Heat flow on Alexandrov spaces*

Kazumasa Kuwada†

(Graduate school of humanities and sciences, Ochanomizu university)

This talk is based on [3], a joint work with N. Gigli and S. Ohta.

Let (X, d) be a compact Alexandrov space of curvature bounded from below by $k \in \mathbb{R}$ (e.g. a compact Riemannian manifold). Suppose $n = \dim_H X \in \mathbb{N}$. Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure, which will be regarded as a canonical base measure on X. In what follows, we consider two different ways to define "heat distribution" on X.

First we deal with the Dirichlet energy. By using properties of the distance d, we can define a weak gradient ∇f as well as a canonical Hilbertian metric $\langle \nabla f, \nabla f \rangle$ (see [5]). By using these notions, we define the energy functional $\mathcal{E}$ on $L^2(X)$ by

$$\mathcal{E}(f, f) := \int_X \langle \nabla f, \nabla f \rangle d\mathcal{H}^n.$$

and a first order L^2-Sobolev space $W^{1,2}(X)$. Note that $\mathrm{Lip}(X)$ is dense in $W^{1,2}(X)$ and $\langle \nabla f, \nabla f \rangle^{1/2}$ coincides with the local Lipschitz constant $|\nabla_d f|$ $\mathcal{H}^n$-a.e. for $f \in \mathrm{Lip}(X)$. Moreover, $(\mathcal{E}, W^{1,2}(X))$ becomes a strongly local regular Dirichlet form. We denote the associated generator and semigroup by Δ and T_t respectively. It is shown in [5] that T_t has a positive Hölder-continuous density $p_t(x, y)$. Thus we can define $T_t \mu$ for any $\mu \in \mathcal{P}(X)$.

Next we consider the gradient flow of the relative entropy on $\mathcal{P}(X)$. Recall that d_2^W stands for the L^2-Wasserstein distance on $\mathcal{P}(X)$. Note that $(\mathcal{P}(X), d_2^W)$ becomes a geodesic metric space. For $\mu \in \mathcal{P}(X)$, we define the relative entropy by

$$\mathrm{Ent}(\mu) := \int_X \rho \log \rho \, d\mathcal{H}^n$$

when $d\mu = \rho d\mathcal{H}^n$ and $\mathrm{Ent}(\mu) = \infty$ otherwise. For $\mu \in \mathcal{P}(X)$ with $\mathrm{Ent}(\mu) < \infty$, we define the local slope as

$$|\nabla_- \mathrm{Ent}|(\mu) := \limsup_{\nu \to \mu} \frac{\max\{\mathrm{Ent}(\mu) - \mathrm{Ent}(\nu), 0\}}{W_2(\mu, \nu)}.$$

We say that an absolutely continuous curve $(\mu_t)_{t \ge 0}$ in $(\mathcal{P}(X), d_2^W)$ is a gradient flow of Ent if $\mathrm{Ent}(\mu_t) < \infty$ for $t \ge 0$ and

$$\mathrm{Ent}(\mu_t) = \mathrm{Ent}(\mu_s) + \frac{1}{2} \int_t^s |\dot{\mu}_r|^2 \, dr + \frac{1}{2} \int_t^s |\nabla_- \mathrm{Ent}|^2(\mu_r) \, dr$$

for all $0 \le t < s$, where $|\dot{\mu}_s| := \lim_{h \to 0} \frac{1}{h} d_2^W(\mu_{s+h}, \mu_s)$. This is one of possible formulations of "$\partial_r \mu_r = -\nabla \mathrm{Ent}(\mu_r)$" in this nonsmooth setting. To study more about it, we consider the "curvature-dimension condition" $\mathsf{CD}(K, \infty)$ given as follows: For $K \in \mathbb{R}$, we say that $(X, d, \mathcal{H}^n)$ enjoys $\mathsf{CD}(K, \infty)$ when

$$\mathrm{Ent}(\nu_t) \le (1-t)\mathrm{Ent}(\nu_0) + t\mathrm{Ent}(\nu_1) - Kt(1-t) d_2^W(\nu_0, \nu_1)$$

for any minimal geodesic ν_t in $(\mathcal{P}(X), d_2^W)$. The condition $\mathsf{CD}(K, \infty)$ is known as a generalization of the presence of a lower Ricci curvature bound by K. It is shown in [7] that $(X, d, \mathcal{H}^n)$ satisfies

*研究集会「確率論シンポジウム」(2010 年 12 月 20 日—12 月 23 日 於数理解析研究所) 講演予稿

†URL: http://www.math.ocha.ac.jp/kuwada e-mail: kuwada.kazumasa@ocha.ac.jp

$\mathsf{CD}(K,\infty)$ with $K=(n-1)k$. Under $\mathsf{CD}(K,\infty)$, we can apply the general theory of gradient flows on a metric space to show the existence and the uniqueness ([2]; see [1] also). Moreover, gradient flows μ_t and μ'_t of Ent satisfies the following contraction property (see [6]):

$$d_2^W(\mu_t,\mu'_t)\le \mathrm{e}^{-Kt}d_2^W(\mu_0,\mu'_0). \tag{C_2}$$

Thus we can extend the notion of the gradient flow of Ent for any initial condition $\mu_0\in\mathcal{P}(X)$.

Under these formulations, our main theorem asserts the following:

Theorem 1 [3, Theorem 3.1] *Given $\mu_0\in\mathcal{P}(X)$, let μ_t be the gradient flow of the relative entropy. Then $\mu_t=T_t\mu_0$.*

As a direct consequence of Theorem 1, we obtain (C_2) for $T_t\mu_0$ and $T_t\mu'_0$ instead of μ_t and μ'_t. By combining it with the result in [4] and a known regularity of T_t, we obtain the following:

Theorem 2 [3, Theorems 4.3,4.4 and 4.6] *Suppose that $\mathsf{CD}(K,\infty)$ holds.*

(i) *Let $f\in W^{1,2}(X)$ and $t>0$. Then $T_tf\in\mathrm{Lip}(X)$ and*

$$|\nabla_d T_tf|(x)\le \mathrm{e}^{-Kt}T_t(|\nabla f|^2)(x)^{1/2}$$

holds for all $x\in X$. In particular, the following Bakry-Émery L^2-gradient estimate holds:

$$|\nabla T_tf|(x)\le \mathrm{e}^{-Kt}T_t(|\nabla f|^2)(x)^{1/2}\quad \text{for a.e. } x.$$

(ii) *$p_t(x,\cdot)\in\mathrm{Lip}(X)$ and $T_tf\in\mathrm{Lip}(X)$ for all $x\in X$ and $f\in L^1(X)$.*

(iii) *Let f be an L^2-eigenfunction of Δ. Then $f\in\mathrm{Lip}(X)$.*

(iv) *A weak Γ_2-condition*

$$\frac{1}{2}\int_X \Delta g\langle\nabla f,\nabla f\rangle d\mathcal{H}^n-\int_X g\langle\nabla\Delta f,\nabla f\rangle d\mathcal{H}^n\ge K\int_X g\langle\nabla f,\nabla f\rangle d\mathcal{H}^n$$

holds for $f\in D(\Delta)$ with $\Delta f\in W^{1,2}(X)$ and $g\in D(\Delta)\cap L^\infty_+(X)$ with $\Delta g\in L^\infty(X)$.

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