Central limit theorem for random walks on nilpotent covering graphs with weak asymmetry

Ryuya NAMBA (Okayama University)

(Jointwork with Satoshi ISHIWATA (Yamagata) and Hiroshi KAWABI (Keio))

Long time behaviors of random walks (RWs) on an infinite graph is a well-studied topic in geometry, harmonic analysis and graph theory, to say nothing of probability theory. It is known that geometric features such as the *periodicity* and the *volume growth* of the underlying graph affect long time behaviors of RWs. By putting an emphasis on them, Ishiwata, Kawabi and Kotani [1] considered a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on a Γ-crystal lattice X, a covering graph of a finite graph whose covering transformation group Γ is abelian. Through a discrete analogue of the harmonic map from X into a Euclidean space $\Gamma \otimes \mathbb{R}$, they established two kinds of functional central limit theorems (CLTs) for $\{w_n\}_{n=0}^{\infty}$. In fact, since a diverging drift term arising from the non-symmetry prevents us from taking the CLT-scaling limit directly, it is difficult to prove such CLTs. To overcome the difficulty, two schemes were introduced in [1]. One is to replace the usual transition operator by the transition-shift operator to "delete" the diverging drift term. The other is to introduce a family of non-symmetric RWs on X to "weaken" the diverging drift term. (The latter scheme is also applied in the study of the hydrodynamic limit of *weakly asymmetric* simple exclusion processes.)

Let Γ be a finitely generated nilpotent group. In [2], we considered a non-symmetric RW $\{w_n\}_{n=0}^{\infty}$ on a Γ-*nilpotent covering graph*, a generalization of both crystal lattices and Cayley graphs of a finitely generated group of polynomial volume growth. By extending the former scheme to the nilpotent case, we established a functional CLT for $\{w_n\}_{n=0}^{\infty}$ in [2]. The main purpose of this talk is to extend the latter scheme to the nilpotent case and to establish another functional CLT for $\{w_n\}_{n=0}^{\infty}$. This talk is based on our recent preprint [3].

Let $X = (V, E)$ be a Γ-nilpotent covering graph. Here V is the set of all vertices and E the set of all oriented edges in X. For $e \in E$, we denote the origin, terminus and inverse edge of e by $o(e), t(e)$ and $\overline{e}$, respectively. We set $E_x := \{e \in E \,|\, o(e) = x\}$ for $x \in V$. Let $p : E \longrightarrow (0, 1]$ be a Γ-invariant transition probability and $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$ a RW on X starting from $x \in V$ associated with p. Through the covering map $\pi : X \longrightarrow X_0$, we also consider the RW $(\Omega_{\pi(x)}(X_0), \mathbb{P}_{\pi(x)}, \{\pi(w_n)\}_{n=0}^{\infty})$ and the corresponding transition probability is also denoted by $p : E_0 \longrightarrow (0, 1]$. We denote by $m : V_0 \longrightarrow (0, 1]$ the normalized invariant measure on X_0 and also write $m : V \longrightarrow (0, 1]$ for the Γ-invariant lift of m to X. Let $\mathrm{H}_1(X_0, \mathbb{R})$ be the first homology group of X_0. We define the *homological direction* of the RW on X_0 by $\gamma_p := \sum_{e \in E_0} p(e)m\big(o(e)\big)e \in \mathrm{H}_1(X_0, \mathbb{R})$. We call the RW on X_0 (m-)*symmetric* if $p(e)m\big(o(e)\big) = p(\overline{e})m\big(t(e)\big)$ for $e \in E_0$. Otherwise, it is called (m-)*non-symmetric*. Note that the RW on X_0 is (m-)symmetric if and only if $\gamma_p = 0$.

Thanks to the celebrated theorem of Malćev, we find a connected and simply connected nilpotent Lie group G such that Γ is isomorphic to a cocompact lattice in G. The nilpotent Lie group G is equipped with the canonical dilations $(\tau_\varepsilon)_{\varepsilon \geq 0}$, which gives a scalar multiplication on

G. By realizing X into G, CLTs for RWs on X can be discussed. Let $\mathfrak{g}$ be the corresponding Lie algebra of G and $\mathfrak{g}^{(1)} \cong G/[G,G]$ the generating part of $\mathfrak{g}$. We take a canonical surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0,\mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ by using the general theory of covering spaces. Thanks to the map $\rho_{\mathbb{R}}$ and the discrete Hodge–Kodaira theorem, a flat metric g_0 associated with the transition probability p, called the *Albanese metric*, is induced on $\mathfrak{g}^{(1)}$. A periodic realization $\Phi_0 : X \longrightarrow G$ is said to be *modified harmonic* if

$$\sum_{e\in E_x} p(e)\log\Big(\Phi_0\big(o(e)\big)^{-1}\cdot\Phi_0\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(1)}} = \rho_{\mathbb{R}}(\gamma_p) \qquad (x\in V).$$

The quantity $\rho_{\mathbb{R}}(\gamma_p)\in\mathfrak{g}^{(1)}$ is called the *asymptotic direction*, which also appears in the law of large numbers for $\mathfrak{g}^{(1)}$-valued RW $\{\log\big(\Phi_0(w_n)\big)\big|_{\mathfrak{g}^{(1)}}\}_{n=0}^{\infty}$. It should be noted that $\gamma_p=0$ implies $\rho_{\mathbb{R}}(\gamma_p)=\mathbf{0}_{\mathfrak{g}}$, however, the converse does not hold in general.

For the given transition probability p, we introduce a family of Γ-invariant transition probabilities $(p_\varepsilon)_{0\le\varepsilon\le1}$ on X by $p_\varepsilon(e):=p_0(e)+\varepsilon q(e)$ for $e\in E$, where

$$p_0(e):=\frac{1}{2}\Big(p(e)+\frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big),\quad q(e):=\frac{1}{2}\Big(p(e)-\frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big).$$

Namely, the family $(p_\varepsilon)_{0\le\varepsilon\le1}$ is given by the linear interpolation between the given transition probability $p=p_1$ and the (m-)symmetric probability p_0. Moreover, the homological direction γ_{p_ε} equals $\varepsilon\gamma_p$ for every $0\le\varepsilon\le1$, which plays a key role in the proof of main theorems.

We now fix a reference point $x_*\in V$ such that $\Phi_0^{(0)}(x_*)=\mathbf{1}_G$, where $\mathbf{1}_G$ is the unit element of G. We write $g_0^{(\varepsilon)}$ for the Albanese metric on $\mathfrak{g}^{(1)}$ associated with p_ε and $\Phi_0^{(\varepsilon)}: X\longrightarrow G$ be the (p_ε-)modified harmonic realization for every $0\le\varepsilon\le1$. We set $\mathcal{Y}_{k/n}^{(\varepsilon,n)}(c):=\tau_{n^{-1/2}}\big(\Phi_0^{(\varepsilon)}(w_k(c))\big)$ for $n\in\mathbb{N}$, $k=0,1,\dots,n$, $c\in\Omega_{x_*}(X)$ and $0\le\varepsilon\le1$. We then define a G-valued continuous stochastic process $\mathcal{Y}^{(\varepsilon,n)}=(\mathcal{Y}_t^{(\varepsilon,n)})_{0\le t\le1}$ by the geodesic interpolation of $\{\mathcal{Y}_{k/n}^{(\varepsilon,n)}\}_{k=0}^{n}$ with respect to the Carnot–Carathéodory metric on G. We take an orthonormal basis $\{V_1,V_2,\dots,V_{d_1}\}$ of $(\mathfrak{g}^{(1)},g_0^{(0)})$ and consider a stochastic differential equation (SDE)

$$dY_t=\sum_{i=1}^{d_1}V_i(Y_t)\circ dB_t^i+\rho_{\mathbb{R}}(\gamma_p)(Y_t)\,dt,\qquad Y_0=\mathbf{1}_G,$$

where $(B_t)_{0\le t\le1}=(B_t^1,B_t^2,\dots,B_t^{d_1})_{0\le t\le1}$ is an $\mathbb{R}^{d_1}$-standard Brownian motion starting from $B_0=\mathbf{0}$. Let $(Y_t)_{0\le t\le1}$ be the G-valued diffusion process which solves the SDE above.

We now state our main result as follows:

Theorem. *Under several natural assumptions on* $\{\Phi_0^{(\varepsilon)}\}_{0\le\varepsilon\le1}$, *the sequence* $\{\mathcal{Y}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ *converges in law to a G-valued diffusion process Y in* $C^{0,\alpha\text{-Höl}}([0,1];G)$ *as* $n\to\infty$ *for all* $\alpha<1/2$.

References

[1] S. Ishiwata, H. Kawabi and M. Kotani: J. Funct. Anal. **272** (2017), pp.1553–1624.
[2] S. Ishiwata, H. Kawabi and R. Namba: preprint (2018), `arXiv:1806.03804`.
[3] S. Ishiwata, H. Kawabi and R. Namba: preprint (2018), `arXiv:1808.08856`.