Wong-Zakai approximation of solutions to reflecting stochastic differential equations on domains in Euclidean spaces

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1 Preliminary and main theorem

This talk is based on [1]. Let D be a connected domain in $\mathbb{R}^d$. We define the set $\mathcal{N}_x$ of inward unit normal vectors at the boundary point $x \in \partial D$ by

$$\mathcal{N}_x = \cup_{r>0} \mathcal{N}_{x,r} \tag{1}$$

$$\mathcal{N}_{x,r} = \left\{ \mathbf{n} \in \mathbb{R}^d \mid |\mathbf{n}| = 1, B(x - r\mathbf{n}, r) \cap D = \emptyset \right\}, \tag{2}$$

where $B(z,r) = \{y \in \mathbb{R}^d \mid |y - z| < r\}$, $z \in \mathbb{R}^d$, $r > 0$. Note that

$$\mathbf{n} \in \mathcal{N}_{x,r} \Longleftrightarrow \text{For any } y \in \bar{D}, \quad (y - x, \mathbf{n}) + \frac{1}{2r}|y - x|^2 \geq 0. \tag{3}$$

Let us recall what Skorohod problem is.

Definition 1 (Skorohod Problem)**.** *Let $w = w(t)$ $(0 \leq t \leq T)$ be a continuous path on $\mathbb{R}^d$ with $w(0) \in \bar{D}$. The pair of paths (ξ, ϕ) on $\mathbb{R}^d$ is a solution of a Skorohod problem associated with w if the following properties hold.*

(i) *$\xi = \xi(t)$ $(0 \leq t \leq T)$ is a continuous path in $\bar{D}$ with $\xi(0) = w(0)$.*

(ii) *It holds that $\xi(t) = w(t) + \phi(t)$ for all $0 \leq t \leq T$.*

(iii) *$\phi = \phi(t)$ $(0 \leq t \leq T)$ is a continuous bounded variation path on $\mathbb{R}^d$ such that $\phi(0) = 0$ and*

$$\phi(t) = \int_0^t \mathbf{n}(s) d\|\phi\|_{[0,s]} \tag{4}$$

$$\|\phi\|_{[0,t]} = \int_0^t 1_{\partial D}(\xi(s)) d\|\phi\|_{[0,s]}. \tag{5}$$

where $\mathbf{n}(t) \in \mathcal{N}_{\xi(t)}$ if $\xi(t) \in \partial D$.

In the above, $\|\phi\|_{[0,t]}$ stands for the total variation of ϕ on $[0,t]$. When the solution ξ is unique, we denote $\xi = \Gamma(w)$ and we call the mapping Γ a Skorohod map. We also denote $L(w) = \Gamma(w) - w (= \phi)$ which corresponds to the local time at the boundary ∂D.

Example 2. Let $D = (0, \infty) \subset \mathbb{R}$. Then

$$\phi(t) = -\min_{0 \leq s \leq t} (w(s) \wedge 0) = \max_{0 \leq s \leq t} \{(-w(s)) \vee 0\}. \tag{6}$$

In particular,

$$|\Gamma(w)_t - \Gamma(w')_t| \leq 2 \max_{0 \leq s \leq t} |w(s) - w'(s)|. \tag{7}$$

This implies the Skorohod map on the half space D in $\mathbb{R}^d$ is Lipschitz continuous in $C([0,T] \to \mathbb{R}^d)$. Also an explicit form of Γ in the case of $D = (0, a)$ was obtained in [5].

In this talk, we consider domains whose boundary may not be smooth. More precisely, we consider the following conditions (A), (B), (C), (C') on domains following [10] and [6].

Definition 3. (1) *Condition* (A) (*uniform exterior sphere condition*). *There exists a constant* $r_0 > 0$ *such that*

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset \quad \text{for any } x \in \partial D. \tag{8}$$

(2) *Condition* (B). *There exist constants* $\delta > 0$ *and* $\beta \geq 1$ *satisfying:*
for any $x \in \partial D$ *there exists a unit vector* l_x *such that*

$$(l_x, \mathbf{n}) \geq \frac{1}{\beta} \qquad \text{for any } \mathbf{n} \in \cup_{y \in B(x,\delta)\cap \partial D} \mathcal{N}_y. \tag{9}$$

(3) *Condition* (C). *There exists a* C_b^2 *function* f *on* $\mathbb{R}^d$ *and a positive constant* γ *such that for any* $x \in \partial D$, $y \in \bar{D}$, $\mathbf{n} \in \mathcal{N}_x$ *it holds that*

$$(y - x, \mathbf{n}) + \frac{1}{\gamma}\left((Df)(x), \mathbf{n}\right)|y - x|^2 \geq 0. \tag{10}$$

(4) *Condition* (C'). *There exists a* C_b^2 *function* f *on* $\mathbb{R}^d$ *and a positive constant* γ' *such that for any* $x \in \partial D$ *and* $\mathbf{n} \in \mathcal{N}_x$ *it holds that*

$$((Df)(x), \mathbf{n}) \geq \gamma'. \tag{11}$$

Remark 4. (1) (A) with any $r_0 > 0$ holds for any convex domain. (B) holds for any bounded convex domain in $\mathbb{R}^d$ and convex domain in $\mathbb{R}^2$.

(2) (A) and (C') $\Longrightarrow$ (C) with $\gamma = 2r_0\gamma'$

(3) (C')$\Longrightarrow$ (B)

(4) (C) holds for any convex domain with $f \equiv 0$.

(5) (A) and (C') hold on any bounded domain with piecewise smooth boundary and convex angles.

Theorem 5 ([10])**.** *Assume conditions* (A) *and* (B). *Then there exists a unique solution to the Skorohod problem for any continuous path* w. *Moreover the Skorohod mapping* $\Gamma : w \mapsto \xi$ *is* $1/2$-*Hölder continuous map in the uniform norm: Consider two Skorohod equations* $\xi = w + \phi$, $\xi' = w' + \phi'$. *Then*

$$\begin{aligned}|\xi(t) - \xi'(t)|^2 \leq &\left\{|w(t) - w'(t)|^2 + 4\left(\|\phi\|_{[0,t]} + \|\phi'\|_{[0,t]}\right)\max_{0\leq s\leq t}|w(s) - w'(s)|\right\}\\ &\exp\left\{\left(\|\phi\|_{[0,t]} + \|\phi'\|_{[0,t]}\right)/r_0\right\}, \qquad 0 \leq t \leq T.\end{aligned} \tag{12}$$

Let us explain the meaning of reflecting SDE. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{F}_t$ be the right-continuous filtration with the property that $\mathcal{F}_t$ contains all null sets of $(\Omega, \mathcal{F}, P)$. Let $B = B(t)$ be an $\mathcal{F}_t$-Brownian motion on $\mathbb{R}^n$. Let $\sigma \in C(\mathbb{R}^d \to \mathbb{R}^n \otimes \mathbb{R}^d)$, $b \in C(\mathbb{R}^d \to \mathbb{R}^d)$ be continuous mappings. We consider an SDE with reflecting boundary condition on $\bar{D}$:

$$X(t) = x + \int_0^t \sigma(X(s))dB(s) + \int_0^t b(X(s))ds + \Phi(t), \tag{13}$$

where $x \in \bar{D}$. We denote this SDE by SDE(σ, b) simply. A pair of $\mathcal{F}_t$-adapted continuous processes $(X(t), \Phi(t))$ is called a solution to (13) if the following holds. Let

$$Y(t) = x + \int_0^t \sigma(X(s))dB(s) + \int_0^t b(X(s))ds \tag{14}$$

Then $(X(\cdot, \omega), \Phi(\cdot, \omega))$ is a solution of the Skorohod problem associated with $Y(\cdot, \omega)$ for almost all $\omega \in \Omega$. If $X(t)$ is a measurable function of $\{B(s) \mid 0 \leq s \leq t\}$, then the solution (X, Φ) is called a strong solution of the SDE. The following result is due to Saisho who improved the result in Lions-Sznitman [6] and Tanaka [13].

Theorem 6. *Assume D satisfies conditions* (A) *and* (B) *and σ and b are bounded and global Lipschitz maps. Then there exists a unique strong solution to* (13).

Assume $\sigma \in C_b^2$ and $b \in C_b^1$. Let $N \in \mathbb{N}$. We define the Wong-Zakai approximation X^N to the solution X of the SDE$(\sigma, \tilde{b})$, where $\tilde{b}(x) = b(x) + \frac{1}{2}\mathrm{tr}(D\sigma)[\sigma(x)\cdot](\cdot)$ as the solution to the reflecting ODE:

$$X^N(t) = x + \int_0^t \sigma(X^N(s))dB^N(s) + \int_0^t b(X^N(s))ds + \Phi^N(t), \tag{15}$$

where

$$B^N(t) = B(t_{k-1}^N) + \frac{\Delta_N B_k}{\Delta_N}(t - t_{k-1}^N) \qquad t_{k-1}^N \le t \le t_k^N, \tag{16}$$

$$\Delta_N B_k = B(t_k^N) - B(t_{k-1}^N), \qquad \Delta_N = T/N, \qquad t_k^N = \frac{kT}{N}. \tag{17}$$

The solution to the above reflecting ODE exists uniquely by Theorem 6. Actually, we can prove the following.

Proposition 7 ([1])**.** *Assume the same assumptions as in Theorem* 6*. Let $w = w(t)$ be a continuous bounded variation path on $\mathbb{R}^n$. Then there exists a unique continuous bounded variation path $x(t)$ on $\mathbb{R}^d$ satisfying the reflecting ODE:*

$$x(t) = x + \int_0^t \sigma(x(s))dw(s) + \int_0^t b(x(s))ds + \Phi(t). \tag{18}$$

Proof. This can be proved by the Euler-Peano approximation of x, the continuity of Γ and Arzela and Ascoli's theorem. □

The following is our main theorem.

Theorem 8 ([1])**.** *Assume $\sigma \in C_b^2$, $b \in C_b^1$ and conditions* (A), (B) *and* (C)*. Let X be the solution to* SDE$(\sigma, \tilde{b})$*, where $\tilde{b} = b + \frac{1}{2}\mathrm{tr}(D\sigma)(\sigma)$. Let $0 < \theta < 1$. There exists a positive constant $C_{T,\theta}$ such that for all $N \in \mathbb{N}$,*

$$E\left[\max_{0\le t\le T} |X^N(t) - X(t)|^2\right] \le C_{T,\theta}\Delta_N^{\theta/6}. \tag{19}$$

To prove this theorem, we need Euler-Peano approximation.

2 Euler-Peano approximation

For $0 \le k \le N$, set $t_k^N = kT/N$. Let us define $X_E^N(t)$ $(0 \le t \le T)$ as the solution to the Skorohod problem inductively which is given by $X_E^N(0) = x \in \mathbb{R}^d$ and

$$\begin{aligned} X_E^N(t) = X_E^N(t_{k-1}^N) &+ \sigma(X_E^N(t_{k-1}^N))(B(t) - B(t_{k-1}^N)) + b(X_E^N(t))(t - t_k^N) \\ &+ \Phi_E^N(t) - \Phi_E^N(t_{k-1}^N) \qquad t_{k-1}^N \le t \le t_k^N. \end{aligned} \tag{20}$$

In other words, X_E^N satisfies

$$X_E^N(t) = x + \int_0^t \sigma(X_E^N(\pi_N(s)))dB(s) + \int_0^t b(X_E^N(\pi_N(s)))ds + \Phi_E^N(t), \tag{21}$$

where $\pi_N(t) = \max\{t_k \mid t_k \le t\}$. Define

$$Y_E^N(t) = x + \int_0^t \sigma(X_E^N(\pi_N(s)))dB(s) + \int_0^t b(X_E^N(\pi_N(s)))ds. \tag{22}$$

Then by the definition of the solution of the SDE, it holds that

$$X_E^N(t) = \Gamma\left(Y_E^N\right)(t). \tag{23}$$

We have

Theorem 9. *Assume the same assumptions as in Theorem* 6 *and condition* (C)*. Then for any* $p \geq 1$*, there exists* $C_p > 0$ *such that*

$$E\left[\max_{0\leq t\leq T} |X_E^N(t) - X(t)|^{2p}\right] \leq C_p \Delta_N^p. \tag{24}$$

To prove this theorem, we need the following estimates. In the following, we use the oscillation and the total variation of the path:

$$\|w\|_{\infty,[s,t]} = \max_{s\leq u\leq v\leq t} |w(u) - w(v)|, \tag{25}$$

$$\|w\|_{[s,t]} = \sup_{\Delta} \sum_{k=1}^{N} |w(t_k) - w(t_{k-1})|, \tag{26}$$

where $\Delta = \{s = t_0 < \cdots < t_N = t\}$ is a partition of the interval $[s, t]$.

Lemma 10. *Assume the same assumptions as in Theorem* 6*. Let* $p \geq 1$*. There exists a positive constant* C_p *such that*

$$E[\|X\|_{\infty,[s,t]}^{2p}] \leq C_p|t - s|^p, \tag{27}$$

$$E[|\Phi\|_{[s,t]}^{2p}] \leq C_p|t - s|^p. \tag{28}$$

$$E[\|X_E^N\|_{\infty,[s,t]}^{2p}] \leq C_p|t - s|^p, \tag{29}$$

$$E\left[\|\Phi_E^N\|_{[s,t]}^{2p}\right] \leq C_p|t - s|^p. \tag{30}$$

To prove the above moment estimates, we need an estimate on the total variation of the local time terms Φ and Φ_E^N.

Lemma 11 ([10, 1])**.** *Assume* (A) *and* (B)*. Let us fix* $0 < \theta < 1$*. Then there exist positive constants* C_1, C_2, C_3 *such that*

$$\|\phi\|_{[s,t]} \leq C_1 \left(1 + \|w\|_{\mathcal{H},[s,t],\theta}^{C_2} (t - s)\right) e^{C_3\|w\|_{\infty,[s,t]}} \|w\|_{\infty,[s,t]}, \tag{31}$$

where

$$\|w\|_{\mathcal{H},[s,t],\theta} = \sup_{s\leq u<v\leq t} \frac{|w(v) - w(u)|}{|u - v|^{\theta}}. \tag{32}$$

The constants C_i *depend only on* r_0, β, δ *in conditions* (A) *and* (B)*.*

Proof. This is a quantative version of the estimate in Proposition 3.1 and Theorem 4.2 in [10]. □

3 Proof of main theorem

For simplicity, we may denote $\Delta_N B_k$, Δ_N, t_k^N by ΔB_k, Δ, t_k. By the definition, it holds that

$$\begin{aligned} X^N(t) = X^N(t_{k-1}) &+ \int_{t_{k-1}}^{t} \sigma(X^N(s)) \frac{\Delta B_k}{\Delta} ds + \int_{t_{k-1}}^{t} b(X^N(s)) ds \\ &+ \Phi^N(t) - \Phi^N(t_{k-1}) \qquad t_{k-1} \leq t \leq t_k. \end{aligned} \tag{33}$$

Clearly, $X^N(t_{k-1})$ is $\mathcal{F}_{t_{k-1}}$-measurable. Let

$$Y^N(t)=x+\int_0^t\sigma(X^N(s))dB^N(s)+\int_0^t b(X^N(s))ds. \tag{34}$$

Then $X^N=\Gamma(Y^N)$ and $\Phi^N=L(Y^N)$.

We have the following moment estimates.

Lemma 12. *Assume $\sigma\in C_b^2$, $b\in C_b^1$ and coditions* (A) *and* (B)*. Let $p\ge 1$. There exists a positive number C_p which is independent of N such that for all $0\le s\le t\le T$,*

$$E[\|Y^N\|_{\infty,[s,t]}^{2p}]\le C_p|t-s|^p, \tag{35}$$

$$E[\|X^N\|_{\infty,[s,t]}^{2p}]\le C_p|t-s|^p, \tag{36}$$

$$E[|\Phi^N\|_{[s,t]}^{2p}]\le C_p|t-s|^p. \tag{37}$$

In the proof of the above moment estimates, we use estimates for the total variation of local time term in Lemma 11. Further, we need the following Lemma 13 which is used for the estimate of the total variation of Φ^N on the small interval I, where $|I|=O(\Delta_N)$. The estimate in Lemma 11 is not good for such estimates, because the estimate in Lemma 11 contains the exponential term[1].

Lemma 13 ([1])**.** *Assume condition* (A) *and the existence of the solution ξ to the Skorohod problem for a continuous bounded variation path w. Then the total variation of the solution ξ has the estimate:*

$$\|\xi\|_{[s,t]}\le 2(\sqrt{2}+1)\|w\|_{[s,t]} \tag{38}$$

Remark 14. Under the admissibility of the domain, Lions and Sznitman proved that $\|\phi\|_{[s,t]}\le\|w\|_{[s,t]}$ which implies $\|\xi\|_{[s,t]}\le 2\|w\|_{[s,t]}$. Our proof does not use the admissibility but use the following estimate in Lemma 2.3 (i) in [10].

$$|\xi(t)-\xi(s)|^2\le|w(t)-w(s)|^2+\frac{1}{r_0}\int_s^t|\xi(u)-\xi(s)|^2d\|\phi\|_{[0,u]}+2\int_s^t(w(t)-w(u),d\phi(u)). \tag{39}$$

The following is a key lemma for the proof of L^p convergence of Wong-Zakai approximation.

Lemma 15 ([1])**.** *Assume the same assumption as in Theorem 8. Let X_E^N be the Euler-Peano approximation to SDE$(\sigma,\tilde{b})$, where $\tilde{b}=b+\frac{1}{2}\mathrm{tr}(D\sigma)(\sigma)$. Then for any $0<\theta<1$, there exists a positive constant C_θ such that for all N,*

$$\sup_{0\le k\le N}E\left[|X^N(t_k^N)-X_E^N(t_k^N)|^2\right]\le C_\theta\cdot\Delta_N^{\theta/2}. \tag{40}$$

Remark 16. The rate of convergence in (40) is, roughly speaking, half of that of the Wong-Zakai approximation to the SDE without reflection term. However, this bound may not be so bad. Assume (A) and (B) hold. By (12), we obtain

$$E[|\Gamma(B)_t-\Gamma(B^N)_t|^2]\le C\Delta_N^{\theta/2}, \tag{41}$$

where $0<\theta<1$. By examining the proof in [10], one can replace the term $\|\phi\|_{[0,t]}+\|\phi'\|_{[0,t]}$ in (12) by $\|\phi-\phi'\|_{[0,t]}$. I am not sure whether or not this estimate gives better estimates than the above. Also we note that this rate of convergence appeared in the study of the Euler approximation in [11, 12]. Actually Słomiński obtained more precise estimates containing $\log N$ in the Euler and Euler-Peano approximation. Of course, if D is a half space (or convex polyhedron, see [3]) in a Euclidean space, then Γ is Lipschitz continuous and the upper bound in (41) is $O(\Delta_N^\theta)$. Also, it seems that the calculation in [2] also gives the convergence speed $O(\Delta_N^\theta)$ for Wong-Zakai approximations of general reflecting SDEs in the half space case. However, to my knowledge, I do not know examples of reflecting SDE for which the slow convergence speed $\Delta_N^{\theta/2}$ really appear.

[1]When D is convex, the exponential term vanishes. So we may not need Lemma 13 in the convex case.

Lemma 17. *Assume the same assumptions in Lemma* 15 *and consider the same SDE. Let* $0 < \theta < 1$*. Then there exists a positive constant* $C_{T,\theta}$ *such that*

$$E\left[\max_{0\le t\le T}|X^N(t)-X^N_E(t)|^2\right]\le C_{p,T,\theta}\Delta_N^{\theta/6}. \tag{42}$$

Proof. This can be proved by Lemma 15 and the moment estimates for X^N_E, X^N (Lemma 10 and Lemma 12).

□

Proof of main theorem. The proof follows from Theorem 9 and Lemma 17. □

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