JUMP-TYPE HUNT PROCESSES GENERATED BY LOWER BOUNDED SEMI-DIRICHLET FORMS

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(joint work with Masatoshi Fukushima)

Let E be a locally compact separable metric space, m a positive Radon measure on E with full topological support and let k be a non-negative Borel function on $E \times E$ off the diagonal.

k_s and k_a denote the symmetric part and anti-symmetric part of k, respectively:

$$k_s(x,y) = \frac{1}{2}\big(k(x,y) + k(y,x)\big), \quad k_a(x,y) = \frac{1}{2}\big(k(x,y) - k(y,x)\big), \quad x, y \in X.$$

We allow $k_s(x,y)m(dy)$ to have similar singularities to the Lévy measure of a Lévy process, while k_a is assumed to be less singular than k_s.

We then construct an associated regular lower bounded semi-Dirichlet form η on $L^2(E;m)$ producing a Hunt process X^0 on E whose jump behaviours are governed by k.

The inner product and the norm in $L^2(E;m)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $\mathcal{F}$ be a dense linear subspace of $L^2(E;m)$ such that $u \wedge 1 \in \mathcal{F}$ whenever $u \in \mathcal{F}$. A bilinear fomr η on $\mathcal{F}$, which is not necessarily symmetric, is called a *lower bounded closed form* if the following three conditions are satisfied: We set $\eta_\beta(u,v) = \eta(u,v) + \beta(u,v),\ u, v \in \mathcal{F}$. There exists a $\beta_0 \geq 0$ such that

(B.1) (lower boundedness); for any $u \in \mathcal{F},\ \ \eta_{\beta_0}(u,u) \geq 0$.

(B.2) (sector condition); for any $u, v \in \mathcal{F}$,

$$\big|\eta(u,v)\big| \leq K\sqrt{\eta_{\beta_0}(u,u)} \cdot \sqrt{\eta_{\beta_0}(v,v)},$$

for some constant $K \geq 1$.

(B.3) (completeness); the space $\mathcal{F}$ is complete with respect to the norm $\eta_\alpha^{1/2}(\cdot,\cdot)$ for some, or equivalently, for all $\alpha > \beta_0$.

For a lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E;m)$, there exist unique semigroups $\{T_t; t > 0\},\ \{\widehat{T}_t; t > 0\}$ of linear operators on $L^2(E;m)$ satisfying

$$(T_t f, g) = (f, \widehat{T}_t g),\ f, g \in L^2(E;m),\ ||T_t|| \leq e^{\beta_0 t},\ ||\widehat{T}_t|| \leq e^{\beta_0 t},\ t > 0, \tag{1}$$

such that their Laplace transforms G_α and $\widehat{G}_\alpha$ are determined for $\alpha > \beta_0$ by

$$G_\alpha f,\ \widehat{G}_\alpha f \in \mathcal{F}, \quad \eta_\alpha(G_\alpha f, u) = \eta_\alpha(u, \widehat{G}_\alpha f) = (f, u), \quad f \in L^2(E;m),\ u \in \mathcal{F}.$$

$\{T_t; t > 0\}$ is said to be *Markovian* if $0 \le T_t f \le 1,\ t > 0$, whenever $f \in L^2(E;m),\ 0 \le f \le 1$. It was shown by H. Kunita in 1969 that the semigroup $\{T_t; t > 0\}$ is Markovian if and only if

$$Uu \in \mathcal{F} \quad \text{and} \quad \eta(Uu, u - Uu) \ge 0 \quad \text{for any} \quad u \in \mathcal{F}, \tag{2}$$

where Uu denotes the unit contraction of u: $Uu = (0 \vee u) \wedge 1$. A lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E;m)$ satisfying (1.2) will be called a *lower bounded semi-Dirichlet form* on $L^2(E;m)$. The term 'semi' is added to indicate that the dual semigroup $\{\widehat{T}_t; t > 0\}$ may not be Markovian although it is positivity preserving. For a lower bounded semi-Dirichlet form η which is regular in the sense stated below, if the associated dual semigroup $\{\widehat{T}_t; t > 0\}$ were Markovian, or equivalently, if m were excessive, then η is necessarily a non-negative definite closed form, namely, β_0 in conditions **(B.1), (B.3)** (resp. **(B.2)**) can be retaken to be 0 (resp. 1).

A lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$ is said to be *regular* if $\mathcal{F} \cap C_0(E)$ is uniformly dense in $C_0(E)$ and η_α-dense in $\mathcal{F}$ for $\alpha > \beta_0$, where $C_0(E)$ denotes the space of continuous functions on E with compact support. S. Carrillo-Menendez in 1975 constructed a Hunt process properly associated with any regular lower bounded semi-Dirichlet form on $L^2(E;m)$ by reducing the situation to the case where η is non-negative definite.

Later on, the non-negative definite semi-Dirichlet form was investigated by Z.-M. Ma, L. Oberbeck and M. Röckner(1994) and P. Fitzsimmons(2001) specifically in a general context of the quasi-regular Dirichlet form and the special standard process. However, in producing the forms η from non-symmetric kernels k corresponding to a considerably wide class of jump type Hunt processes in finite dimensions whose dual semigroups need not be Markovian, we will be forced to allow positive β_0.

For $u \in C_0^{\mathrm{lip}}(E)$ and $n \in \mathbb{N}$, the integral

$$\mathcal{L}^n u(x) := \int_{\{y \in E : d(x,y) > 1/n\}} \big(u(y) - u(x)\big) k(x,y) m(dy), \quad x \in E, \tag{3}$$

makes sense under some conditions on the kernel k and the measure m. We prove that the finite limit

$$\eta(u,v) = -\lim_{n \to \infty} \int_E \mathcal{L}^n u(x) v(x) m(dx), \quad \text{for} \quad u, v \in C_0^{\mathrm{lip}}(E), \tag{4}$$

exists, η extends to $\mathcal{F}^0 \times \mathcal{F}^0$ and $(\eta, \mathcal{F}^0)$ is a lower bounded semi-Dirichlet form on $L^2(E;m)$. $(\eta, \mathcal{F}^0)$ is also regular, then it gives rise to an associated Hunt process $X^0 = (X_t^0, P_x^0)$ on E. We call X^0 the *minimal Hunt process* associated with the form η. (4) indicates that the limit of $\mathcal{L}^n$ in n plays a role of a pre-generator of X^0 informally.

If we define the kernel k^* by

$$k^*(x,y) := k(y,x) \qquad x, y \in E, \quad x \ne y, \tag{5}$$

and the form η^* by (3) and (4) with k^* in place of k, we have the same conclusions as above for η^*. In particular, there exists a minimal Hunt process X^{0*} associated with the form η^*.

For an arbitrary open subset $D \subset E$ and its closure $\overline{D}$, we also construct a Hunt process $X^{D,0}$ on D and a Hunt process $X^{\overline{D}}$ on $\overline{D}$, respectively, in analogous manners.

We exhibits an example in which η is not non-negative definite so that the dual semigroup is not Markovian although it is of positivity preserving.

$X^{D,0}$ is shown to be the part process of $X^{\overline{D}}$ on D. When D is relatively compact, we also show that $X^{D,0}$ is censored in the sense that it is of no killing inside D and killed only when the path approaches to the boundary.

When E is a d-dimensional Euclidean space and m is the Lebesgue measure, a typical example of X^0 is the stable-like process with a variable exponent that will be also identified with the solution of a martingale problem up to an η-polar starting points. Stable-like processes have been studied by R. Bass, A. Negoro, T. Tsuchiya among others.

Approachability to the boundary ∂D in finite time of its censored process $X^{D,0}$ on a bounded open subset D will be examined in terms of the polarity of ∂D for the symmetric stable processes with indices that bound the variable exponent $\alpha(x)$. The censored symmetric stable processes have been studied by K. Bogdan, K. Burdzy and Z.-Q. Chen(2003).

References

[B88b] R. Bass, Uniqueness in law for pure jump Markov processes, *Probab. Theory Related Fields*, **79** (1988), 271–287

[BBC03] K. Bogdan, K. Burdzy and Z.-Q. Chen, Censored stable processes, *Probab. Theory Relat. Fields*, **127** (2003), 89-152

[C75] S. Carrillo-Menendez, Processus de Markov associé à une forme de Dirichlet non-symétrique, *Z. Wahrsch. verw. Geb.*, **33** (1975), 139-154

[Fi01] P.J. Fitzsimmons, On the quasi-regular semi-Dirichlet forms, *Potential Analysis*, **15** (2001), 151-185

[F71] M. Fukushima, Dirichlet spaces and strong Markov processes, *Trans. Amer. Math. Soc.* **162** (1971), 185–224

[FOT94] M. Fukushima, Y. Ōshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 1994; 2nd rev. and ext. ed. 2010

[K69] H. Kunita, Sub Markov semi-groups in Banach lattices, in *Proc. Int. Conf. functional analysis and related topics*, Tokyo, 1969

[MOR94] Z.-M. Ma, L. Oberbeck and M. Röckner, Markov processes associated with semi-Dirichlet forms, *Osaka J. Math.*, **32** (1994), 97-119

[MR92] Z.-M. Ma and M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Universitext. Springer-Verlag, Berlin, 1992

[N94] A. Negoro, Stable-like processes: construction of the transition density and the behavior of sample paths near $t = 0$, *Osaka J. Math.*, **31** (1994), 189–214

[O88] Y. Ōshima, Lectures on Dirichlet spaces, Lecture Notes at Erlangen University, 1988

[T92] M. Tsuchiya, Lévy measure with generalized polar decomposition and the associated SDE with jumps, *Stochastic Stochastic Rep.*, **38** (1992), 95–117