On-diagonal oscillation of the heat kernel on nested fractals

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It is a general belief that heat kernels on fractals exhibit highly oscillatory behaviors as opposed to the classical case of Riemannian manifolds. In this talk, we establish an oscillation of $p_t(x,x)$ as a function in t, where $p_t(x,y)$ denotes the canonical heat kernel on a given fractal.

More precisely, Let $K \subset \mathbb{R}^D$ be a *nested fractal*, e.g. any one of the examples shown below in Figure 1. It is known that there exists a canonical diffusion process, so-called the *Brownian motion*, on K, and it admits a continuous transition density (heat kernel) $p_t(x,y)$ (with respect to the self-similar measure with uniform weight). Moreover, Kumagai [1] has proved the following *sub-Gaussian estimate*

$$c_1 t^{-d_s/2} \exp\left(-\left(\frac{d(x,y)^{d_w}}{c_1 t}\right)^{\frac{1}{d_w-1}}\right) \le p_t(x,y) \le c_2 t^{-d_s/2} \exp\left(-\left(\frac{d(x,y)^{d_w}}{c_2 t}\right)^{\frac{1}{d_w-1}}\right) \tag{1}$$

for $t \in (0,1]$; here d is a suitably defined geodesic metric on K, $c_1, c_2 \in (0,\infty)$ are some constants, and $d_s \in (0,2)$ and $d_w \in (2,\infty)$ are called the *spectral dimension* and the *walk dimension of* K, respectively. In particular, for any $x \in K$ we have

$$c_1 \le t^{d_s/2} p_t(x,x) \le c_2, \qquad t \in (0,1]. \tag{2}$$

Then it is natural to ask whether the limit

$$\lim_{t \downarrow 0} t^{d_s/2} p_t(x,x) \tag{3}$$

exists or not. The main theorem of this talk asserts that this limit does not exist for *"generic"* points $x \in K$ and, in the case of the N-polygasket with $N \ge 3$ **odd**, for ***any*** $x \in K$. (See Figure 1-(b); note that the 3-polygasket is nothing but the (2-dimensional level-2) Sierpinski gasket.)

We need some definitions to state precisely the main theorem of this talk. Let $\{F_i\}_{i\in S}$ be the family of contraction maps defining our fractal K, and let V_0 be the set of boundary points of K (marked by solid circles in the examples in Figure 1). We set $V_m := \bigcup_{i\in S} F_i(V_{m-1})$ for $m \in \mathbb{N}$, inductively, and $V_* := \bigcup_{m\in\mathbb{N}} V_m$. Let $\pi : S^{\mathbb{N}} \to K$ be the canonical projection, which is defined by $\{\pi(\omega_1\omega_2\omega_3\dots)\} = \bigcap_{m\in\mathbb{N}} F_{\omega_1} \circ \dots \circ F_{\omega_m}(K)$, and let $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ be the usual shift operator given by $\sigma(\omega_1\omega_2\omega_3\dots) := \omega_2\omega_3\omega_4\dots$.

Definition 1 For $x, y \in \mathbb{R}^D$ with $x \ne y$, let $g_{xy} : \mathbb{R}^D \to \mathbb{R}^D$ be the reflection in the hyperplane $H_{xy} := \{z \in \mathbb{R}^D \mid |z-x| = |z-y|\}$. We define

$$\mathcal{S} := \{z \in K \mid g_{x_1y_1} g_{x_2y_2} \cdots g_{x_{2n-1}y_{2n-1}}(z) = z \text{ for some } n \in \mathbb{N} \text{ and } x_i, y_i \in V_0,\ x_i \ne y_i\}, \tag{4}$$

$$\mathcal{S}_* := \{z \in K \mid \lim_{m\to\infty} \operatorname{dist}(\pi(\sigma^m(\omega)), \mathcal{S}) = 0 \text{ for any } \omega \in \pi^{-1}(z)\}. \tag{5}$$

By a version of Borel-Cantelli lemma, we easily see that $\mathcal{S}_*$ is *"measure-theoretically small"* in the following sense.

Proposition 2 *$\mathcal{S}$ is compact and* $\operatorname{int}_K \mathcal{S} = \emptyset$*. Moreover, if ν is a σ-ergodic Borel probability measure on $S^{\mathbb{N}}$ with $\nu \circ \pi^{-1}(\mathcal{S}) < 1$, then $\nu \circ \pi^{-1}(\mathcal{S}_*) = 0$.*

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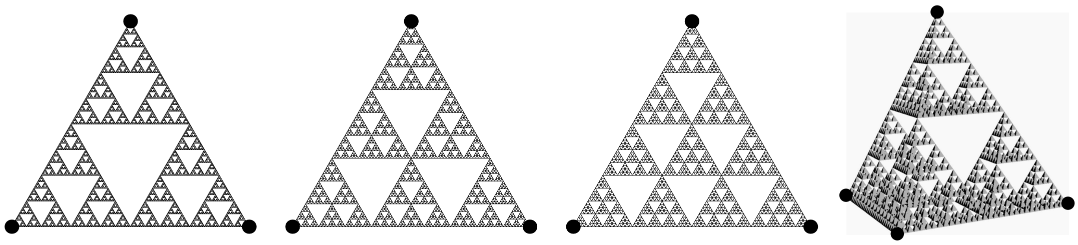
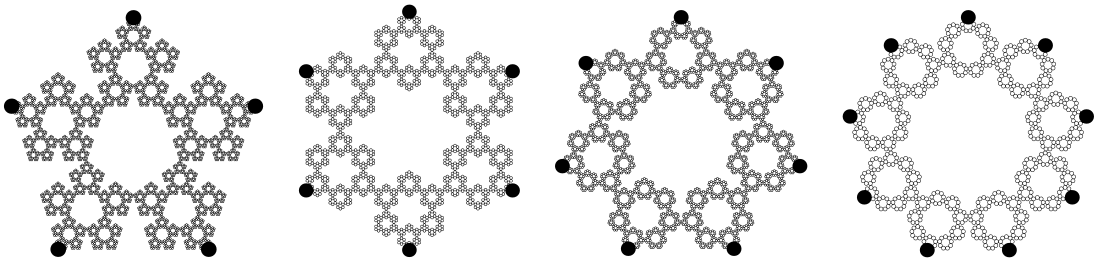
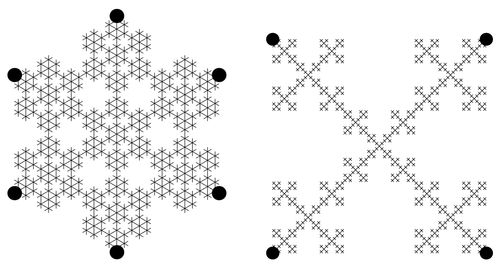


(a) *Sierpinski gaskets. From the left, two-dimensional level-l Sierpinski gaskets ($l = 2, 3, 4$) and three-dimensional level-2 Sierpinski gasket.*



(b) *N-polygaskets. From the left, pentagasket ($N = 5$), hexagasket ($N = 6$), heptagasket ($N = 7$) and nonagasket ($N = 9$).*



(c) *Some other nested fractals. From the left, snowflake and the Vicsek set.*

Figure 1: *Examples of nested fractals. In each fractal, the set V_0 of its boundary points is marked by solid circles.*

The following is the main theorem of this talk.

Theorem 3 (K.) (i) *Assume $\#V_0 \geq 3$ (, so that $V_0 \subset \mathcal{S}$ and $V_* \subset \mathcal{S}_*$). Then the limit* (3) *does not exist for any $x \in K \setminus \mathcal{S}_*$. If in addition the limit* (3) *does not exist for any $\mathcal{S} \setminus V_0$, then neither does it for any $x \in K \setminus V_*$.*
(ii) *The limit* (3) *does not exist for any $x \in V_*$ when K is either the D-dimensional level-l Sierpinski gasket with $D \geq 2$, $l \geq 2$* (see Figure 1-(a)) *or the N-polygasket with $N \geq 3$, $N/4 \notin \mathbb{N}$* (see Figure 1-(b)).

Since $\boldsymbol{\mathcal{S} \subset V_*}$ when K is the N-polygasket with $N \geq 3$ **odd**, Theorem 3 immediately yields the following corollary.

Corollary 4 (K.) *The limit* (3) *does not exist for **any $x \in K$** when K is the N-polygasket with $N \geq 3$ **odd**.* (Note that the 3-polygasket is the (two-dimensional level-2) Sierpinski gasket.)

Note that $\mathcal{S} \not\subset V_*$ for all the other examples in Figure 1. Therefore for them we cannot conclude the non-existence of the limit (3) for $x \in \mathcal{S}_*$ by our method although it is quite likely.

References

[1] T. Kumagai, Estimates of transition densities for Brownian motion on nested fractals, *Probab. Theory Related Fields* **96** (1993), 205–224.