## Cooperation principle and disappearance of chaos in random complex dynamics

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# 1 Introduction

The details of this talk are included in the author's papers [1,2]. In the usual iteration of a single rational map $h$ with $\deg(h) \geq 2$, we always have the "chaotic part" in the Riemann sphere. However, in this talk, we show that in the i.i.d. random complex dynamics of polynomials, for a generic probability measure $\tau$ on the space of polynomial maps, **(1)** the chaos of the averaged system disappears, due to the **automatic cooperation** of the generator maps, **(2)** there exists a stability of the limit state w.r.t. the perturbation of $\tau$, and **(3)** the orbit of a Hölder continuous function under the transition operator $M_\tau$ converges exponentially fast to the finite-dimensional space $\mathcal{U}_\tau$ of finite linear combinations of unitary eigenvectors of $M_\tau$.

Moreover, in the limit state, under certain conditions, we have complex analogues of the devil's staircase. Note that the devil's staircase can be regarded as the function of probability of tending to $+\infty$ with respect to the random dynamics on $\mathbb{R}$ such that at every step we take $h_1$ with probability 1/2 and $h_2$ with probability 1/2.

# 2 Preliminaries

**Definition 2.1.** We denote by $\hat{\mathbb{C}}(:= \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1 \cong S^2)$ the Riemann sphere and denote by $d$ the spherical distance on $\hat{\mathbb{C}}$. We set Rat:=$\{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h$ is a non-const. rational map$\}$ endowed with the distance $\eta$ defined by $\eta(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$.We set $\text{Rat}_+ := \{g \in \text{Rat} \mid \deg(g) \geq 2\}$. We set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial map}, \deg(g) \geq 2\}$ endowed with the relative topology from Rat. Note that Rat and $\mathcal{P}$ are semigroups where the semigroup operation is functional composition. A subsemigroup $G$ of Rat is called a **rational semigroup**. A subsemigroup $G$ of $\mathcal{P}$ is called a **polynomial semigroup**.

**Definition 2.2.** Let $G$ be a rational semigroup. We set
$F(G) := \{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{ of } z \text{ s.t. } G \text{ is equicontinuous on } U\}$. This is called the **Fatou set** of $G$. We set $J(G) := \hat{\mathbb{C}} \setminus F(G)$. This is called the **Julia set** of $G$. If $G$ is generated by a subset $\Lambda$ of Rat, then we write $G = \langle \Lambda \rangle$.

**Definition 2.3.** For a metric space $X$, we denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on $X$ endowed with the topology such that
"$\mu_n \to \mu$" $\Leftrightarrow$ "for each bounded continuous function $\varphi : X \to \mathbb{R}$, $\int_X \varphi \, d\mu_n \to \int_X \varphi \, d\mu$."

**Remark 2.4.** If $X$ is a compact metric space, then $\mathfrak{M}_1(X)$ is a compact metrizable space.

From now on, we take a $\tau \in \mathfrak{M}_1(\text{Rat})$ and we consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \text{Rat}$ according to $\tau$. This determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and for each Borel measurable subset $A$ of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ from $x$ to $A$ is defined as $p(x, A) = \tau(\{g \in \text{Rat} \mid g(x) \in A\})$.

**Definition 2.5.** Let $\tau \in \mathfrak{M}_1(\text{Rat})$. **(1)** We set $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi \text{ is conti.}\}$ endowed with the sup. norm $\|\cdot\|_\infty$. **(2)** Let $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ be the operator defined by $M_\tau(\varphi)(z) := \int_{\text{Rat}} \varphi(g(z)) \, d\tau(g), \ \forall \varphi \in C(\hat{\mathbb{C}}), \forall z \in \hat{\mathbb{C}}$. **(3)** Let $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the dual of $M_\tau$. That is, for each $\rho \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and for each $\varphi \in C(\hat{\mathbb{C}})$, $\int \varphi \, d(M_\tau^*(\rho)) := \int M_\tau(\varphi) \, d\rho$. (Remark: $M_\tau^*$ can

be regarded as the "averaged map" of $\operatorname{supp}\tau$). **(4)** We set
$F_{meas}(\tau) := \{\mu \in \mathfrak{M}_1(\hat{\mathbb{C}}) \mid \exists$ nbd $B$ of $\mu$ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ s.t. $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is equiconti. on $B\}$. **(5)** Let $\mathcal{U}_\tau$ be the space of all finite linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where an eigenvector is said to be unitary if the absolute value of the corresponding eigenvalue is 1. **(6)** Let $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0$ as $n \to \infty\}$. **(7)** Let $\tilde{\tau} := \otimes_{j=1}^{\infty}\tau \in \mathfrak{M}_1((\mathrm{Rat})^{\mathbb{N}})$. **(8)** Let $G_\tau$ be the rational semigroup generated by $\operatorname{supp}\tau$. **(9)** We say that a non-empty compact subset $K$ of $\hat{\mathbb{C}}$ is a **minimal set** of $G_\tau$ in $\hat{\mathbb{C}}$ if $K$ is minimal in $\{L \subset \hat{\mathbb{C}} \mid \emptyset \neq L$ is compact, $\forall g \in G_\tau, g(L) \subset L\}$ w.r.t. $\subset$ . Moreover, we set $\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) := \{L \mid L$ is a minimal set of $G_\tau$ in $\hat{\mathbb{C}}\}$. **(10)** For a minimal set $L$ of $G_\tau$ in $\hat{\mathbb{C}}$ and a point $z \in \hat{\mathbb{C}}$, we set $T_{L,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}} \mid d(\gamma_n \cdots \gamma_1(z), L) \to 0$ as $n \to \infty\})$. This is **the probability of tending to $L$ starting with the initial value** $z \in \hat{\mathbb{C}}$.

**Definition 2.6.** Let $G$ be a rational semigroup. We set $J_{\ker}(G) := \bigcap_{h\in G} h^{-1}(J(G))$. This is called the **kernel Julia set** of $G$.

Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ be such that supp $\tau$ is compact. If there exists an $f_0 \in \mathcal{P}$ and a non-empty open subset $U$ of $\mathbb{C}$ such that $\{f_0 + c \mid c \in U\} \subset \operatorname{supp}\tau$, then $J_{\ker}(G_\tau) = \emptyset$. Thus, we can say that mostly $J_{\ker}(G_\tau) = \emptyset$.

# 3 Results

**Theorem 3.1 (Theorem A, Cooperation Principle and Disappearance of Chaos).** *Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ be such that $\operatorname{supp}\tau$ is compact. Suppose $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$.*
(note: if $\exists g \in \operatorname{supp}\tau$ with $\deg(g) \geq 2$, then $J(G_\tau) \neq \emptyset$.)
*Then, we have all of the following* (1)–(8).

(1) $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$. (Note: For this statement (1), "$J(G_\tau) \neq \emptyset$" is not needed.)

(2) $\forall z \in \hat{\mathbb{C}}\ \ \exists \mathcal{A}_z \subset (\mathrm{Rat})^{\mathbb{N}}$ *with* $\tilde{\tau}(\mathcal{A}_z) = 1$ *s.t. the following* $(*)$ *holds.*

   $(*)$ $\forall \gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$, $\exists \delta = \delta(z, \gamma) > 0$ *s.t.* $\mathrm{diam}\gamma_n \cdots \gamma_1(B(z, \delta)) \to 0$ *as* $n \to \infty$, *where* diam *denotes the diameter w.r.t. the spherical distance.*

(3) $\mathcal{B}_{0,\tau}$ *is a closed subspace of* $C(\hat{\mathbb{C}})$ *and* $C(\hat{\mathbb{C}}) = \mathcal{U}_\tau \oplus \mathcal{B}_{0,\tau}$.

(4) $1 \leq \dim_{\mathbb{C}} \mathcal{U}_\tau < \infty$.

(5) *For each* $\varphi \in \mathcal{U}_\tau$ *and for each connected component* $U$ *of* $F(G_\tau)$, $\varphi|_U$ *is constant.*

(6) $\exists \alpha \in (0, 1)$ *s.t.* $\forall \varphi \in \mathcal{U}_\tau$, $\varphi$ *is* $\alpha$*-Hölder continuous on* $\hat{\mathbb{C}}$.

(7) $1 \leq \sharp\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.

(8) *Let* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$. *Then* $M_\tau(T_{L,\tau}) = T_{L,\tau}$ *and* $T_{L,\tau} \in \mathcal{U}_\tau$. *Moreover, for each* $z \in \hat{\mathbb{C}}$, $\sum_{L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}})} T_{L,\tau}(z) = 1$.

**Remark 3.2.** Theorem A describes new phenomena which cannot hold in the usual iteration dynamics of a single $g \in \mathrm{Rat}$ with $\deg(g) \geq 2$. For example, $F_{meas}(\delta_g) \neq \mathfrak{M}_1(\hat{\mathbb{C}})$.

**Definition 3.3.** Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ be such that $\operatorname{supp}\tau$ is compact. We say that $\tau$ is **mean stable** if there exist non-empty open subsets $U, V$ of $F(G_\tau)$ and a number $n \in \mathbb{N}$ such that all of the following (1)–(3) hold. **(1)** $\overline{V} \subset U \subset \overline{U} \subset F(G_\tau)$. **(2)** $\forall \gamma = (\gamma_1, \gamma_2, \ldots) \in (\operatorname{supp}\tau)^{\mathbb{N}}$, $(\gamma_n \circ \cdots \circ \gamma_1)(\overline{U}) \subset V$. **(3)** $\forall z \in \hat{\mathbb{C}}$, $\exists g \in G_\tau$ s.t. $g(z) \in U$.

**Remark 3.4.** If $\tau$ is mean stable, then $J_{\ker}(G_\tau) = \emptyset$.

**Definition 3.5.** Let $\mathcal{Y}$ be a closed subset of Rat. Let $\mathfrak{M}_{1,c}(\mathcal{Y}) := \{\tau \in \mathfrak{M}_1(\mathcal{Y}) \mid \text{supp}\,\tau \text{ is compact}\}$. Let $\mathcal{O}$ be the topology in $\mathfrak{M}_{1,c}(\mathcal{Y})$ such that $\tau_n \to \tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ if and only if **(1)** $\int \varphi\, d\tau_n \to \int \varphi\, d\tau$ for each bounded continuous function $\varphi : \mathcal{Y} \to \mathbb{R}$, and **(2)** $\text{supp}\,\tau_n \to \text{supp}\,\tau$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of $\mathcal{Y}$.

**Theorem 3.6 (Theorem B, Density of Mean Stable Systems).**

(1) $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau$ *is mean stable*$\}$ *is open and dense in* $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.

(2) $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau$ *is mean stable and* $\sharp \text{supp}\,\tau < \infty\}$ *is dense in* $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.

We remark that in the study of iteration of a single rational map, we have a very famous conjecture which states that hyperbolic rational maps are dense in the space of rational maps. Theorem B solves this kind of problem in the study of random dynamics of complex polynomials.

**Theorem 3.7.** *The set* $\{\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+) \mid \tau$ *is mean stable* $\} \cup \{\rho \in \mathfrak{M}_{1,c}(\text{Rat}_+) \mid \text{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}, J(G_\rho) = \hat{\mathbb{C}}\}$ *is dense in* $(\mathfrak{M}_{1,c}(\text{Rat}_+), \mathcal{O})$.

**Definition 3.8.** For a $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ with $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$, let $\pi_\tau : C(\hat{\mathbb{C}}) \to \mathcal{U}_\tau$ be the canonical projection coming from Theorem A.

**Theorem 3.9 (Theorem C, Stability).** *Suppose* $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ *is mean stable and* $J(G_\tau) \neq \emptyset$. *Then there exists a neighborhood* $\Omega$ *of* $\tau$ *in* $(\mathfrak{M}_{1,c}(\text{Rat}), \mathcal{O})$ *such that all of the following* (1)(2)(3) *hold.* **(1)** *For each* $\nu \in \Omega$, $\nu$ *is mean stable* (*thus* Theorem A *for* $\nu$ *holds*). **(2)** *The maps* $\nu \mapsto \pi_\nu$ *and* $\nu \mapsto \mathcal{U}_\nu$ *are continuous on* $\Omega$. **(3)** *The map* $\nu \mapsto \sharp\text{Min}(G_\nu, \hat{\mathbb{C}})$ *is constant on* $\Omega$.

**Definition 3.10.** For each $\alpha \in (0,1)$, we set $C^\alpha(\hat{\mathbb{C}}) := \{\varphi \in C(\hat{\mathbb{C}}) \mid \|\varphi\|_\alpha < \infty\}$,
where $\|\varphi\|_\alpha := \sup_{z \in \hat{\mathbb{C}}} |\varphi(z)| + \sup_{x,y \in \hat{\mathbb{C}}, x \neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha$. ($\alpha$-Hölder norm.)

**Theorem 3.11 (Theorem D, Exponential Rate of Convergence).** *Let* $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$. *Suppose that* $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, *and for each minimal set* $L$ *of* $G_\tau$ *in* $\hat{\mathbb{C}}$, $L \subset F(G_\tau)$. (Note: if $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then all of the above assumptions hold.)
*Then* $\exists \alpha \in (0,1)\ \exists C > 0\ \exists \lambda \in (0,1)$ s.t.
*for each* $\varphi \in C^\alpha(\hat{\mathbb{C}})$ *and for each* $n \in \mathbb{N}$, $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi\|_\alpha$.

Figure 1: The Julia set of $G = \langle h_1, h_2 \rangle$, where $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2, h_2 := g_2^2$.

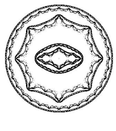
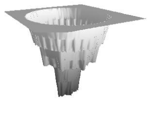


Figure 2: The graph of $z \mapsto T_{\infty,\tau}(z)$, where, letting $(h_1, h_2)$ be the element in Figure 1, we set $\tau := \sum_{j=1}^2 (1/2)\delta_{h_j}$. A devil's coliseum (a complex analogue of the devil's staircase). $\tau$ is mean stable. The set of varying points of $T_{\infty,\tau}$ is equal to Figure 1.




**References**

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