Diffusion Processes in Thin Tubes and their Limits on Graphs

Seiichiro Kusuoka
Research Fellow of the Japan Society for the Promotion of Science (PD)
(Joint work with Sergio Albeverio)

We concern diffusion processes running on tubular domains with Dirichlet (i.e. absorbing-like) boundary conditions, and obtain the limit processes where the thin tubular domains shrink to graphs. Problems of this type are important for models of phenomenons. Partial differential equations on graphs are often considered for fluid mechanics in thin tubes, for example circuits of blood vessels, or circuits of neurons. These equations are simplified by regarding the thin tubes as graphs. However, if we look at them with microscope, they consist of tubes.

These problems have been studied before intensively in the case of Neumann boundary conditions by probabilistic tools and analytic tools. In the case of Dirichlet boundary conditions, there are difficulties, and there have been less works concerned with this case. Now we use probabilistic methods, and discuss the case of shrinking by potentials. Shrinking by potentials are associated with the case of Dirichlet boundary conditions. Our goal is to determine the limit process on a given graph. However, by locality, the behavior of diffusion processes associated with differential operators is determined in a given point by the behavior in neighborhoods of it. Thus, it is enough to consider the case of "N-spider" (the definition appears below).

Consider an n-dimensional Euclidean space $\mathbb{R}^n$, let $d(\cdot,\cdot)$ be the distance function in $\mathbb{R}^n$, and O be the origin. Let $\{e_i\}_{i=1}^N$ be N different unit vectors in $\mathbb{R}^n$ and $I_i := \{se_i : s \in [0,\infty)\}$. Consider an N-spider graph Γ defined by $\Gamma := \bigcup_{i=1}^N I_i$. Γ is also called an N-star graph. Let A be the set in $\mathbb{R}^n$ given by

$$A := \bigcup_{i,j:\ i\neq j} \{x \in \mathbb{R}^n :\ x\cdot e_i = x\cdot e_j\}.$$

For $x \in \mathbb{R}^n \setminus A$, let $\pi(x)$ be the nearest point in Γ from x. Note that $\pi(x)$ is uniquely determined for all $x \in \mathbb{R}^n \setminus A$.

Let u_i be a differentiable function on $[0,1)$ such that

$$u_i(0) = 0, \quad u_i' \geq 0, \quad \text{and} \quad -\lim_{R\uparrow 1} \frac{u_i(R)}{\log(1-R)} = +\infty.$$

for $i = 1,2,\ldots,N$. (u_i determines the potential acting in the thin tube around I_i). Let c_i be a positive number for $i = 1,2,\ldots,N$ and $\kappa := \max\left\{\sqrt{2}c_i/\sqrt{1-\langle e_i, e_j\rangle} : i,j = 1,2,\ldots,N\right\}$. c_i has the interpretation of width of the tube around I_i. Let U be a function on $\mathbb{R}^n$ with values in $[0,\infty]$, and assume

$$\begin{aligned}
U(x) &= u_i(c_i^{-1}d(x,\Gamma)), && x \in \{x \in \mathbb{R}^n : \pi(x) \in I_i, d(x,I_i) < c_i, |x| \geq \kappa\}\\
U(x) &= +\infty, && x \in \{x \in \mathbb{R}^n : \pi(x) \in I_i, d(x,I_i) \geq c_i, |x| \geq \kappa\},\\
U(x) &< +\infty, && x \in \{x \in \mathbb{R}^n : |x| \leq \kappa/2\},
\end{aligned}$$

$\Omega := \{x : U(x) < \infty\}$ is a simply connected and unbounded domain, $\partial\Omega$ is a C^2-manifold, and $U|_\Omega$ is a C^1-function in Ω. This structure Ω is sometimes called a "fattened" N-spider. In addition, we assume

$$-\lim_{m\to\infty} \frac{U(x_m)}{\log(d(x_m,\partial\Omega))} = +\infty$$

for any sequence $\{x_m\}$ which converges to a point $x \in \partial\Omega$.

Let $\Omega^\varepsilon := \varepsilon\Omega$, and $U^\varepsilon(x) = U(\varepsilon^{-1}x)$ for $x \in \mathbb{R}^n$ for all $\varepsilon > 0$. Note that $U^\varepsilon(x) \in [0,+\infty)$ for $x \in \Omega^\varepsilon$, and $\partial\Omega^\varepsilon$ is a C^2-manifold, and $U^\varepsilon|_{\Omega^\varepsilon}$ is a C^1-function in Ω^ε. Consider a diffusion process X^ε given by the following equation:

$$X^\varepsilon(t) = X^\varepsilon(0) + \int_0^t \sigma(X^\varepsilon(s))dW(s) + \int_0^t b(X^\varepsilon(s))ds - \int_0^t (\nabla U^\varepsilon)(X^\varepsilon(s))ds, \tag{1}$$

where $X^\varepsilon(0)$ is an Ω^ε-valued random variable, W is an n-dimensional Wiener process, $\sigma \in C_b(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$, and $b \in C_b(\mathbb{R}^n; \mathbb{R}^n)$. We can show that any solution X^ε of (1) does not hit $\partial\Omega^\varepsilon$. Let $a(x) := \sigma(x)\sigma^T(x)$ and assume that a is a uniformly positive definite matrix. Then, (1) has the unique solution.

We show the tightness of $\{X^\varepsilon\}$ in the continuous path space and characterize the limit process by the martingale problem. The behavior of the limit process X on each edge $I_i \setminus O$ is obtained easily by using geometrical methods. Hence, the difficulty is only on the behavior near O, and by observing the boundary condition at O we can characterize X.

Let

$$p_i := \frac{c_i^{n-1} \int_0^1 r^{n-2} e^{-u_i(r)} dr}{\sum_{i=1}^N c_i^{n-1} \int_0^1 r^{n-2} e^{-u_i(r)} dr}.$$

The weights of the boundary condition at O are determined by the values $\{p_i\}$. We remark that when u_i is independent of i, then we have $p_i := c_i^{n-1} / \left(\sum_{i=1}^N c_i^{n-1} \right)$, hence the weights $\{p_i\}$ are determined by the ratio of the area of the cross-section around the edge I_i. Define a second-order differential operator L_i on I_i by

$$\mathcal{L}_i := \frac{1}{2} |\sigma(x) e_i|^2 \partial_{e_i}^2 + b(x) e_i \partial_{e_i}$$

for $i = 1, 2, \ldots, N$. Define the second-order differential operator $\mathcal{L}$ on $C_0(\Gamma)$:

$$\begin{aligned}
\mathscr{D}(\mathcal{L}) &:= \Bigg\{ f \in C_0(\Gamma) : \ f|_{I_i \setminus O} \in C_b^2(I_i \setminus O) \text{ for all } i = 1, 2, \ldots, N, \\
&\qquad \lim_{s \downarrow 0} \mathcal{L}_i f(s e_i) \text{ has a common value in } i = 1, 2, \ldots, N, \\
&\qquad \sum_{i=1}^N p_i \left(\lim_{s \downarrow 0} (\partial_{e_i} f)(s e_i) \right) = 0 \Bigg\}, \\
\mathcal{L} f(x) &:= \mathcal{L}_i f(x), \quad x \in I_i \setminus O, \\
\mathcal{L} f(O) &:= \lim_{s \downarrow 0} \mathcal{L}_i f(s e_i).
\end{aligned}$$

Note that $\mathcal{L} f(O)$ does not depend on the selection of $i = 1, 2, \ldots, N$. The boundary condition of $\mathcal{L}$ at O is the weighted Kirchhoff boundary condition.

Theorem 1. *Consider diffusion processes X^ε defined by (1). Assume that $\sigma(O) = I_n$ and the law of X^ε converges to a probability measure μ_0 on Γ. Then, X^ε converges weakly on $C([0, +\infty); \mathbb{R}^n)$ to the diffusion process X as $\varepsilon \downarrow 0$, where X is determined by the conditions that the law of $X(0)$ is equal to μ_0 and*

$$E\left[f(X(t)) - f(X(s)) - \int_s^t \mathcal{L} f(X(u)) du \middle| \mathscr{F}_s \right] = 0$$

for $t \geq s \geq 0$ and $f \in \mathscr{D}(\mathcal{L})$, where $(\mathscr{F}_t)$ is the filtration generated by X. Therefore, $\mathcal{L}$ is the generator of X.

This argument is also available for diffusion processes in thin tubes with reflecting on the boundary of the tubes. This case is associated with Neumann boundary condition. In this case, the weights of boundary condition at O are $\{\widehat{p}_i\}$ where $\widehat{p}_i := c_i^{n-1} / \sum_{i=1}^N c_i^{n-1}$.

References

[1] Sergio Albeverio and Seiichiro Kusuoka, Diffusion Processes in Thin Tubes and their Limits on Graphs, submitting.