Height fluctuations of the one-dimensional KPZ equation

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The Kardar-Parisi-Zhang equation is a well-known equation (nonlinear stochastic partial differential equation) to describe surface growth phenomena. For the one-dimensionanl case it reads

$$\partial_t h(x,t) = \tfrac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu \partial_x^2 h(x,t) + \sqrt{D}\eta(x,t). \tag{1}$$

Here $x \in \mathbb{R}$ is a space coordinate, $t \geq 0$ is time and $h \in \mathbb{R}$ is the surface height at t and x. In addition $\eta(x,t)$ is a Gaussian white noise with $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$. $\lambda, \nu, D > 0$ are the parameters of the equation. ν represents the strength of the diffusive relaxation, λ the nonliearity and D the noise.

In the long time asymptotic regime, a renormalization group argument suggests that the height fluctuations scale like $O(t^{1/3})$. This exponent $1/3$ is believed to be universal for a wide class of growth processes (KPZ universality).

The KPZ equation itself has not been studied so much compared to similar lattice models like the asymmetric simple exclusion process (ASEP). One reason would be that there was a difficulty of the definition of the equation itself due to the irregular behaviors of $h(x,t)$.

In the mean time, the studies of fluctuation propeties of ASEP and related models have progressed greatly using the connection to the techniques from integrable systems such as random matrix theory and Bethe ansatz.

In this presentation, we show that the fluctuations of the height of the KPZ equation is written as an integral of the Fredholm determinant [1–4]. This is done by combining a plausible regulaization procedure of the equation proposed by Bertini and Giacomin using the Cole-Hopf transformation some time ago, and a contour integration formula for the distribution of a particle position in ASEP by Tracy and Widom.

We consider the droplet growth in which the macroscopic height is given by

$$h(x,t) = \begin{cases} -x^2/2\lambda t & \text{for } |x| \leq \lambda t/\delta\,, \\ (\lambda/2\delta^2)t - |x|/\delta & \text{for } |x| > \lambda t/\delta, \end{cases}$$

with $\delta \ll 1$. This corresponds to the narrow wedge initial conditions:

$$h(x,0) = -|x|/\delta.$$

Let ξ_t have the probability density,

$$\rho_t(s) = \int_{-\infty}^{\infty} \gamma_t \mathrm{e}^{\gamma_t(s-u)} \exp\left[-\mathrm{e}^{\gamma_t(s-u)}\right] \\ \times \Big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\Big)\mathrm{d}u$$

where $\gamma_t = 2^{-1/3}\alpha^{4/3}t^{1/3}$, $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = K_{\mathrm{Ai}}(x,y) + \int_0^{\infty} \mathrm{d}\lambda(\mathrm{e}^{\gamma_t\lambda} - 1)^{-1} \\ \times \big(\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \mathrm{Ai}(x-\lambda)\mathrm{Ai}(y-\lambda)\big)\,.$$

Thm. For $h(x,t)$ described by the KPZ equation (1) with the above initial conditions, one has

$$(\lambda/2\nu)h(x,t/2\nu) \stackrel{d}{=} -x^2/2t - \tfrac{1}{12}\gamma_t^3 + 2\log\alpha + \gamma_t\xi_t \tag{2}$$

where $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$.

Let ξ_{TW} obeys the GUE Tracy-Widom distributions;

$$\mathbb{P}[\xi_{\mathrm{TW}} \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$

where K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^{\infty} \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda).$$

This distribution describes the largest eigenvalue distribution of GUE random matrices in the limit of large matrix dimension. It also appears as the limiting distribution for the height fluctuations of several growth models. By using the formula (2) it is easy to show $\lim_{t\to\infty}\xi_t = \xi_{\mathrm{TW}}$ in distribution. With this one could say that the KPZ equation is in the KPZ universality class.

The presentation is based on collaborations with H. Spohn, T. Imamura.

参考文献

[1] T. Sasamoto and H. Spohn, *Exact height distributions for the KPZ equation with narrow wedge initial condition.*, Nucl. Phys. B **834** (2010), 523–542.

[2] ———, *The crossover regime for the weakly asymmetric simple exclusion process*, J. Stat. Phys. **140** (2010), 209–231.

[3] ———, *Universality of the one-dimensional KPZ equation.*, Phys. Rev. Lett. **834** (2010), 523–542.

[4] G. Amir, I. Corwin, and J. Quastel, *Probability distribution of the free energy of the continuum directed random polymer in* 1 + 1 *dimensions, arXiv:1003.0443.*