The Laplacian on some round Sierpiński carpets and Weyl's asymptotics for its eigenvalues

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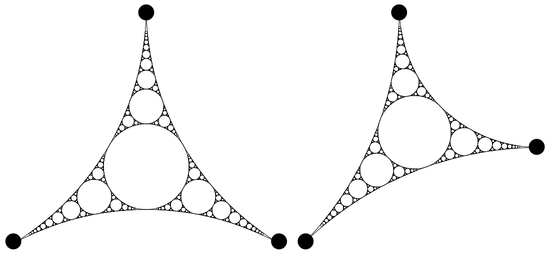

Fig. 1. Apollonian gaskets $K_{\alpha,\beta,\gamma}$

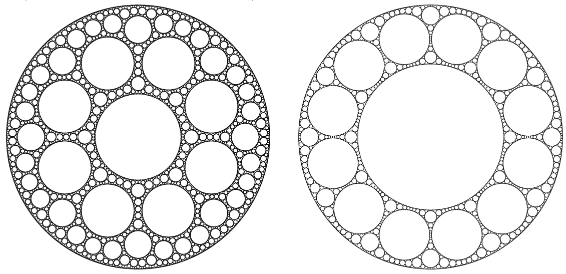

Fig. 2. Round Sierpiński carpets $\partial_\infty G_8$, $\partial_\infty G_{12}$

The purpose of this talk is to present the speaker's recent results on the construction of a "canonical" Laplacian on round Sierpiński carpets invariant with respect to certain Kleinian groups (i.e., discrete subgroups of the group $\mathrm{M\ddot{o}b}(\widehat{\mathbb{C}})$ of (orientation preserving or reversing) Möbius transformations on $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$) and on Weyl's asymptotics for its eigenvalues. Here a *round Sierpiński carpet* refers to a subset of $\widehat{\mathbb{C}}$ homeomorphic to the standard Sierpiński carpet whose complement consists of disjoint open disks in $\widehat{\mathbb{C}}$.

1. Preceding results for the Apollonian gasket

The construction of the Laplacian is based on the speaker's preceding study of the simplest case of the *Apollonian gasket* $K_{\alpha,\beta,\gamma}$. This is a compact fractal subset of $\mathbb{C}$ obtained from an ideal triangle, i.e., the closed subset of $\mathbb{C}$ enclosed by mutually *externally* tangent three circles, with the radii of the circles $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ and with the set $V_0^{\alpha,\beta,\gamma}$ of its three vertices (see Fig. 1). Set $\mathcal{C}(K_{\alpha,\beta,\gamma}) := \{f \mid f : K_{\alpha,\beta,\gamma} \to \mathbb{R},\ f \text{ is continuous}\}$.

Theorem 1.1 (K., cf. [5]). *There exist a finite Borel measure μ on $K_{\alpha,\beta,\gamma}$ with full support and an irreducible, strongly local, regular symmetric Dirichlet form $(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$ on $L^2(K_{\alpha,\beta,\gamma}, \mu)$ such that for any affine function $h : \mathbb{C} \to \mathbb{R}$, $h|_{K_{\alpha,\beta,\gamma}} \in \mathcal{F}_{\alpha,\beta,\gamma}$ and*

$$\mathcal{E}^{\alpha,\beta,\gamma}(h|_{K_{\alpha,\beta,\gamma}}, v) = 0 \qquad \text{for any } v \in \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma}) \text{ with } v|_{V_0^{\alpha,\beta,\gamma}} = 0 \tag{1.1}$$

(i.e., $h|_{K_{\alpha,\beta,\gamma}}$ is $\mathcal{E}^{\alpha,\beta,\gamma}$*-harmonic* on $K_{\alpha,\beta,\gamma} \setminus V_0^{\alpha,\beta,\gamma}$)*. Moreover, $\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma} \times \mathcal{C}_{\alpha,\beta,\gamma}}$ are unique (up to positive constant multiples of $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma} \times \mathcal{C}_{\alpha,\beta,\gamma}}$).*

Theorem 1.2 (K.). $\mathcal{C}^{\mathrm{LIP}}_{\alpha,\beta,\gamma} := \{u|_{K_{\alpha,\beta,\gamma}} \mid u : \mathbb{C} \to \mathbb{R},\ u \text{ is Lipschitz}\} \subset \mathcal{C}_{\alpha,\beta,\gamma}$ *and*

$$\mathcal{E}^{\alpha,\beta,\gamma}(u, v) = \sum\nolimits_{C \in \mathcal{A}_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C \langle \nabla_C u, \nabla_C v \rangle \, d\mathrm{vol}_C \quad \text{for any } u, v \in \mathcal{C}^{\mathrm{LIP}}_{\alpha,\beta,\gamma}, \tag{1.2}$$

where $\mathcal{A}_{\alpha,\beta,\gamma}$ denotes the set of all the arcs appearing in the construction of $K_{\alpha,\beta,\gamma}$, $\mathrm{rad}(C)$ the radius of C, ∇_C the gradient on C and vol_C the length measure on C.

Theorem 1.3 (K.). *As the measure μ in Theorem 1.1, $\mu^{\alpha,\beta,\gamma} := \sum_{C \in \mathcal{A}_{\alpha,\beta,\gamma}} \mathrm{rad}(C)\mathrm{vol}_C$ can be taken. Moreover, the Laplacian associated with $(K_{\alpha,\beta,\gamma}, \mu^{\alpha,\beta,\gamma}, \mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$ has discrete spectrum and its eigenvalues $\{\lambda_n^{\alpha,\beta,\gamma}\}_{n \in \mathbb{N}}$ (with each repeated according to multiplicity) satisfy, with[1] $d_{\mathsf{AG}} := \dim_{\mathrm{H}} K_{\alpha,\beta,\gamma}$ and some $c_0 \in (0, \infty)$ independent of α, β, γ,*

$$\lim\nolimits_{\lambda \to \infty} \lambda^{-d_{\mathsf{AG}}/2} \#\{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \le \lambda\} = c_0 \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}) \in (0, \infty). \tag{1.3}$$

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[1] $\dim_{\mathrm{H}}$ and $\mathcal{H}^d$ denote Hausdorff dimension and the d-dimensional Hausdorff measure, respectively.

2. Kleinian groups G_m with limit set a round Sierpiński carpet

Let $m \in \mathbb{N}$, $m > 6$. Since $\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} < \pi$ there exists a geodesic triangle with inner angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$, which is unique up to hyperbolic isometry, in the Poincaré disk model $\mathbb{B}^2 := \{z \in \mathbb{C} \mid |z| < 1\}$ of the hyperbolic plane. More specifically, set $\ell_1 := \mathbb{R}$, $\ell_3 := \{te^{\pi i/m} \mid t \in \mathbb{R}\}$ and choose $t, r \in (0,\infty)$ so that $\ell_2 := \{z \in \mathbb{C} \mid |z - te^{\pi i/m}| = r\}$ is orthogonal to $\partial\mathbb{B}^2 := \{z \in \mathbb{C} \mid |z| = 1\}$ and intersects ℓ_1 with angle $\frac{\pi}{3}$; there is a unique such choice of t, r by virtue of $\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} < \pi$. Let $\triangle_0$ denote the closed geodesic triangle formed by ℓ_1, ℓ_2, ℓ_3 and define a subgroup Γ_m of $\text{Möb}(\widehat{\mathbb{C}})$ by $\Gamma_m := \langle \{\text{Inv}_{\ell_k}\}_{k=1}^3 \rangle$, where Inv_ℓ denotes the inversion (reflection) in a circle or a straight line ℓ. Then *Poincaré's polygon theorem* (see, e.g., [2, Section 8]) tells us that $\mathbb{B}^2 = \bigcup_{\tau\in\Gamma_m} \tau(\triangle_0)$, where $\tau(\triangle_0)$ and $\sigma(\triangle_0)$ intersect only on their boundaries for any $\tau, \sigma \in \Gamma_m$ with $\tau \neq \sigma$.

Next, choose $r_m \in (0,1)$ so that $S := \{z \in \mathbb{C} \mid |z| = r_m\}$ intersects ℓ_2 with angle $\frac{\pi}{3}$; it is elementary to see that there is a unique such choice of r_m. Then it turns out (see, e.g., [1]) that the subgroup G_m of $\text{Möb}(\widehat{\mathbb{C}})$ defined by $G_m := \langle \Gamma_m, \text{Inv}_S \rangle$ is a Kleinian group and that $\partial_\infty G_m := \overline{\bigcup_{g\in G_m} g(\partial\mathbb{B}^2)}$ is the *limit set* of G_m (i.e., the minimum non-empty closed G_m-invariant subset of $\widehat{\mathbb{C}}$) and is in fact a round Sierpiński carpet (being homeomorphic to the standard Sierpiński carpet follows from [6]).

Set $K_0 := (\partial_\infty G_m) \cap \mathbb{B}^2$, $\mathcal{G} := \{g \in \text{Möb}(\widehat{\mathbb{C}}) \mid g^{-1}(\infty) \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}\}$ and $K_g := g(K_0)$ for $g \in \mathcal{G}$. Also set $\mathcal{D}_g := \{gh(\widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}) \mid h \in G_m\} \setminus \{g(\widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2})\}$, so that $\mathcal{D}_g$ is a family of disjoint open disks in $\mathbb{C}$ with $K_g = g(\mathbb{B}^2) \setminus \bigcup_{D\in\mathcal{D}_g} D$. Now we adopt (1.2) as the *definition* of the Dirichlet form on K_g and similarly for the volume measure on K_g.

Definition 2.1 (K.)**.** Let $g \in \mathcal{G}$ and set $\mathcal{C}_g := \{u|_{K_g} \mid u : \mathbb{C} \to \mathbb{R},\ u \text{ is Lipschitz}\}$. We define a Borel measure ν^g on K_g and a symmetric bilinear form $\mathcal{E}^g : \mathcal{C}_g \times \mathcal{C}_g \to \mathbb{R}$ by

$$\nu^g := \sum\nolimits_{D\in\mathcal{D}_g} \text{rad}(\partial D)\text{vol}_{\partial D}, \ \ \mathcal{E}^g(u,v) := \sum\nolimits_{D\in\mathcal{D}_g} \text{rad}(\partial D) \int_{\partial D} \langle \nabla_{\partial D} u, \nabla_{\partial D} v \rangle \, d\text{vol}_{\partial D}.$$

Proposition 2.2 (K.)**.** *On $L^2(K_g, \nu^g)$, $(\mathcal{E}^g, \mathcal{C}_g)$ is closable and its closure $(\mathcal{E}^g, \mathcal{F}_g)$ is a strongly local regular Dirichlet form whose associated Laplacian has discrete spectrum.*

Since G_m is convex cocompact (hence Gromov hyperbolic), $d_m := \dim_{\text{H}} K_g \in (1,2)$ and $\mathcal{H}^{d_m}(K_g) \in (0,\infty)$ by [4, Theorem 7]. The following is the main result of this talk.

Theorem 2.3 (K.)**.** *There exists $c_m \in (0,\infty)$ such that for any $g \in \mathcal{G}$, the eigenvalues $\{\lambda_n^g\}_{n\in\mathbb{N}}$ (with each eigenvalue repeated according to its multiplicity) of the (non-negative definite) Laplacian associated with $(K_g, \nu^g, \mathcal{E}^g, \mathcal{F}_g)$ satisfy*

$$\lim\nolimits_{\lambda\to\infty} \lambda^{-d_m/2} \#\{n \in \mathbb{N} \mid \lambda_n^g \le \lambda\} = c_m \mathcal{H}^{d_m}(K_g). \tag{2.1}$$

Theorem 2.3 is proved by applying *Kesten's renewal theorem* [3, Theorem 2] to a certain Markov chain on the *space of "all possible Euclidean shapes of K_g"* defined by $H\backslash\mathcal{G} := \{Hg \mid g \in \mathcal{G}\}$, where H denotes the group of Euclidean similarities of $\mathbb{C}$.

References

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