A hypercontractive family of the Ornstein–Uhlenbeck semigroup and its connection with Φ-entropy inequalities*

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Given a positive integer d, let γ_d be the d-dimensional standard Gaussian measure. For every $p > 0$, define $L^p(\gamma_d)$ to be the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\|f\|_p^p := \int_{\mathbb{R}^d} |f(x)|^p \, \gamma_d(dx) < \infty$ (we abuse the notation even when $p < 1$). We denote by $Q = \{Q_t\}_{t \geq 0}$ the Ornstein–Uhlenbeck semigroup acting on $L^1(\gamma_d)$: for $f \in L^1(\gamma_d)$ and $t \geq 0$,

$$(Q_t f)(x) := \int_{\mathbb{R}^d} f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \gamma_d(dy), \quad x \in \mathbb{R}^d.$$

It is well known that Q enjoys the hypercontractivity: if $f \in L^p(\gamma_d)$ for some $p > 1$, then

$$\|Q_t f\|_{q(t)} \leq \|f\|_p \quad \text{for all } t \geq 0, \tag{HC}$$

where $q(t) = e^{2t}(p-1) + 1$. The hypercontractivity (HC) was firstly observed by Nelson [6] and found later by Gross [4] to be equivalent to the (Gaussian) logarithmic Sobolev inequality†:

$$\int_{\mathbb{R}^d} |f|^2 \log |f| \, d\gamma_d \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_d + \|f\|_2^2 \log \|f\|_2 \,, \tag{LSI}$$

which holds true for any weakly differentiable function f in $L^2(\gamma_d)$ with $|\nabla f| \in L^2(\gamma_d)$. It is also known (see [1, Proposition 4]) that (HC) is equivalent to the exponential hypercontractivity: for any $f \in L^1(\gamma_d)$ with $e^f \in L^1(\gamma_d)$, it holds that

$$\|\exp(Q_t f)\|_{e^{2t}} \leq \|e^f\|_1 \quad \text{for all } t \geq 0. \tag{eHC}$$

One of the objectives of this talk is to show, by employing stochastic analysis, that two hypercontractivities (HC) and (eHC) are unified into

Theorem 1 ([5], Theorem 1.1)**.** *Let a positive function c in $C^1((0,\infty))$ satisfy*

$$c' > 0 \text{ and } c/c' \text{ is concave on } (0,\infty), \tag{C}$$

and set

$$u(t,x) = \int_0^x c(y)^{e^{2t}} dy, \quad t \geq 0,\ x > 0. \tag{1}$$

Then for any nonnegative, measurable function f on $\mathbb{R}^d$ such that

$$u(0,f) \in L^1(\gamma_d),$$

we have

$$v\left(t, \|u(t,Q_t f)\|_1\right) \leq v\left(0, \|u(0,f)\|_1\right) \quad \text{for all } t \geq 0. \tag{uHC}$$

Here for every $t \geq 0$, the function $v(t,\cdot)$ is the inverse function of $u(t,x)$, $x > 0$.

Typical examples of functions c fulfilling the condition (C) are x^{p-1} with $p > 1$ and e^x; in fact, they both satisfy $(c/c')'' = 0$. These two choices of c in Theorem 1 lead to (HC) and (eHC), respectively. Another example of c will be given in the talk.

Recall the well-known fact that differentiating the left-hand side of (HC) at $t = 0$ yields (LSI); the same argument enables us to obtain from (uHC) the following generalization of (LSI):

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*This talk is based on [5].

†The Gaussian logarithmic Sobolev inequality goes back to Stam [7].

Corollary 1 ([5], Corollary 3.1)**.** *For a function c satisfying the assumptions in Theorem 1, set*

$$G(x) = \int_0^x c(y)\,dy \qquad \text{and} \qquad H(x) = \int_0^x c(y)\log c(y)\,dy$$

for $x > 0$. Then for any $f \in C_b^1(\mathbb{R}^d)$ with $\inf_{x\in\mathbb{R}^d} f(x) > 0$, we have

$$\int_{\mathbb{R}^d} H(f)\,d\gamma_d \le \frac{1}{2}\int_{\mathbb{R}^d} c'(f)\,|\nabla f|^2\,d\gamma_d + H\circ G^{-1}\left(\|G(f)\|_1\right). \tag{gLSI}$$

Here G^{-1} is the inverse function of G.

It is shown in [5, Proposition 3.3] that for every function f as in Corollary 1, the validity of (gLSI) for any positive $c \in C^1((0,\infty))$ satisfying (C), is necessary and sufficient for that of Φ-*entropy inequalities* for any $\Phi \in C^2((0,\infty))$ satisfying the condition that

$$\Phi'' > 0 \text{ and } 1/\Phi'' \text{ is concave on } (0,\infty). \tag{P}$$

In a general setting of *Markov triple* (E,μ,Γ) with associated Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ as treated in detail in [2, Chapters 4–7], the triple is said to satisfy the Φ-entropy inequality with constant $R > 0$ if

$$\int_E \Phi(f)\,d\mu - \Phi\left(\int_E f\,d\mu\right) \le \frac{R}{2}\int_E \Phi''(f)\Gamma(f,f)\,d\mu \tag{ΦI}$$

for any positive $f \in \mathcal{D}(\mathcal{E})$. In the setting of the present talk, namely in the case $E = \mathbb{R}^d$, $\mu = \gamma_d$ and $\Gamma(f,f) = |\nabla f|^2$, it is known (see, e.g., [3, Section 4]) that (ΦI) holds with $R = 1$ under the condition (P). In this talk, we show that in the present setting, the hypercontractive family (uHC) indexed by positive c's satisfying (C), is equivalent to the family of Φ-entropy inequalities (ΦI) with $R = 1$ indexed by Φ's satisfying (P). The same reasoning applies to the general setting of Markov triple as well. For instance, if a probability measure μ on $E = \mathbb{R}^d$ is in the form $\mu(dx) = e^{-V(x)}dx$ with $V \in C^2(\mathbb{R}^d)$ whose Hessian matrix satisfies $y\cdot\mathrm{Hess}_V(x)y \ge \rho|y|^2$, $x, y \in \mathbb{R}^d$, for some $\rho > 0$, then the Φ-entropy inequality (ΦI) for $\Gamma(f,f) = |\nabla f|^2$ is known (cf. [3, Corollary 2.1]) to hold with $R = 1/\rho$, which entails that (uHC) holds true for the semigroup generated by $\Delta - \nabla V\cdot\nabla$, with exponent e^{2t} in (1) replaced by $e^{2\rho t}$.

If time permits, we will also present a unification of (eHC) and the reverse hypercontractivity of the Ornstein–Uhlenbeck semigroup Q.

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