可算マルコフシフトの大偏差原理とその連分数展開への応用

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Denote by X the set of all one-sided infinite sequences over the set $\mathbb{N}$ of positive integers, namely $X = \{x = (x_1, x_2, \ldots): x_i \in \mathbb{N},\ i \in \mathbb{N}\}$, endowed with the product topology of the discrete topology on $\mathbb{N}$. Define the left shift $\sigma\colon X \to X$ by $(\sigma x)_i = x_{i+1}$ ($i \in \mathbb{N}$). For each $x \in X$ and $n \in \mathbb{N}$ define *an n-cylinder* by

$$[x_1, \ldots, x_n] = \{y = (y_i) \in X : x_i = y_i \text{ for every } i \in \{1, \ldots, n\}\}.$$

Let $\phi\colon X \to \mathbb{R}$ be a function. A Borel probability measure μ_ϕ on X is *Bowen's Gibbs measure for the potential* ϕ [1, 4, 5] if there exist constants $c_0 > 0$, $c_1 > 0$ and $P \in \mathbb{R}$ such that for every $x \in X$ and every $n \in \mathbb{N}$,

$$c_0 \le \frac{\mu_\phi[x_1, \ldots, x_n]}{\exp\left(-Pn + \sum_{i=0}^{n-1} \phi(\sigma^i(x))\right)} \le c_1.$$

Let $\mathcal{M}$ denote the space of Borel probability measures on X endowed with the weak*-topology. We are concerned with the following three sequences $\{\Delta_n\}$, $\{\Xi_n\}$, $\{\Upsilon_{y,n}\}$ of Borel probability measures on $\mathcal{M}$:

For each $x \in X$ and $n \in \mathbb{N}$ define $\delta_x^n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i x}$, with $\delta_{\sigma^i x}$ the unit point mass at $\sigma^i x$. Denote by Δ_n the distribution of the $\mathcal{M}$-valued random variable $x \mapsto \delta_x^n$ on the probability space (X, μ_ϕ);

For each integer $n \in \mathbb{N}$ define

$$\Xi_n = \left(\sum_{x \in \mathrm{Per}_n \sigma} \exp S_n\phi(x)\right)^{-1} \sum_{x \in \mathrm{Per}_n \sigma} \exp S_n\phi(x) \delta_{\delta_x^n},$$

$$\Upsilon_{y,n} = \left(\sum_{x \in \sigma^{-n} y} \exp S_n\phi(x)\right)^{-1} \sum_{x \in \sigma^{-n} y} \exp S_n\phi(x) \delta_{\delta_x^n},$$

where $\mathrm{Per}_n\sigma = \{x \in X : \sigma^n x = x\}$, $\sigma^{-n} y = \{x \in X : \sigma^n x = y\}$ and $y \in X$ is fixed.

Theorem A. ([6, Theorem A]). *Let $\phi\colon X \to \mathbb{R}$ be a measurable function and μ_ϕ a Bowen's Gibbs measure for the potential ϕ. Then $\{\Delta_n\}$, $\{\Xi_n\}$, $\{\Upsilon_{y,n}\}$ are exponentially tight and satisfy the Large Deviation Principle with the same convex good rate function I. All their weak*-limit points are supported on subsets of the set $I^{-1}(0)$.*

Under the hypotheses and notation of Theorem A, we call $\nu \in \mathcal{M}$ *a minimizer* if $I(\nu) = 0$. We give a sufficient condition for the uniqueness of minimizer. For a function $\phi\colon X \to \mathbb{R}$ put

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{x_1 \cdots x_n} \sup_{[x_1, \ldots, x_n]} \exp \sum_{i=0}^{n-1} \phi \circ \sigma^i,$$

where the sum runs over all n-cylinders. For $\gamma \in (0, 1]$ we introduce a metric d_γ on X by setting $d_\gamma(x, y) = \exp\left(-\gamma \inf\{i \in \mathbb{N} : x_i \ne y_i\}\right)$, with the convention $e^{-\infty} = 0$. A function $\phi\colon X \to \mathbb{R}$ is *Hölder continuous* if there exist $C > 0$ and $\gamma \in (0, 1]$ such that for every $k \in \mathbb{N}$ and all $x, y \in [k]$, $|\phi(x) - \phi(y)| \le C d_\gamma(x, y)$.

Theorem B. *Let $\phi\colon X \to \mathbb{R}$ be a Hölder continuous function such that $\beta_\infty := \inf\{\beta \in \mathbb{R}\colon P(\beta\phi) < \infty\} < 1$. Then there exists a unique shift-invariant Bowen's Gibbs measure for the potential ϕ. It is the unique equilibrium state for ϕ, i.e., the unique measure which attains the supremum*

$$\sup\left\{h(\nu) + \int \phi d\nu\colon \nu \in \mathcal{M} \text{ is shift-invariant and } \int \phi d\nu > -\infty\right\}$$

($h(\nu)$ being the entropy of ν with respect to σ)), and it is the unique minimizer of the rate function I in Theorem A. The $\{\Delta_n\}$, $\{\Xi_n\}$, $\{\Upsilon_{y,n}\}$ converge in the weak-topology to the unit point mass at the minimizer.*

We apply Theorem B to the Gauss map $G\colon (0,1] \to [0,1)$ given by $G(x) = 1/x - \lfloor 1/x \rfloor$. For $x \in (0,1) \setminus \mathbb{Q}$, define $(a_i(x))_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ by $a_i(x) = \left\lfloor \frac{1}{G^{i-1}(x)} \right\rfloor$, and put

$$[a_1(x); a_2(x); \cdots; a_n(x)] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cdots + \cfrac{1}{a_n(x)}}}.$$

Then $x = \lim_{n\to\infty}[a_1(x); a_2(x); \cdots; a_n(x)]$. The map $\pi\colon x \in (0,1)\setminus\mathbb{Q} \to (a_i(x))_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ is a homeomorphism, and commutes with G and the left shift. Hence, the study of the behavior of $a_1(x), a_2(x), a_3(x), \ldots$ translates to that of the dynamics of G.

Define $\phi := -\log|DG| \circ \pi^{-1}$. Then $\beta_\infty = 1/2$ [3]. For each $\beta > 1/2$ the potential $\beta\phi$ satisfies the conditions in Theorem B. Denote by μ_β the G-invariant Borel probability measure which corresponds to the unique shift-invariant Bowen's Gibbs measure for the potential $\beta\phi$.

Corollary. (Equidistribution of weighted periodic points). *For every $\beta > 1/2$ the following convergence in the weak*-topology holds:*

$$\frac{1}{\sum_{x\in\mathrm{Per}_n(G)} |DG^n(x)|^{-\beta}} \sum_{x\in\mathrm{Per}_n(G)} |DG^n(x)|^{-\beta}\delta_x^n \longrightarrow \mu_\beta \quad (n\to\infty).$$

The convergence for $\beta = 1$ was first proved in [2] by directly showing the tightness of the sequence of measures. The μ_1 is the Gauss measure: $d\mu_1 = \frac{1}{\log 2}\frac{dx}{1+x}$.

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