Heat kernel and mixing time convergence for sequences of simple random walks on graphs

D. A. Croydon*

The main conclusion of this work, which is a joint project with Ben Hambly (University of Oxford) and Takashi Kumagai (Kyoto University), is that the mixing times of the simple random walks on a sequence of graphs converge to the mixing time of a limiting diffusion whenever the corresponding state spaces, invariant measures and heat kernels converge. Whilst this result is intuitively obvious, proving it demands the introduction of a framework in which we can express the relevant conditions in a precise and useful way. For this purpose, I will explain how to define a 'generalised Gromov-Hausdorff metric' that extends the usual notion of Gromov-Hausdorff convergence of compact metric spaces to include associated measures and heat kernel-type functions. In addition to this, I will describe how our results can be applied to a number of examples, including some simple lattice models, self-similar fractal graphs with random weights, critical Galton-Watson trees, the Erdős-Rényi random graph at criticality and the range of a random walk in high dimensions.

In order to be more specific, it is helpful to first present some notation for graphs and the random walks upon them. Suppose $G=(V(G),E(G))$ is a finite connected graph with at least two vertices, where $V(G)$ denotes the vertex set and $E(G)$ the edge set of G, and d_G is a metric on $V(G)$. Let $\mu^G:V(G)^2\rightarrow\mathbb{R}_+$ be a symmetric weight function that satisfies $\mu^G_{xy}>0$ if and only if $\{x,y\}\in E(G)$. The discrete time simple random walk on the weighted graph G is then the Markov chain $((X^G_m)_{m\geq 0},\mathbf{P}^G_x,x\in V(G))$ with transition probabilities $(P_G(x,y))_{x,y\in V(G)}$ given by $P_G(x,y):=\mu^G_{xy}/\mu^G_x$, where $\mu^G_x:=\sum_{y\in V(G)}\mu^G_{xy}$. If π^G is a measure on $V(G)$ defined by setting, for $A\subseteq V(G)$, $\pi^G(A):=\sum_{x\in A}\mu^G_x/\sum_{x\in V(G)}\mu^G_x$, then π^G is the invariant probability measure for X^G. The transition density of X^G, with respect to π^G, is given by $(p^G_m(x,y))_{x,y\in V(G),m\geq 0}$, where

$$p^G_m(x,y):=\frac{\mathbf{P}^G_x(X_m=y)}{\pi^G(\{y\})}.$$

Due to parity concerns for bipartite graphs, it is convenient to consider a smoothed version of this function, $(q^G_m(x,y))_{x,y\in V(G),m\geq 0}$, obtained by set-

*Dept of Statistics, University of Warwick, Coventry, CV4 7AL, United Kingdom; d.a.croydon@warwick.ac.uk.

ting

$$q_m^G(x,y) := \frac{p_m^G(x,y) + p_{m+1}^G(x,y)}{2},$$

and define the L^p-mixing time of G by

$$t_{\mathrm{mix}}^p(G) := \inf\left\{ m > 0 : \sup_{x\in V(G)} \|q_m^G(x,\cdot) - 1\|_{L^p(\pi^G)} \leq 1/4 \right\}.$$

Finally, in the case that a sequence of graphs $(G^N)_{N\geq 1}$ is being considered, π^{G^N} will be abbreviated to π^N, and q^{G^N} to q^N.

As for the prototypical limiting objects, (F, d_F) is a compact metric space and π is a non-atomic Borel probability measure on F with full support. Moreover, $(q_t(x,y))_{x,y\in F, t>0}$ is the jointly continuous transition density of a conservative π-symmetric Hunt process $X^F = (X_t^F)_{t\geq 0}$ on F. (Some additional mild conditions will be imposed on π and $(q_t(x,y))_{x,y\in F,t>0}$ to avoid various trivialities, but are omitted here for brevity.) The transition density $(q_t(x,y))_{x,y\in F,t>0}$ is said to converge to stationarity in an L^p sense for some $p \in [1,\infty]$ if it holds that

$$\lim_{t\to\infty} \|q_t(x,\cdot) - 1\|_{L^p(\pi)} = 0,$$

for every $x \in F$. If this previous condition is satisfied, then it is possible to check that the L^p-mixing time of F,

$$t_{\mathrm{mix}}^p(F) := \inf\left\{ t > 0 : \sup_{x\in F} \|q_t(x,\cdot) - 1\|_{L^p(\pi)} \leq 1/4 \right\},$$

is a finite quantity.

Given this setup, the first goal of my talk will be to explain how we can make precise the following condition regarding the convergence of a sequence of graphs $(G^N)_{N\geq 1}$ and associated quantities to F, π and $(q_t(x,y))_{x,y\in F,t>0}$. As noted above, this involves a generalised Gromov-Hausdorff metric, the definition of which is based upon the fundamental Gromov-Hausdorff idea of considering isometric embeddings of the relevant objects into a common space, where distances can be measured, and then optimising over all such embeddings.

Assumption 1. *$(G^N)_{N\geq 1}$ is a sequence of finite connected graphs with at least two vertices for which there exists a sequence $(\gamma(N))_{N\geq 1}$ such that, for any compact interval $I \subset (0,\infty)$,*

$$\left(\left(V(G^N), d_{G^N}\right), \pi^N, \left(q_{\gamma(N)t}^N(x,y)\right)_{x,y\in V(G^N), t\in I} \right)$$

converges to

$$\left((F, d_F), \pi, (q_t(x,y))_{x,y\in F, t\in I}\right)$$

in a generalised Gromov-Hausdorff sense.

I will subsequently describe how this assumption can be applied to yield the convergence of the L^p-mixing times of the graphs in the sequence $(G^N)_{N\geq 1}$. The only additional assumption made is that the limiting heat kernel converges to stationarity in an L^p sense, which guarantees that the limiting L^p-mixing time is well-defined.

Theorem 1. *Suppose that Assumption 1 is satisfied. If* $p \in [1,\infty]$ *is such that the transition density* $(q_t(x,y))_{x,y\in F,t>0}$ *converges to stationarity in an* L^p *sense, then* $t^p_{\mathrm{mix}}(F) \in (0,\infty)$ *and*

$$\gamma(N)^{-1} t^p_{\mathrm{mix}}(G^N) \to t^p_{\mathrm{mix}}(F).$$

To complete my talk, I will discuss the application of this theorem to the examples mentioned in the opening paragraph. In particular, I will comment upon how to check Assumption 1 by using the local limit theorem proved in [1], which demonstrates the appropriate heat kernel convergence occurs whenever the laws of the simple random walks converge to the law of the limiting diffusion and a certain tightness condition is satisfied.

References

[1] D. A. Croydon and B. M. Hambly, *Local limit theorems for sequences of simple random walks on graphs*, Potential Anal. **29** (2008), no. 4, 351–389.