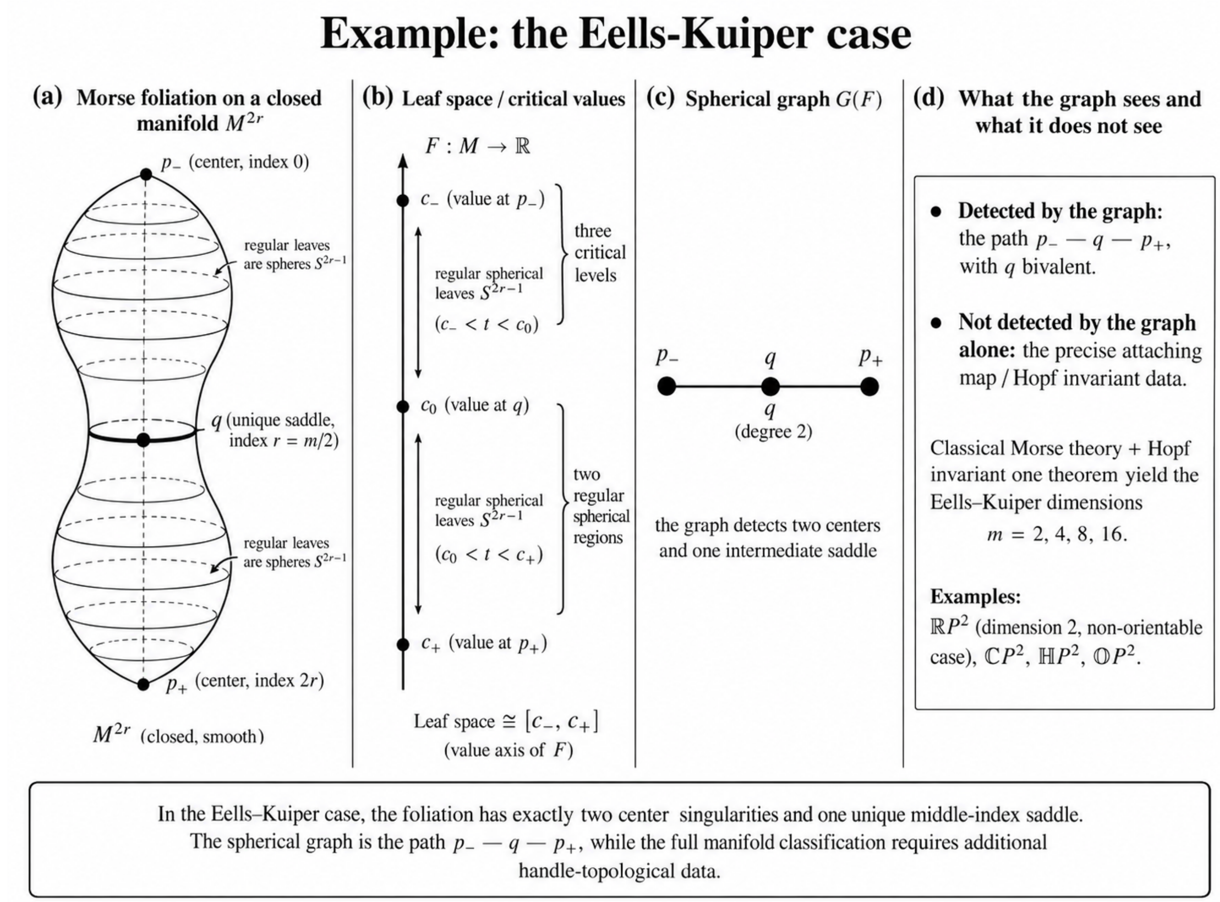

Figure 13: The Eells–Kuiper case. The foliation has two centers and one unique intermediate-index saddle. The leaf-space picture has three critical values, and the associated spherical graph is the path $p_- - - q - - p_+$. The graph detects the two centers and the existence of the bivalent saddle, but the full Eells–Kuiper classification requires additional handle-topological data.

We record a broader situation in which the same graph can be defined without assuming $k > \ell$.

Proposition 9.2 [General graph formula]. *Let $\mathcal{F}$ be a transversely oriented Morse foliation on a closed connected m-manifold, $m \geq 3$, without saddle-connections. Suppose that every connected component of*

$$M \setminus \mathcal{S}(\mathcal{F})$$

is foliated by spheres and that every separatrix S associated with a saddle q is closed in the singular-leaf sense,

$$\overline{S} = S \cup \{q\}.$$

Then the graph $\mathcal{G}(\mathcal{F})$ is a finite connected multigraph, possibly with loops and multiple edges. Every center vertex has degree 1; every saddle of index 1 or $m-1$ has degree 3; and every saddle of index r, $2 \leq r \leq m-2$, has degree 2.

Let k denote the number of centers and let

$$a = \#\{q \text{ saddle} : \mathrm{Ind}(q) = 1 \text{ or } m-1\}.$$

Then

$$\boxed{k = a + 2 - 2\beta_1(\mathcal{G}(\mathcal{F})).}$$

In particular,

$$k \equiv a \pmod 2, \qquad k \leq a + 2.$$

Proof. By lemma 4.5, every component of $M \setminus \mathcal{S}(\mathcal{F})$ is a spherical product region $S^{m-1} \times (0,1)$. The condition $\overline{S} = S \cup \{q\}$ prevents non-trivial separatrix accumulation, so each end of such a product region approaches a well-defined singular vertex. Since $\mathrm{Sing}(\mathcal{F})$ is finite and the local Morse models have only finitely many complementary sectors, only finitely many edges occur. The resulting graph is connected because M is connected and the spherical regions together with the singular-separatrix skeleton exhaust M.

The local Morse models also give the stated degrees: one at a center, three at an index-1 or index-$(m-1)$ saddle, and two at an intermediate-index saddle.

Let

$$b = \#\{q \text{ saddle} : 2 \leq \mathrm{Ind}(q) \leq m - 2\}$$

and write

$$\beta = \beta_1(\mathcal{G}(\mathcal{F})).$$

If V and E denote the numbers of vertices and edges of the connected graph, then

$$V = k + a + b.$$

The degree formula gives

$$2E = k + 3a + 2b,$$

while connectedness gives

$$E = V - 1 + \beta.$$

Combining these identities yields

$$k = a + 2 - 2\beta,$$

which is the asserted formula. The parity relation and the inequality follow immediately. □

Remark 9.3. Intermediate-index saddles do not appear in the identity of proposition 9.2 because they are bivalent vertices and therefore only subdivide edges. The formula also shows that each independent cycle of the graph lowers the number of centers by two relative to the tree case.

The additional hypothesis $\overline{S} = S \cup \{q\}$ is used to prevent non-trivial accumulation of separatrices and to ensure that the ends of the spherical product regions determine genuine vertices of the graph. In the case $k > \ell$ of theorem 9.1, no separate closure hypothesis is required, because the elimination procedure itself reduces the foliation to one of the two basic models and the graph is reconstructed by reversing the trivial-pair eliminations.

9.1 The three-critical-point case

The graph also gives a useful way of isolating the combinatorial part of the Eells–Kuiper situation. Suppose that a closed manifold M^m admits a Morse function with exactly three critical points. The associated Morse foliation has two centers, say p_- and p_+, and one saddle q. Its graph is therefore the path

$$p_- \;---\; q \;---\; p_+.$$

In particular, the graph detects that the unique saddle is bivalent and hence of intermediate index. It does not, however, determine the precise Morse index of q or the attaching map of the corresponding handle.

Let $r = \mathrm{Ind}(q)$. The Morse function gives a handle decomposition with one handle in each of the dimensions 0, r, and m. Poincaré duality, for instance with $\mathbb{Z}_2$ coefficients, forces

$$r = m - r,$$

so that $m = 2r$. The remaining restriction is not graph-theoretic: the classical Eells–Kuiper analysis of the middle-dimensional handle attachment [E-K1, E-K2], together with the Hopf-invariant-one theorem [Adams], yields

$$r \in \{1, 2, 4, 8\}, \qquad m \in \{2, 4, 8, 16\}.$$

Thus the graph provides the combinatorial skeleton of the three-critical-point case, whereas the full Eells–Kuiper classification depends on the topology of the middle-dimensional handle attachment. In the standing framework of this section $m \geq 3$, so only the dimensions 4, 8, and 16 occur; the classical two-dimensional case $m = 2$, which gives $\mathbb{RP}^2$, lies outside the local valence discussion used here.

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