Asymptotic behavior in $\mathbb{Z}^d$ of the critical two-point functions for long-range statistical-mechanical models in high dimensions

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1. Motivation Self-avoiding walk (SAW) on $\mathbb{Z}^d$ is a model for linear polymers. The SAW two-point function is defined as

$$G_p^{\mathrm{SAW}}(x) := \sum_{\omega:o\to x} p^{|\omega|} \prod_{j=1}^{|\omega|} D(\omega_j - \omega_{j-1}) \prod_{s<t} (1 - \delta_{\omega_s,\omega_t}),$$

where $p \geq 0$ is the fugacity, $|\omega|$ is the length of a path $\omega = (\omega_0, \omega_1, \dots, \omega_{|\omega|})$, and $D : \mathbb{Z}^d \to [0, 1]$ is the $\mathbb{Z}^d$-symmetric 1-step distribution for the underlying random walk (RW). The contribution from the 0-step walk is $\delta_{o,x}$ by convention. If the product $\prod_{s<t}(1 - \delta_{\omega_s,\omega_t})$ is absent, $G_p^{\mathrm{SAW}}(x)$ is simply equal to the RW Green's function $G_p^{\mathrm{RW}}(x)$, and therefore the radius of convergence $p_{\mathrm{c}}^{\mathrm{SAW}}$ is not less than $1 \equiv p_{\mathrm{c}}^{\mathrm{RW}}$. It is known that the susceptibility $\chi_p^{\mathrm{SAW}} := \sum_{x\in\mathbb{Z}^d} G_p^{\mathrm{SAW}}(x)$ is finite if and only if $p < p_{\mathrm{c}}^{\mathrm{SAW}}$ and diverges as $p \nearrow p_{\mathrm{c}}^{\mathrm{SAW}}$. For RW, $\chi_p^{\mathrm{RW}} = (1 - p)^{-1}$ for $p < 1$.

We also consider percolation and the Ising model that are defined by D. For percolation, each bond $\{u, v\}$, which is a pair of vertices in $\mathbb{Z}^d$, is either occupied or vacant independently of the other bonds, and the probability that $\{u, v\}$ is occupied equals $pD(v - u)$, where $p \geq 0$ is the percolation parameter. The percolation two-point function $G_p^{\mathrm{perc}}(x)$ is the probability that there is a path of occupied bonds from o to x. We also use the notation $G_p^{\mathrm{Ising}}(x)$ to denote the Ising two-point function (the precise definition will be shown at my presentation). When we discuss those models simultaneously, we omit the superscripts.

We are interested in asymptotic behavior of the critical two-point function $G_{p_{\mathrm{c}}}(x)$ as $|x| \to \infty$. For the finite-range case, such as $D(x) \propto \mathbb{1}_{\{0<|x|\leq L\}}$ for some $L \in [1, \infty)$, it has been proved [3, 4, 6] that, for SAW and the Ising model with $d > 4$ and for percolation with $d > 6$, if d or L is sufficiently large, then there is a model-dependent constant A ($\equiv 1$ for RW) such that, as $|x| \to \infty$,

$$G_{p_{\mathrm{c}}}(x) \sim \frac{A}{p_{\mathrm{c}}} \frac{a_d}{\sigma^2 |x|^{d-2}}, \tag{1}$$

where $a_d := \frac{d}{2}\Gamma(\frac{d-2}{2})\pi^{-d/2}$ and $\sigma^2 := \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) = O(L^2)$.

For the long-range case, such as $D(x) \propto (|x| \vee L)^{-d-\alpha}$ for some $L \in [1, \infty)$ and $\alpha > 0$, which is in the domain of attraction of α-stable distributions, it has been proved [5] that, for SAW and the Ising model with $d > 2(\alpha \wedge 2)$ and for percolation with $d > 3(\alpha \wedge 2)$, if L is sufficiently large, then

$$\hat{G}_p(k) := \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} G_p(x) \asymp \frac{1}{p_{\mathrm{c}} - p + p(1 - \hat{D}(k))} \qquad \text{uniformly in } p < p_{\mathrm{c}}.$$

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This is already close to RW: $\hat{G}_p^{\rm RW}(k) = (1 - p\hat{D}(k))^{-1}$ for $p < 1$. However, it does not immediately imply (1) or something similar to the Riesz potential when $\alpha < 2$, in particular. This is why we initiated this research.

2. Main result It is known (e.g., [1]) that there are $v_\alpha = O(L^{\alpha\wedge 2})$ and $\epsilon > 0$ such that the Fourier transform of $D(x) \propto (|x| \vee L)^{-d-\alpha}$ obeys

$$1 - \hat{D}(k) = v_\alpha |k|^\alpha \times \begin{cases} 1 + O(|k|^\epsilon) & (\alpha \neq 2), \\ \log \frac{1}{|k|} + O(1) & (\alpha = 2). \end{cases}$$

The following is the main statement of the ongoing project with L.-C. Chen [2].

Theorem. Let $\alpha \neq 2$. For RW with any $d > \alpha \wedge 2$ and any L, for SAW and the Ising model with $d > 2(\alpha \wedge 2)$ and $L \gg 1$, and for percolation with $d > 3(\alpha \wedge 2)$ and $L \gg 1$, there is a model-dependent constant A ($\equiv 1$ for RW) such that, as $|x| \to \infty$,

$$G_{p_{\rm c}}(x) \sim \frac{A}{p_{\rm c}} \frac{g_\alpha}{v_\alpha |x|^{d-\alpha\wedge 2}}, \qquad \text{where} \quad g_\alpha := \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}. \tag{2}$$

We note that, when $\alpha > 2$, the factor g_α/v_α in (2) simply equals a_d/σ^2 in (1). Therefore, the above theorem for $\alpha > 2$ concludes the same result as in [3, 4, 6] for the finite-range case.

At my presentation, I will explain the key steps for the proof of the above theorem.

References

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