JENSEN'S INEQUALITY ON CONVEX SPACES

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1. p-Uniform Convexity

Definition 1.1 (p-Uniformly Convex Space; cf. Naor-Silberman [4])**.** A metric space (Y,d) is called *p-uniformly convex with parameter $k>0$* if (Y,d) is a geodesic space and for any three points $x,y,z\in Y$, any minimal geodesic $\gamma:=(\gamma_t)_{t\in[0,1]}$ in Y with $\gamma_0=x$, $\gamma_1=y$, and all $t\in[0,1]$,

$$d^p(z,\gamma_t)\le(1-t)d^p(z,x)+td^p(z,y)-\frac{k}{2}t(1-t)d^p(x,y). \tag{1.1}$$

By definition, putting $z=\gamma_t$, we see $k\in]0,2]$ and $p\in[2,\infty[$. The inequality (1.1) yields the (strict) convexity of $Y\ni x\mapsto d^p(z,x)$ for a fixed $z\in Y$. Any closed convex subset of a p-uniformly convex space is again a p-uniformly convex space with the same parameter. Any L^p space over a measurable space is p-uniformly convex with parameter $k=\frac{8}{4^pp^2}(\frac{p-1}{p})^{p-1}$ provided $p>2$, and it is 2-uniformly convex with parameter $k=2(p-1)$ provided $1<p\le 2$. Every CAT(0)-space is a p-uniformly convex space with parameter $k=\frac{8}{4^pp^2}(\frac{p-1}{p})^{p-1}$ for $p\ge 2$ (we can take $k=2$ for $p=2$), because $\mathbb{R}^2$ is isometrically embedded into $L^p([0,1])$ for $p>1$ (see [1],[4]) and any L^p-space of maps into CAT(0)-space is again p-uniformly convex for $p\ge 2$. Ohta [5] proved that for $\kappa>0$ any CAT(κ)-space Y with $\mathrm{diam}(Y)<R_\kappa/2$ is a 2-uniformly convex space with parameter $\{(\pi-2\sqrt{\kappa}\varepsilon)\tan\sqrt{\kappa}\varepsilon\}$ for any $\varepsilon\in]0,R_\kappa/2-\mathrm{diam}(Y)]$.

Lemma 1.1 (Projection Map to Convex Set)**.** *Let (Y,d) be a complete p-uniformly convex space with parameter $k\in]0,2]$. The the following hold:*

1. *Let F be a closed convex subset of (Y,d). Then, for each $x\in Y$, there exists a unique element $\pi_F(x)\in F$ such that $d(x,F)=d(\pi_F(x),x)$ holds. We call $\pi_F:Y\to F$ the projection map to F.*
2. *Let F be as above. Then π_F satisfies*

$$d^p(z,\pi_F(z))+\frac{k}{2}d^p(\pi_F(z),w)\le d^p(z,w),\quad \textit{for } z\in Y, w\in F. \tag{1.2}$$

Definition 1.2 (Vertical Geodesics)**.** Let (Y,d) be a geodesic space. Take a geodesic η with a point p_0 on it and another geodesic γ through p_0. We say that γ *is vertical to η at p_0* (write $\gamma\perp_{p_0}\eta$ in short) if for any $x\in\gamma$ and $y\in\eta$, $d(x,p_0)\le d(x,y)$ holds.

Let (Y,d) be a complete p-uniformly convex space with parameter $k\in]0,2]$. We consider the following conditions:

- **(A)** For any closed convex set F in (Y,d), the projection map $\pi_F:Y\to F$ satisfies $d(\pi_F(x),y)\le d(x,y)$ for $x\in Y$, $y\in F$.
- **(B)** Let γ and η be minimal geodesic segments such that γ intersects η at p_0. Then $\gamma\perp_{p_0}\eta$ imlies $\eta\perp_{p_0}\gamma$.

Lemma 1.2. **(B)** *implies* **(A)**.

Lemma 1.3 (Stability of **(B)** under Product)**.** *Let (Y_i, d_i) $(i = 1, 2)$ be p-uniformly convex spaces with the common parameter $k \in]0, 2]$ satisfying* **(B)** *and (Y, d) the p-product metric space defined by the following:*

$$\begin{cases} Y := Y_1 \overset{p}{\times} Y_2 := \{(x_1, x_2) \mid x_1 \in Y_1, x_2 \in Y_2\}, \\ d(x, y) = \left(d_1^p(x_1, y_1) + d_2^p(x_2, y_2)\right)^{1/p} \qquad \textit{for } x = (x_1, x_2),\ y = (y_1, y_2) \in Y. \end{cases}$$

Then (Y, d) is a p-uniformly convex space with parameter $k \in]0, 2]$ and it satisfies **(B)**.

2. p-Barycenter

Let (Y, d) be a metric space and for $p > 0$ let $\mathcal{P}^p(Y)$ be the family of all Borel probability measures on Y having p-th moment.

Definition 2.1 (p-Barycenter; cf. [4])**.** Fix $p \geq 2$. For $\mu \in \mathcal{P}^p(Y)$, if $z \mapsto \int_Y d^p(z, x)\mu(dx)$ has a minimizer $b_p(\mu) \in Y$, then we call $b_p(\mu)$ the *p-barycenter, or p-center of mass* of $\mu \in \mathcal{P}^p(Y)$. For $\mu \in \mathcal{P}^{p-1}(Y)$ and $w \in Y$, we consider the following function F_w:

$$F_w(z) := \int_Y (d^p(z, x) - d^p(w, x))\mu(dx). \tag{2.1}$$

We easily see

$$|F_w(z)| \leq p d(z, w) \int_Y (d(z, x) + d(w, x))^{p-1}\mu(dx) < \infty.$$

If $Y \ni z \mapsto F_w(z)$ admits a minimizer $b_p(\mu)$ independent of w in the sense that $F_w(z) \geq F_w(b_p(\mu))$ if and only if $F_v(z) \geq F_v(b_p(\mu))$ for all $z, w, v \in Y$, we call it *p-barycenter, or p-center of mass* of $\mu \in \mathcal{P}^{p-1}(Y)$. If the p-barycenter of $\mu \in \mathcal{P}^p(Y)$ exists, then it is a p-barycenter of $\mu \in \mathcal{P}^{p-1}(Y)$.

Lemma 2.1 ([4],[3])**.** *Let (Y, d) be complete p-uniformly convex space with some parameter $k \in]0, 2]$. Then $\mu \in \mathcal{P}^{p-1}(Y)$ admits the unique p-barycenter.*

For any metric space (Y, d), we easily see $b_p(\delta_x) = x$ for $x \in Y$.

Proposition 2.1. *Let (Y, d) be a complete p-uniformly convex space with a parameter $k \in]0, 2]$. For $x, y \in Y$ and $t \in [0, 1]$, we have*

$$b_p((1-t)\delta_x + t\delta_y) = \gamma_{xy}\left(\frac{t^{\frac{1}{p-1}}}{t^{\frac{1}{p-1}} + (1-t)^{\frac{1}{p-1}}}\right),$$

where γ_{xy} is the unique minimal segment joining x to y.

Remark 2.1. Let (Y, d) be a complete p-uniformly convex space with a parameter $k \in]0, 2]$.

(1) Let (Y, d) be a complete CAT(0)-space. In this case, we can consider p-barycenter $b_p(\mu)$ of $\mu \in \mathcal{P}^{p-1}(Y)$ for any $p \geq 2$. For $p > 2$, $b_p(\mu) \neq b_2(\mu)$ in general.

(2) For any probability measure $\mu \in \mathcal{P}^1(H)$ over a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$, $b_2(\mu)$ is nothing but the mean $\int_H x\mu(dx)$ of μ, that is,

$$\langle b_2(\mu), h \rangle_H = \int_H \langle x, h \rangle_H \mu(dx) \quad \text{ for } h \in H.$$

But for $\mu \in \mathcal{P}^{p-1}(H)$ with $p > 2$, $\alpha = b_p(\mu)$ is the unique solution of $\int_H |x - \alpha|^{p-2}(x - \alpha)\mu(dx) = 0$.

3. Jensen's inequalty on p-uniformly convex spaces

Our Jensen's inequality is the following:

Theorem 3.1 (Jensen's Inequality on p-Uniformly Convex Space)**.**
Let (Y, d) be a complete p-uniformly convex with some parameter $k \in]0, 2]$. Suppose that (Y, d) satisfies **(B)**. *Let φ be a lower semi-continuous convex function on Y and $\mu \in \mathcal{P}^{p-1}(Y)$. Suppose $\varphi \in L^{p-1}(Y; \mu)$. Then we have*

$$\varphi(b_p(\mu)) \leq b_p(\varphi_*\mu). \tag{3.1}$$

Here $\varphi_*\mu$ is the image (or push-forward) measure of μ by φ on $\mathbb{R}$ and $b_p(\varphi_*\mu)$ is the p-barycenter of $\varphi_*\mu \in \mathcal{P}^{p-1}(\mathbb{R})$ by regarding that $\mathbb{R}$ is a p-uniformly convex space.

Corollary 3.1 (Jensen's Inequality on 2-Uniformly Convex Space)**.**
Let (Y, d) be a complete 2-uniformly convex with some parameter $k \in]0, 2]$. Suppose that (Y, d) satisfies **(B)**. *Let φ be a lower semi-continuous convex function on Y and $\mu \in \mathcal{P}^1(Y)$. Suppose $\varphi \in L^1(Y; \mu)$. Then we have*

$$\varphi(b_2(\mu)) \leq \int_Y \varphi(x)\mu(dx). \tag{3.2}$$

Corollary 3.2 (Fundamental Contraction Property)**.** *Let (Y, d) be a complete 2-uniformly convex with some parameter $k \in]0, 2]$. Suppose that (Y, d) is a NPC space in the sense of Busemann and satsifies* **(B)**. *Let $\mu, \nu \in \mathcal{P}^1(Y)$. Then*

$$d(b_2(\mu), b_2(\nu)) \leq d_{W^1}(\mu, \nu),$$

where $d_{W^1}(\mu, \nu)$ is the Kantorovich-Rubinstein/L^1-Wasserstein distance between μ and ν defined by

$$d_{W^1}(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{Y \times Y} d(x, y)\pi(dxdy).$$

Here $\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(Y \times Y) \mid \pi(A \times Y) = \mu(A), \pi(Y \times B) = \nu(B)$ for $A, B \in \mathcal{B}(Y)\}$.

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