多変量 ARMA 過程の有限予測係数に対する閉形式表示

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Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit circle and the closed unit disk, in $\mathbb{C}$, respectively. Let $d \in \mathbb{N}$. In this talk, a d-variate *ARMA* (autoregressive moving-average) process $\{X_k : k \in \mathbb{Z}\}$ is a $\mathbb{C}^d$-valued, centered, weakly stationary process with spectral density w of the form

$$w(e^{i\theta}) = h(e^{i\theta})h(e^{i\theta})^*, \qquad \theta \in [-\pi, \pi) \tag{1}$$

with $h : \mathbb{T} \to \mathbb{C}^{d\times d}$ satisfying the following condition:

$$\begin{aligned}&\text{the entries of } h(z) \text{ are rational functions in } z \text{ that have}\\ &\text{no poles in } \overline{\mathbb{D}}, \text{ and } \det h(z) \text{ has no zeros in } \overline{\mathbb{D}}.\end{aligned} \tag{C}$$

It is known that there exists $h_\sharp : \mathbb{T} \to \mathbb{C}^{d\times d}$ that satisfies (C) and

$$w(e^{i\theta}) = h(e^{i\theta})h(e^{i\theta})^* = h_\sharp(e^{i\theta})^* h_\sharp(e^{i\theta}), \qquad \theta \in [-\pi, \pi), \tag{2}$$

and $h_\sharp$ is unique up to a constant unitary factor. We may take $h_\sharp = h$ for the univariate case $d = 1$ but not so for $d \geq 2$, and this is one of the main difficulties when we deal with multivariate processes. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. We can write $h(z)^{-1}$ in the form

$$h(z)^{-1} = -\rho_0 - \sum_{\mu=1}^{K}\sum_{j=1}^{m_\mu} \frac{1}{(1-\overline{p}_\mu z)^j}\rho_{\mu,j} - \sum_{j=1}^{m_0} z^j \rho_{0,j}, \tag{3}$$

where

$$\begin{cases} K \in \mathbb{N} \cup \{0\}, \\ p_\mu \in \mathbb{D}\setminus\{0\} \quad (\mu = 1, \ldots, K), \quad p_\mu \neq p_\nu \quad (\mu \neq \nu), \\ m_\mu \in \mathbb{N} \quad (\mu = 1, \ldots, K), \quad m_0 \in \mathbb{N}\cup\{0\}, \\ \rho_{\mu,j} \in \mathbb{C}^{d\times d} \quad (\mu = 0, 1, \ldots, K,\ j = 1, \ldots, m_\mu), \quad \rho_0 \in \mathbb{C}^{d\times d}, \\ \rho_{\mu,m_\mu} \neq 0 \quad (\mu = 0, 1, \ldots, K). \end{cases} \tag{4}$$

Here the convention $\sum_{k=1}^{0} = 0$ is adopted in the sums on the right-hand side of (3).

The next theorem shows that $h_\sharp^{-1}$ of a vector ARMA process has the same m_0 and the same poles with the same multiplicities as h^{-1}.

Theorem 1. *For m_0, K and $(p_1, m_1), \ldots, (p_L, m_L)$ in (3) with (4), $h_\sharp^{-1}$ has the form*

$$h_\sharp(z)^{-1} = -\rho_0^\sharp - \sum_{\mu=1}^{K}\sum_{j=1}^{m_\mu} \frac{1}{(1-\overline{p}_\mu z)^j}\rho_{\mu,j}^\sharp - \sum_{j=1}^{m_0} z^j \rho_{0,j}^\sharp, \tag{5}$$

where

$$\begin{cases} \rho_{\mu,j}^\sharp \in \mathbb{C}^{d\times d} \quad (\mu = 0, 1, \ldots, K,\ j = 1, \ldots, m_\mu), \quad \rho_0^\sharp \in \mathbb{C}^{d\times d}, \\ \rho_{\mu,m_\mu}^\sharp \neq 0 \ (\mu = 0, 1, \ldots, K). \end{cases} \tag{6}$$

We are concerned with the *finite predictor coefficients* $\phi_{n,j} \in \mathbb{C}^{d\times d}$ $(j = 1, \dots, n)$ of a d-variate ARMA process $\{X_k\}$, defined by

$$P_{[-n,-1]}X_0 = \phi_{n,1}X_{-1} + \cdots + \phi_{n,n}X_{-n}, \tag{7}$$

where, for $n \in \mathbb{N}$, $P_{[-n,-1]}X_0$ stands for the best linear predictor of the future value X_0 based on the finite past $\{X_{-n}, \dots, X_{-1}\}$.

The next theorem gives a closed-form expression for $\phi_{n,j}$.

Theorem 2. *Suppose that $m_\mu = 1$ $(\mu = 1, \dots, K)$ and $m_0 = 0$. Then, for $n \geq 1$ and $j = 1, \dots, n$, we have*

$$\phi_{n,j} = c_0 a_j + c_0 \mathbf{p}_0^{\mathrm{T}} (I_{dM} - \tilde{G}_n G_n)^{-1} (\Pi_n \Theta)^* \{\Lambda^{\mathrm{T}} \Pi_n \Theta \Xi_j \rho + \Xi_{n-j+1} \tilde{\rho}\}, \tag{8}$$

where $a_j = \sum_{\mu=1}^{K} \overline{p}_\mu^j \rho_{\mu,1}$ for $j \geq 1$, $\mathbf{p}_0^{\mathrm{T}} = (I_d, \dots, I_d) \in \mathbb{C}^{d\times dK}$,

$$\Theta = \begin{pmatrix} p_1 h_\sharp(p_1)\rho_{1,1}^* & & & \text{\Large 0} \\ & p_2 h_\sharp(p_2)\rho_{2,1}^* & & \\ & & \ddots & \\ \text{\Large 0} & & & p_K h_\sharp(p_K)\rho_{K,1}^* \end{pmatrix} \in \mathbb{C}^{dK\times dK},$$

$$\Lambda = \begin{pmatrix} \frac{1}{1-p_1\overline{p}_1} I_d & \frac{1}{1-p_1\overline{p}_2} I_d & \cdots & \frac{1}{1-p_1\overline{p}_K} I_d \\ \frac{1}{1-p_2\overline{p}_1} I_d & \frac{1}{1-p_2\overline{p}_2} I_d & \cdots & \frac{1}{1-p_2\overline{p}_K} I_d \\ \vdots & \vdots & & \vdots \\ \frac{1}{1-p_K\overline{p}_1} I_d & \frac{1}{1-p_K\overline{p}_2} I_d & \cdots & \frac{1}{1-p_K\overline{p}_K} I_d \end{pmatrix} \in \mathbb{C}^{dK\times dK},$$

$$\Pi_n = \begin{pmatrix} p_1^n I_d & & & \text{\Large 0} \\ & p_2^n I_d & & \\ & & \ddots & \\ \text{\Large 0} & & & p_K^n I_d \end{pmatrix} \in \mathbb{C}^{dK\times dK}, \qquad n \geq 0,$$

$$\Xi_n = \begin{pmatrix} \frac{\overline{p}_1^n}{1-p_1\overline{p}_1} I_d & \frac{\overline{p}_2^n}{1-p_1\overline{p}_2} I_d & \cdots & \frac{\overline{p}_K^n}{1-p_1\overline{p}_K} I_d \\ \frac{\overline{p}_1^n}{1-p_2\overline{p}_1} I_d & \frac{\overline{p}_2^n}{1-p_2\overline{p}_2} I_d & \cdots & \frac{\overline{p}_K^n}{1-p_2\overline{p}_K} I_d \\ \vdots & \vdots & & \vdots \\ \frac{\overline{p}_1^n}{1-p_K\overline{p}_1} I_d & \frac{\overline{p}_2^n}{1-p_K\overline{p}_2} I_d & \cdots & \frac{\overline{p}_K^n}{1-p_K\overline{p}_K} I_d \end{pmatrix} \in \mathbb{C}^{dK\times dK}, \qquad n \geq 1,$$

$$\rho = (\rho_{1,1}^{\mathrm{T}}, \rho_{2,1}^{\mathrm{T}}, \dots, \rho_{K,1}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{C}^{dK\times d},$$

$$\tilde{\rho} = \left(\overline{\rho_{1,1}^\sharp}, \overline{\rho_{2,1}^\sharp}, \dots, \overline{\rho_{K,1}^\sharp}\right)^{\mathrm{T}} \in \mathbb{C}^{dK\times d}$$

and $G_n = \Pi_n \Theta \Lambda$, $\tilde{G}_n = (\Pi_n \Theta)^* \Lambda^{\mathrm{T}} \in \mathbb{C}^{dK\times dK}$.

The assumptions in Theorem 2 are just for simplicity of presentation. For the general result, see [1].

参考文献

[1] Inoue, A. (2018). Closed-form expression for finite predictor coefficients of vector ARMA processes, https://arxiv.org/pdf/1805.04820.pdf